Schur-type Theorems for $k$-Triangular Lattice Group-Valued Set Functions with Respect to Filter Convergence

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Abstract

We prove some Schur and limit theorems for lattice group-valued $k$-triangular set functions with respect to filter convergence, by means of sliding hump-type techniques. As consequences, we deduce some Vitali-Hahn-Saks and Nikodým-type theorems. Furthermore, we pose some open problems.

Keywords: (filter) $(D)$-convergence, $k$-triangular set function, limit theorem

1 Introduction

In the literature there have been many studies about limit theorems for set functions, with values in abstract structures. A comprehensive survey can be found, for instance, in [3] (see also its bibliography). Here we deal with some versions of these kinds of theorems for $k$-triangular set functions with values in a lattice group $R$ (for an overview about the case $R = \mathbb{R}$ see also [9, 12] and their bibliographies). In [1,7] it is dealt with the $M$-measures, that is monotone set functions $m$ with $m(\emptyset) = 0$, continuous from above and from below and compatible with respect to finite suprema and infima, which are a particular class of 1-triangular set functions and
have several applications, for example to intuitionistic fuzzy events and observables (see also [1,10]). Observe that there are 1-triangular set functions which are not necessarily monotone, for instance the measuroids (see also [11]). In the context of k-subadditive and/or k-triangular lattice group-valued set functions, in [5] some Schur, Vitali-Hahn-Saks and Nikodým convergence theorems are proved with respect to the classical \((D)\)-convergence, while in [4] some limit theorems are given with respect to filter convergence. Here we extend earlier results to the setting of k-triangular set functions and filter convergence, by using sliding hump-type techniques. For technical reasons, we use \((D)\)-convergence, since by virtue of the Fremlin Lemma it is possible to replace a sequence of regulators with a single \((D)\)-sequence. Finally, we pose some open problems.

2 Preliminaries

We begin with recalling the following basic notions (for an overview, see [3] and its bibliography).

Let \(Q\) be a countable set. A filter \(\mathcal{F}\) of \(Q\) is a nonempty collection of subsets of \(Q\) with \(\emptyset \notin \mathcal{F}\), \(A \cap B \in \mathcal{F}\) whenever \(A, B \in \mathcal{F}\), and such that \(B \in \mathcal{F}\) whenever \(A \in \mathcal{F}\) and \(B \supset A\).

A filter of \(Q\) is free iff it contains the filter \(\mathcal{F}_{\text{cofin}}\) of all cofinite subsets of \(Q\). Given a free filter \(\mathcal{F}\) of \(Q\), we say that a subset of \(Q\) is \(\mathcal{F}\)-stationary iff it has nonempty intersection with every element of \(\mathcal{F}\). We denote by \(\mathcal{F}^*\) the family of all \(\mathcal{F}\)-stationary subsets of \(Q\).

If \(I \in \mathcal{F}^*\), then the trace \(\mathcal{F}(I)\) of \(\mathcal{F}\) on \(I\) is the family \(\{F \cap I : F \in \mathcal{F}\}\). Note that \(\mathcal{F}(I)\) is a filter of \(I\).

A free filter \(\mathcal{F}\) of \(Q\) is said to be diagonal iff for every sequence \((A_n)_n\) in \(\mathcal{F}\) and each \(I \in \mathcal{F}^*\) there exists a set \(J \subset I, J \in \mathcal{F}^*\) such that \(J \setminus A_n\) is finite for any \(n\).

Given an infinite set \(I \subset Q\), a blocking of \(I\) is a countable partition \(\{D_h : h \in \mathbb{N}\}\) of \(I\) into nonempty finite subsets.

A filter \(\mathcal{F}\) of \(Q\) is block-respecting iff for every \(I \in \mathcal{F}^*\) and for each blocking \(\{D_h : h \in \mathbb{N}\}\) of \(I\) there is a set \(J \in \mathcal{F}^*, J \subset I\) with \(\#(J \cap D_h) = 1\) for any \(h\), where \(\#\) denotes the number of elements of the set into brackets (see also [2]).

