

Efficiency of Basis Elements and Fuzzy Weighted Graph

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Abstract

Graph theory can be applied in the planning and scheduling of large complicated projects. The nodes of graph are taken as the milestones that are uniquely placed in the work place. Many problems of such work place depends on the efficiency of the nodes in the respective work place. This paper explains how to augment the efficiency of the devices used at each nodes using fuzzy set theory

Keywords: Fuzzy set, fuzzy weighted graph, maximum efficiency, level set, alpha

1 Introduction

Graph theory has been proved useful in solving complex real life problems, in particular the constituents of those problems themselves are complex in nature. Moreover, many of these systems are dynamic and with incomplete information where various parameters in the systems are uncertain or vague. Hence, such complex dynamic systems is worth investigating. In this paper, we attempt to model complex dynamic systems with several work spaces where uncertainty present in various constituents using graph theoretical tools. The

work place are taken as nodes in a graph and edges the connection between them. We use the concept of metric dimension for solving the optimization of complex networks. We uniquely place Robots (machines) with the assumption that it can sense the shortest path between nodes. The optimum number of robots for locating each node in the work place is known as metric dimension [9, 5, 3]. The set with minimum robots is called metric basis. Networks are composed of finitely (infinitely) many nodes and edges connecting between them. Navigation of Robots plays a cardinal role in solving such complex networks. Mechanical efficiency of a robot is determined by the rate at which the energy and power(input) is transferred into force and movement(output). The efficiency of robots can be calculated with respect to two quantities namely measured performance and the performance of an ideal machine.

$$Efficiency = \frac{Measuredperformance}{Idealperformance} = \frac{Mechanicaladvantage}{velocity} \times 100$$

A frictionless machine would have an efficiency of 100 percentage. But in real life situation a machine cannot achieve 100 percentage efficiency. The terms like young, tall, good or high are fuzzy and the degree of the uncertainty may itself be capable of rigorous expression. There is an extent of uncertainty. Thus fuzzy mathematics tools be better suitor. L.A Zadeh [10] presented a mathematical structure to explain the concept of uncertainty in real life.

2 Preliminaries

Definition 2.1.

Let G be a network. In order to locate each node in the network an optimum number of robots are uniquely placed at junctions(nodes) is required. this number is called metric dimension and the set with minimum number of landmarks(robots)is termed as metric basis or minimum resolving set.

Definition 2.2. Notation

Let $W = \{v_1, v_2, \dots, v_k\}$ be an ordered set of vertices of a graph G and let v be a vertex of G . Then $r(v/W) = (d(v, w_1), d(v, w_2), \dots, d(v, w_k))$. Clearly $r(v/W)$ is an element in R^k .

Definition 2.3. *Suppose each vertex in G has distinct co-ordinates corresponding to W then W is termed as resolving set. If W_1, W_2, \dots, W_m are resolving sets of G and if cardinality of W_r is the least among all other cardinality of $W - i$ for $i = 1, 2, \dots, m$ then W_r is called the basis of G and cardinality of W_r is the metric dimension of G , denoted by $dim(G)$ or $\beta(G)$.*

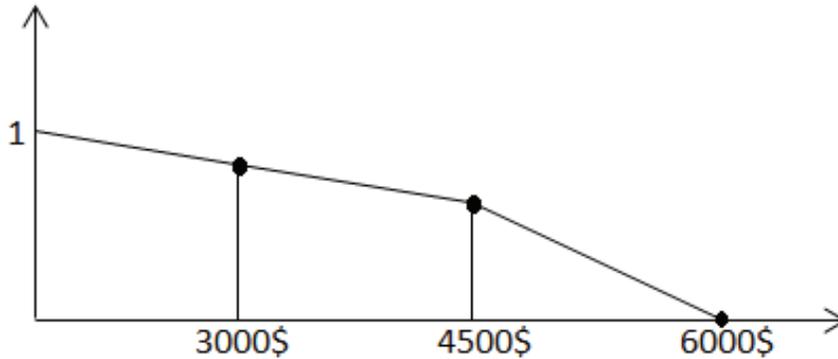


Figure 1:

Definition 2.4. [10]

Let X be a non-empty set. A fuzzy set A in X is $A = \{(x, \mu_A(x)) \text{ for } x \in X$ where $\mu_A : X \rightarrow [0, 1]$ is taken as the degree of membership of the element x in fuzzy set A for each $x \in X$. An example is shown in the following figure:1.

Definition 2.5. [10]

Let A be a fuzzy subset of X , the support of A is the set of all x in X such that $A(x) > 0$.

Definition 2.6.

A fuzzy subset A of a classical set X is called normal if there exists $x \in X$ such that $A(x) = 1$, otherwise A is subnormal.

Definition 2.7.

Given $\alpha \in (0, 1]$, the α -cut of a fuzzy set A on X is denoted by A_α and is defines as $A_\alpha = \{x \in X / A(x) \geq \alpha\}$.

Let A_1 and A_2 are fuzzy subsets of a classical set X . We say that (i) A_1 is a subset of A_2 if $A_1(t) \leq A_2(t), \forall t \in X$, (ii) intersection of A_1 and A_2 is $(A_1 \cap A_2)(t) = \min\{A_1(t), A_2(t)\} = A_1(t) \wedge A_2(t)$, (iii) union of A_1 and A_2 is $(A_1 \cup A_2)(t) = \max\{A_1(t), A_2(t)\} = A_1(t) \vee A_2(t)$ and (iv) the complement of a fuzzy set A is defined as $\neg A(t) = 1 - A(t)$.

3 Known Results

Theorem 3.1. [4] Let B be a matrix obtained by subtracting adjacency matrix $[a_{ij}]$ of G from a $n \times n$ matrix $[c_{ij}]$ obtained by putting the vertex degrees on the diagonals. Then the number of spanning trees is equal to $\det(B)$.

Theorem 3.2. [4]:*Matrix Tree Theorem*

Let G be a graph without loops and with vertex set $\{v_1, v_2, \dots, v_n\}$. Let a_{ij} be the number of edges with end points v_i and v_j and C be the matrix with entry (i, j) is $-a_{ij}$ when $i \neq j$ and is $d(v_i)$ when $i = j$. Construct a matrix C^* by deleting row 's' and column 't' of C then the number of spanning trees $\tau(G) = (-1)^{s+t} \det(C^*)$.

Theorem 3.3. [4] :*Decomposition theorem* Any fuzzy set A on X can be decomposed as $A = \text{Sup}\{\alpha A_\alpha / 0 < \alpha \leq 1\}$. Then we write $A = \sum \alpha A_\alpha$ or $A = \bigcup \alpha A_\alpha$ where $A(x) = \text{Sup}\{\alpha A_\alpha(x) / 0 < \alpha \leq 1\}$ for every $x \in X$.

4 Definitions

Definition 4.1. Fuzzy weighted graph and a fuzzy set A of W .

We associate a real positive number to the edge $e = (v_i, v_j)$ called weight (efficiency) $\mu(v_i)$ for each of the basis elements $v_i \in W = \{v_1, v_2, \dots, v_m\}$ if v_j is identified by v_i and satisfying the following cases:

Case 1: Let $e = (v_j, v_k)$, v_j, v_k not belonging to W , coordinate of $v_j = \{d(v_j, v_1), d(v_j, v_2), \dots, d(v_j, v_m)\}$ and $\text{Min}\{d(v_j, v_1), d(v_j, v_2), \dots, d(v_j, v_m)\} = (v_j, v_l)$ implies that v_j is identified by the basis element v_l . Let the coordinate of $v_k = \{d(v_k, v_1), d(v_k, v_2), \dots, d(v_k, v_m)\}$ and $\text{Min}\{d(v_k, v_1), d(v_k, v_2), \dots, d(v_k, v_m)\} = (v_k, v_h)$ implies that v_k is identified by v_h . Then $w(e) = \frac{\mu(v_l) + \mu(v_h)}{2}$.

Case 2: Let $e = (v_j, v_k)$ be a pendant edge and v_j, v_k not belonging to W with $\text{deg}(v_k) = 1$. If $\text{Min}\{d(v_j, v_1), d(v_j, v_2), \dots, d(v_j, v_m)\} = (v_j, v_l)$ then $w(e) = \mu(v_l)$.