A Dedekind complete lattice group \(R\) is super Dedekind complete iff for every nonempty set \(A \subset R\), bounded from above, there is a finite or countable subset with the same supremum as \(A\).

A bounded double sequence \((a_{t,l})_{t,l}\) in \(R\) is a \((D)\)-sequence or a regulator iff \((a_{t,l})_l\) is a decreasing sequence and \(\bigwedge_l a_{t,l} = 0\) for any \(t \in \mathbb{N}\).

A lattice group \(R\) is weakly \(\sigma\)-distributive iff \(\bigwedge_{\varphi \in \mathbb{N}^h} (V_{t=1}^\infty a_{t,\varphi(t)}) = 0\) for every \((D)\)-sequence \((a_{t,l})_{t,l}\).
A sequence \((x_n)_n\) in \(R\) is \((D)\)-convergent to \(x\) if and only if there is a \((D)\)-sequence \((a_{t,l})_{t,l}\) in \(R\) such that for every \(\phi \in \mathbb{N}^\mathbb{N}\) there is \(n^* \in \mathbb{N}\) with \(|x_n - x| \leq V_{t=1}^{\infty} a_{t,\phi(t)}\) whenever \(n \geq n^*\), and we write \((D)\lim_n x_n = x\).

Some examples of super Dedekind complete and weakly \(\sigma\)-distributive lattice groups are the space \(\mathbb{N}^\mathbb{N}\) of all permutations of \(\mathbb{N}\) endowed with the usual componentwise order and the space \(L^0(X,\mathcal{B},\nu)\) of all \(\nu\)-measurable functions defined on a measure space \((X,\mathcal{B},\nu)\) with the identification up to \(\nu\)-null sets, where \(\nu\) is a positive, \(\sigma\)-additive and \(\sigma\)-finite extended real-valued measure, endowed with almost everywhere convergence.

A sequence \((x_n)_n\) in \(R\) \((DF)\)-converges to \(x \in R\) if there is a \((D)\)-sequence \((a_{t,l})_{t,l}\) with \(\{n \in \mathbb{N}: |x_n - x| \leq V_{t=1}^{\infty} a_{t,\phi(t)}\} \in \mathcal{F}\) for all \(\phi \in \mathbb{N}^\mathbb{N}\). Observe that, when \(R = \mathbb{R}\), the \((DF)\)-convergence coincides with the usual filter convergence. Moreover, when \(\mathcal{F} = \mathcal{F}_{cof}\), \((DF)\)- and \((D)\)-convergence are equivalent.

We now deal with some basic properties of lattice group-valued set functions. From now on, \(R\) is a Dedekind complete lattice group, \(G\) is an infinite set, \(\Sigma\) is a \(\sigma\)-algebra of subsets of \(G\), \(m: \Sigma \to R\) is a set function and \(k\) is a fixed positive integer.

The semivariation of \(m\) is defined by \(v(m)(A) = \bigvee\{|m(B)|: B \in \Sigma, B \subset A\}\).

We say that \(m\) is \(k\)-triangular (on \(\Sigma\)) iff
\[
\begin{align*}
\text{m}(A) - k \text{ m}(B) & \leq \text{m}(A \cup B) \\
& \leq \text{m}(A) + k \text{ m}(B) \quad \text{whenever } A, B \in \Sigma, A \cap B = \emptyset
\end{align*}
\]  

and \(0 = \text{m}(\emptyset) \leq \text{m}(A)\) for each \(A \in \Sigma\).

Given a set function \(m: \Sigma \to R\) and an algebra \(\mathcal{L} \subset \Sigma\), the semivariation of \(m\) with respect to \(\mathcal{L}\) is defined by \(v_\mathcal{L}(m)(A) = \bigvee\{|m(B)|: B \in \mathcal{L}, B \subset A\}\). Note that \(v(m) = v_\Sigma(m)\).

A set function \(m: \Sigma \to R\) is continuous from above at \(\emptyset\) if and only if for every decreasing sequence \((H_n)_n\) in \(\Sigma\) with \(\bigcap_{n=1}^{\infty} H_n = \emptyset\) we get \(\bigwedge_n v_\mathcal{L}(m)(H_n) = 0\), where \(\mathcal{L}\) is the \(\sigma\)-algebra generated by \((H_n)_n\) in \(H_1\).