Case 3: Let $e = (v_j, v_k)$ be an edge such that $v_j, v_k \in W$, then $w(e) = \text{Max}\{\mu(v_j), \mu(v_k)\}$.

Case 4: Let $e = (v_j, v_k)$ be an edge such that $v_j \in W$ and v_k not belonging to W , then $w(e) = \mu(v_j)$. The fuzzy weighted graph is denoted by $(G, V, \mu_w(v))$

Note: If alternative minima occurred then choose that basis element with greater efficiency.

Illustration: Consider C_4 with $W = (v_1, v_2)$ and $A = \{\mu(v_1), \mu(v_2)\} = \{1.0.5\}$ is the fuzzy set of W where $\mu(v_1)$ and $\mu(v_2)$ represents the efficiency of the basis elements. The following figure:2 gives a weighted fuzzy graph with respect to W and A .

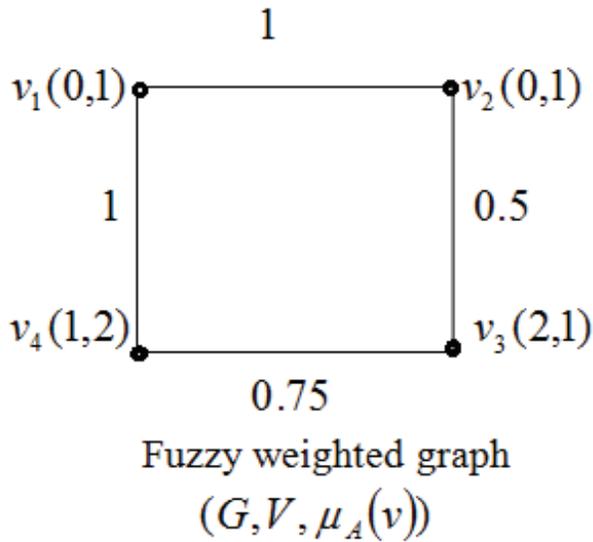


Figure 2: Fuzzy weighted graph

Definition 4.2.

The total efficiency of $(G, V, \mu_w(v))$ of the graph G is the product of the weight of each edge in the graph.

A path is said to be of maximum length if the product of weights of all edges in that path is greater than the product of weight of all the edges in other paths.

Definition 4.3.

A maximum spanning tree of $(G, V, \mu_w(v))$ is spanning tree such that the underlying crisp graph is a tree and such that $\prod_{v \in W} \mu_A(v)$ is maximum.

Definition 4.4.

Let $W = \{v_1, v_2, \dots, v_k\}$ be the basis and $A = \{\mu(v_1), \mu(v_2), \dots, \mu(v_k)\}$ be the fuzzy set of W . Then corresponding to each α -cut we can define a sub graph of $(G, V, \mu_W(v))$ as follows: For $A_{\mu(v_k)}$, the graph itself is the sub graph. For $A_{\mu(v_{k-1})}$ the efficiency of the basis element is zero in $A_{\mu(v_{k-1})}$ and so the sub graph G with respect to $A_{\mu(v_{k-1})}$ is obtained by considering all the edges traversed by the basis elements v_1, v_2, \dots, v_{k-1} and no edge with weight $\mu(v_k)$

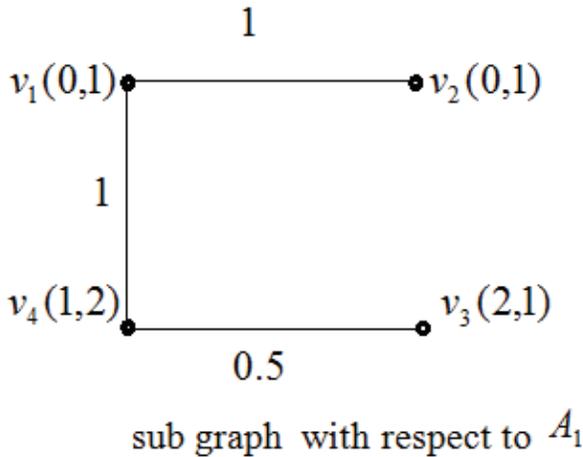


Figure 3: subgraph with respect to A_1

is included. Similarly we can define sub graphs with respect to the other α -cuts.

In the figure:3, $A_1 = \{1, 0\}$, the sub graph with respect to A_1 is given below:

Here the weight of the edge from v_4 to v_3 is $\frac{\mu(v_1) + 0}{2} = 0.5$

5 Main Results

The following theorem explains how to find Robotic assignment spanning tree with maximum efficiency.

Theorem 5.1. *Let G be a connected weighted graph with a basis W with m elements and n vertices. Then with respect to W there exists a RASS with maximum efficiency.*

Proof: *First find the Robotic assignment spanning sub graph H with respect to the basis $W = \{v_1, v_2, \dots, v_m\}$. The graph H can be connected or disconnected. If it is disconnected, let G_1, G_2, \dots, G_k are the components of H . Choose any basis element v_i in W and $d(v_i, v_j) = l = 1$ for $v_j \in V(G)$ not in W . Now add the edges with maximum efficiency from v_i to v_j for each $v_i \in W$ corresponding to $l = 1$. If v_j is identified by more than one basis element through an edge then choose the edge with maximum efficiency for each $v_i \in W$ in such a way that no cycle is formed in H . Now for $d(v_i, v_j) = l = 2$ add all those paths traversed by each basis element with maximum efficiency so that no cycle is formed. Similarly corresponding to each positive l we can enumerate $n - 1$ edges from the given graph in H . Thus H is a spanning tree with maximum efficiency.*

Theorem 5.2. *Let G be a connected weighted graph and T_1 and T_2 be two Robotic assignment spanning trees with respect to the basis W . Let e be an edge in T_1 and not in T_2 then we can find another edge f in T_2 not in T_1 such that the sub graphs $(T_1 \setminus e) \cup f$ and $(T_2 \setminus e) \cup e$ are also Robotic assignment spanning trees.*

Proof: Let $(G, V, \mu_W(v))$ be a fuzzy weighted graph with respect to W and a fuzzy set A of W . Where $W = \{v_1, v_2, \dots, v_k\}$ is the metric basis, $V(G) = \{v_1, v_2, \dots, v_n\}$ is the vertex set and $E(G) = \{e_1, e_2, \dots, e_m\}$, $m \geq n; k \leq n$. Let T_1 and T_2 be two Robotic assignment spanning trees of G . Then T_1 and T_2 has n vertices and $n - 1$ edges. Let e_i be an edge in T_1 which is not in T_2 where $e_i \in E(G)$ for some $i = 1, 2, \dots, m$ and e_j be an edge in T_2 not in T_1 where $e_j \in (EG)$ for some $j \neq i$. We claim that $(T_1 \setminus e_i) \cup e_j$ is also a Robotic spanning tree.

Clearly, e_i and e_j cannot be bridges. For otherwise both $e_i, e_j \in E(T_1), E(T_2)$. This is a contradiction to the given hypothesis. Since e_i is not a bridge, $(T_1 \setminus e_i)$ is a connected acyclic graph with $n - 2$ edges and n vertices. There are two cases:

Case 1: If e_i is a pendant edge then it has one vertex in common with e_j , otherwise $(T_1 \setminus e_i)$ contains $n - 1$ vertices. So $(T_1 \setminus e_i) \cup e_j$ will not be a spanning tree, contradictory to the hypothesis. Let the common vertex of e_i and e_j be v_j . Take $e_i = (v_i, v_j)$ and $e_j = (v_j, v_k)$. Now $T_1 \setminus e_i$ is a tree containing $n - 1$ vertices and between any two vertices there is one and only one path, $(T_1 \setminus e_i) \cup e_j$ is connected and contains no cycle since e_j not belonging to $E(T_1 \setminus e_i)$ Hence $(T_1 \setminus e_i) \cup e_j$ is the Robotic assignment spanning tree containing n vertices.