Note that there exist continuous from above at \(\emptyset\) and \(1\)-triangular set functions, which are neither increasing nor finitely additive (see also [5, Example 2.1]).

A topology \(\tau\) on \(\Sigma\) is a Fréchet-Nikodým topology if the functions \((A, B) \mapsto A \Delta B\) and \((A, B) \mapsto A \cap B\) from \(\Sigma \times \Sigma\) (endowed with the product topology) to \(\Sigma\) are continuous, and for any \(\tau\)-neighborhood \(V\) of \(\emptyset\) in \(\Sigma\) there exists a \(\tau\)-neighborhood \(U\) of \(\emptyset\) in \(\Sigma\) such that, if \(E \in \Sigma\) is contained in some suitable element of \(U\), then \(E \in V\).

Let \(\tau\) be a Fréchet-Nikodým topology on \(\Sigma\). A set function \(m: \Sigma \to R\) is \(\tau\)-continuous on \(\Sigma\) if for each decreasing sequence \((H_n)_n\) in \(\Sigma\), with \(\tau\)-lim \(H_n = \emptyset\), we get \(\bigwedge_n v_\mathcal{L}(m)(H_n) = 0\), where \(\mathcal{L}\) denotes the \(\sigma\)-algebra generated by the sets \(H_n, n \in \mathbb{N}\), in \(H_1\).
We say that the set functions $m_j: \Sigma \rightarrow R$, $j \in \mathbb{N}$, are equibounded iff there is $u \in R$ with $|m_j(A)| \leq u$ for all $j \in \mathbb{N}$ and $A \subset \Sigma$.

The following results will be useful in the sequel.

**Lemma 2.1.** (see also [3, Theorem II.2.2 and Lemma II.2.23]) Let $R$ be any Dedekind complete lattice group, $(a_{j,n})_{j,n}$ be a double sequence in $R$ and $F$ be a diagonal filter of $\mathbb{N}$. If $\lim_{j \in \mathbb{N}} a_{j,n} = 0$ for each $n \in \mathbb{N}$ with respect to a single regulator $(b_{t,l})_{t,l}$ (independent of $n$), then there is a $(D)$-sequence $(c_{t,l})_{t,l}$ (independent of $l$) such that for any $l \in F^*$, there is $J \in F^*$, $J \subset I$, with $(D) \lim_{j \in J} a_{j,n} = 0$ for any $n \in \mathbb{N}$ with respect to $(c_{t,l})_{t,l}$.

**Proposition 2.2.** (see also [5, Proposition 2.3]) If $m: \Sigma \rightarrow R$ is $k$-triangular, then $v(m)$ is $k$-triangular too. Moreover, for any $n \geq 2$ and for every pairwise disjoint sets $E_1, E_2, ..., E_n \in \Sigma$ we get

$$m(E_1) - k \sum_{q=2}^{n} m(E_q) \leq m \left( \bigcup_{q=1}^{n} E_q \right) \leq m(E_1) + k \sum_{q=2}^{n} m(E_q).$$

**Lemma 2.3.** (Fremlin Lemma, see also [8, Lemma 1C], [10, Theorem 3.2.3]) Let $R$ be any Dedekind complete $(\ell)$-group and $(a_{t,l}^{(n)})_{t,l,n} \in \mathbb{N}$, be a sequence of regulators in $R$. Then for any $u \in R$, $u \geq 0$ there is a regulator $(a_{t,l})_{t,l}$ in $R$ with

$$u \land \left( \bigcup_{n=1}^{\infty} \left( \bigvee_{n=1}^{\infty} a_{t,\varphi(t+l+n)}^{(n)} \right) \right) \leq \bigvee_{t=1}^{\infty} a_{t,\varphi(t)}^{(n)} \ \text{for every} \ q \in \mathbb{N} \ \text{and} \ \varphi \in \mathbb{N}^{\mathbb{N}}.$$