Case 2: Suppose e_i is not a pendant edge (not a bridge of G). Then $T_1 \setminus e_i$ is a disconnected graph having two components since there is only one path between any two vertices in $T_1 \setminus e_i$. Let U_1 and U_2 be the components of $T_1 \setminus e_i$. Clearly U_1 and U_2 are trees as $T_1 \setminus e_i$ is a tree. Take $|V(U_1)| = n_1$ and $|V(U_2)| = n_2$ such that $n_1 + n_2 = n$. Let $e_j = (v_j, v_k)$ then $v_j \in U_1$ and $v_k \in U_2$ for otherwise $e_j \in T_1$ which is not possible. Therefore $(T_1 \setminus e_i) \cup (v_j, v_k)$ will be a connected graph containing n vertices. Hence $(T_1 \setminus e_i) \cup e_j$ is a Robotic assignment spanning tree.

By a similar argument we can prove that $(T_2 \setminus e_j) \cup e_i$ is another Robotic spanning tree of G . Hence the result.

Theorem 5.3. *Let $(G, V, \mu_W(v))$ be a connected fuzzy weighted graph with respect to W for which at least two basis elements are adjacent and let A be a fuzzy set of W . Then for the level set of A there exist at least one α -cut of A and a corresponding sub graph for which basis elements traversed less number of $E(G)$ edges to identify all the vertices of G .*

Proof: Without loss of generality assume that $\mu(v_i) > 0$ for every $v_i \in W$ where $W = \{v_1, v_2, \dots, v_k\}$ is the basis and $V(G) = \{v_1, v_2, \dots, v_n\}$, $k \leq n$ is the

vertex set. Let $|E(G)| = e$ and v_k is the basis element adjacent to at least one vertex in W .

Let $A = \{\mu(v_1), \mu(v_2), \dots, \mu(v_k)\}$ be a fuzzy set where $\mu(v_i)$ is the efficiency of each basis element. Clearly $L(A) = A$ since $\mu(v_i) > 0$ and $0 < \mu(v_i) \leq 1$. Assume that $\mu(v_1) < \mu(v_2) \leq \dots \leq \mu(v_k)$. For any $l = 1, \dots, k$, $Ca(v_i) \neq 0$ for $i = 1, 2, \dots, k$, since the graph is connected. That means for $l = 1, \dots, k$, $Ca(v_i) > 0$ or every basis element is adjacent with more than one vertices in $V(G)$. Now consider distinct α -cuts of A with respect to W namely $A_{\mu_{v_1}}, A_{\mu_{v_2}}, \dots, A_{\mu_{v_k}}$. Since $\mu(v_i) > 0$ for $i = 1, 2, \dots, k$, $A_{\mu_{v_k}} = (1, 1, \dots, 1)$. Then the resulting sub graph with respect to $A_{\mu_{v_k}}$ is the graph itself. Now for $A_{\mu_{v_{k-1}}} = (1, 1, \dots, 0)$ the resulting sub graph will contain all edges traversed by the basis elements v_1, v_2, \dots, v_{k-1} such that weight of each edge remains at a positive level.

Claim: Since G is connected v_k must be connected with some vertices in $V(G)$. For $l = 1$, if $Ca(v_k) = r$ then v_k is adjacent with r vertices in $V(G)$. Then the resulting sub graph with respect to α -cut $A_{\mu_{v_{k-1}}}$ will not contain r edges through which v_k is adjacent with r vertices. By the hypothesis v_k is adjacent to one of the vertices in $W = \{v_1, v_2, \dots, v_k\}$ say v_i where $\mu(v_i) < \mu(v_k)$ for $i = 1, 2, \dots, k-1$. Thus, $w(v_k, v_i) = \text{Max}\{\mu(v_k), \mu(v_i)\} = \mu(v_i)$. So the resulting sub graph with respect to $A_{\mu_{v_{k-1}}}$ is connected and the number of edges traversed by the basis elements is less than or equal to $e - r + 1$. Hence the theorem.

6 Conclusions

In this paper we studied with the fuzzy set theoretical tools to describe the irregularity of devices used in large networks. Also the efficiency of the nodes in the network has been increased.

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