**Proposition 2.4.** (see also [3, Proposition IV.1.9]) Let $R$ be a Dedekind complete lattice group, $m: \Sigma \rightarrow R$ be a set function, $(H_n)_{n}$ be a decreasing sequence in $\Sigma$, and set $C_n = H_n \setminus H_{n+1}$, $n \in \mathbb{N}$. For any $A \subset \mathbb{N}$ put $v(A) = m(\bigcup_{n \in A} C_n)$. Let $\mathcal{L}$ be the $\sigma$-algebra generated by the $H_n$‘s in $\mathcal{H}$ and assume that $\bigwedge_n v_{\mathcal{L}}(m)(H_n) = 0$. Then $v$ is continuous from above at $\emptyset$, and for any $n \in \mathbb{N}$ it is

$$v(v([n, +\infty[) = \bigvee (|v(B)|; B \subset [n, +\infty[) \leq \bigvee (|m(C)|; C \in \mathcal{L} \text{ with } C \subset H_n) = v_{\mathcal{L}}(m)(H_n)$$

**3 The main results**

We begin with a Schur-type theorem, which extends [5, Theorem 3.2] and [6,
Lemma 3.1 and Theorem 3.1] to $k$-triangular set functions.

**Theorem 3.1.** Let $R$ be any Dedekind complete ($\ell$)-group, $\mathcal{F}$ be a block-respecting filter of $\mathbb{N}$, $m_j: \mathcal{P}(\mathbb{N}) \to R$, $j \in \mathbb{N}$, be a sequence of continuous from above at $\emptyset$ and equibounded $k$-triangular set functions, and set $\beta_{A,j} := m_j(A)$, $A \in \mathcal{P}(\mathbb{N})$, $j \in \mathbb{N}$. Suppose that:

3.1.1) $(D) \lim_j m_j([n]) = 0$ for any $n \in \mathbb{N}$;

3.1.2) the family $(\beta_{A,j})_{A \in \mathcal{P}(\mathbb{N})}, j \in \mathbb{N}$ $(RDF)$-converges to 0.

Then we get:

3.1.3) $(DF) \lim_j v(m_j)(\mathbb{N}) = 0$.

3.1.4) If $\mathcal{F}$ is also diagonal and $R$ is super Dedekind complete and weakly $\sigma$-distributive, then 3.1.2) implies 3.1.3).

**Proof:** 3.1.3). Let $u := \bigvee_{j \in \mathbb{N}} v(m_j)(\mathbb{N})$. For each $j \in \mathbb{N}$ let $(a_{t,l}^{(j)})_{t,l}$ be a $(D)$-sequence related with continuity from above at $\emptyset$ of $m_j$ and the sequence $(H_n)_n$, where $H_n := [n, +\infty[\cup \mathbb{N}$. For each $\varphi \in \mathbb{N}^\mathbb{N}$ and $j \in \mathbb{N}$ there is $\bar{n} \in \mathbb{N}$ (depending on $\varphi$ and $j$) with

$$v(m_j)([\bar{n}, +\infty[) \leq \bigvee_{t=1}^{\infty} a_{t,\varphi(t)+j}^{(j)}.$$  

By Lemma 2.3 there exists a $(D)$-sequence $(a_{t,l})_{t,l}$ with

$$u \land \left( \bigvee_{q} \left( \bigvee_{j=1}^{q} \left( \bigvee_{t=1}^{\infty} a_{t,\varphi(t)+j}^{(j)} \right) \right) \right) \leq \bigvee_{t=1}^{\infty} a_{t,\varphi(t)}.$$  

for every $\varphi \in \mathbb{N}^\mathbb{N}$. From (4) and (5) it follows that for each $\varphi \in \mathbb{N}^\mathbb{N}$ and $j \in \mathbb{N}$ there exists $\bar{n} \in \mathbb{N}$ with

$$v(m_j)([\bar{n}, +\infty]) \leq \bigvee_{t=1}^{\infty} a_{t,\varphi(t)}.$$  

Let $(b_{t,l})_{t,l}$ be a regulator, satisfying the condition of $(RDF)$-convergence as in 3.1.2). For each $n \in \mathbb{N}$, since $(D) \lim_j m_j([n]) = 0$, there is a $(D)$-sequence $(c_{t,l}^{(n)})_{t,l}$ such that for every $\varphi \in \mathbb{N}^\mathbb{N}$ there is $j \in \mathbb{N}$ with $m_j([n]) \leq \bigvee_{t=1}^{\infty} c_{t,\varphi(t)+n}$ whenever $j \geq \bar{j}$. Thanks to equiboundedness of the $m_j$’s and Lemma 2.3, arguing analogously as in (5), we find a regulator $(c_{t,l})_{t,l}$ such that for every $\varphi \in \mathbb{N}^\mathbb{N}$ and $n \in \mathbb{N}$ there exists $\bar{j} \in \mathbb{N}$ with $m_j(L) \leq k \sum_{n \in L} m_j([n]) \leq k \bigvee_{t=1}^{\infty} c_{t,\varphi(t)}$ whenever $j \geq \bar{j}$ and for any finite subset $L \subset [1, n]$. Set now $d_{t,l} = 5k^2 (a_{t,l} + b_{t,l} + k c_{t,l})$, $t, l \in \mathbb{N}$. We claim that the $(D)$-sequence $(d_{t,l})_{t,l}$ satisfies the condition of $(DF)$-
convergence as in 3.1.3). Otherwise there exists \( \varphi \in \mathbb{N}^\mathbb{R} \) such that \( I^* := \{ j \in \mathbb{N} : v(m_j)(\mathbb{N}) \leq V_{t=1}^\mathbb{R} d_{t,\varphi(t)} \} \notin \mathcal{F} \). From this it follows that every element \( F \) of \( \mathcal{F} \) has nonempty intersection with \( \mathbb{N} \setminus I^* \). This means that the set \( I := \mathbb{N} \setminus I^* \) is \( \mathcal{F} \)-stationary. Note that \( I \) is an infinite set, since \( \mathcal{F} \) is a free filter. Let \( n_0 := 1, a := V_{t=1}^{\mathbb{R}} a_{t,\varphi(t)}, b := V_{t=1}^{\mathbb{R}} b_{t,\varphi(t)}, c := V_{t=1}^{\mathbb{R}} c_{t,\varphi(t)}, d := V_{t=1}^{\mathbb{R}} d_{t,\varphi(t)} \). By continuity from above at \( \emptyset \) of \( m_1 \), there is an integer \( l(1) > 1 \) with \( v(m_1)([l(1), +\infty[) \leq a \). Furthermore, thanks to 3.1.1, there is an integer \( n_1 > l(1) \) with \( m_s(L) \leq k c \) for any \( s \geq n_1 \) and for each finite subset \( L \subset [1, l(1)] \), and thus \( v(m_3)([1, l(1)]) \leq k c \) for all \( s \geq n_1 \).

At the next step, by continuity from above at \( \emptyset \) of \( m_1, ..., m_{n_1} \), there is \( l(n_1) > n_1 \) with \( v(m_r)([l(n_1), +\infty[) \leq a \) for every \( r \leq n_1 \), and by 3.1.1 we find \( n_2 > l(n_1) \) with \( v(m_2)([1, l(n_1)]) \leq k c \) whenever \( s \geq n_2 \). By induction, we find two strictly increasing sequences \( (n_h)_h \) and \( (l(n_h)_h \); such that, for every \( h \in \mathbb{N} \), \( n_{h-1} < l(h) < n_h \), \( v(m_h)([l(n_h), +\infty[) \leq a \) whenever \( r \leq n_h \), and \( v(m_2)([1, l(n_h)]) \leq k c \) for any \( s \geq n_{h+1} \). Without loss of generality, we can choose the \( n_h \)'s with \( I \cap [n_{h-1}, n_h] \neq \emptyset \) for every \( h \in \mathbb{N} \), forming a blocking of \( I \). So there is a set \( J \in \mathcal{F}^* \), \( J \subset I \), \( J = \{ j_0, j_1, j_2, ... \} \), such that \( J \) intersects each interval \( [n_h, n_{h+1}] \) in exactly one point. As \( J \in \mathcal{F}^* \), then at least one of the two sets \( J_1 := \{ j_1, j_3, j_5, ... \} \) and \( J_2 := \{ j_0, j_2, j_4, ... \} \) is \( \mathcal{F} \)-stationary. Without loss of generality, we assume that \( J_1 \in \mathcal{F}^* \).

For every \( h \in \mathbb{N} \), we have
\[
v(m_{j_{2h-1}})([l(n_{2h}), +\infty[) \leq a, \quad v(m_{j_h})([1, l(n_{2h-2})]) \leq k c. \quad (7)
\]
From this, as \( v(m_{j_{2h-1}})(\mathbb{N}) \neq d \), for every \( h \) we get
\[
v(m_{j_{2h-1}})([l(n_{2h-2}), l(n_{2h})]) \neq k (a + b + k c); \quad (8)
\]
otherwise, from (7), (8) and Proposition 2.2 used with \( q = 3 \), \( E_1 = [1, l(n_{2h-2})], E_2 = [l(n_{2h-2}), l(n_{2h})], E_3 = [l(n_{2h}), +\infty[ \), we should have \( v(m_{j_{2h-1}})(\mathbb{N}) \leq d \), a contradiction. By (8) there is \( Q_h \subset [l(n_{2h-2}), l(n_{2h})] \) with
\[
m_{j_{2h-1}}(Q_h) \leq k (a + b + k c). \quad (9)
\]
Note that the \( Q_h \)'s are pairwise disjoint. Set \( H := \bigcup_{h=1}^\mathbb{N} Q_h \). By 3.1.2, there exists \( F \in \mathcal{F} \) with \( m_j(H) \leq b \) for all \( j \in F \), and, since \( J_1 \) is \( \mathcal{F} \)-stationary, there is \( j_1 \in F \cap J_1 \). Let \( h_1 \in \mathbb{N} \) be such that \( j_{2h_1-1} = j_1 \). We have
\[
H = Q_{h_1} \cup (H \cap [l(n_{2h_1}), +\infty[) \cup (H \cap [1, l(n_{2h_1-2})]). \quad (10)
\]
From (10) and (2) used with \( q = 3 \), \( E_1 = Q_{h_1}, E_2 = H \cap [l(n_{2h_1}), +\infty[, E_3 = H \cap [1, l(n_{2h_1-2})] \), we obtain \( m_{j_1}(Q_{h_1}) \leq k (a + b + k c) \), which contradicts (9). So we get 3.1.3.
3.1.4. Let \((b_{t,l})_{t,l}\) be a \((D)\)-sequence, satisfying \((RDF)\)-convergence in 3.1.2). It is easy to see that, for every \(\mathcal{F}\)-stationary set \(J\), the regulator \((b_{t,l})_{t,l}\) satisfies also \((RDF(J))\)-convergence in 3.1.2). Since \(\mathcal{R}\) is super Dedekind complete and weakly \(\sigma\)-distributive, by Lemma 2.1 we find a regulator \((c_{t,l})_{t,l}\), such that for every \(l \in \mathcal{F}^*\) there is \(J \in \mathcal{F}^*, J \subset I\), with

\[
(RD) \lim_{j \in J} m_j([n]) = 0, \quad n \in \mathbb{N},
\]

with respect to \((c_{t,l})_{t,l}\). Let now \((a_{t,l}^{(j)})_{t,l}\), \(n \in \mathbb{N}\), be regulators associated to continuity from above at \(\emptyset\) of the \(m_j\)'s, and \(u\), \((a_{t,l})_{t,l}\), be as in the proof of 3.1.3). Set \(d_{t,l} := 5 k^2(a_{t,l} + b_{t,l} + k c_{t,l})\), \(t, l \in \mathbb{N}\). We claim that the regulator \((d_{t,l})_{t,l}\) satisfies 3.1.4). Otherwise, by proceeding analogously as in the proof of 3.1.3), we find an element \(\varphi \in \mathbb{N}^\mathbb{N}\) and an \(\mathcal{F}\)-stationary set \(I \subset \mathbb{N}\), with \(v(m_j)(\mathbb{N}) \leq \bigvee_{t=1}^\infty d_{t,\varphi(t)}\) for every \(j \in I\). In correspondence with \(I\), there is \(J \in \mathcal{F}^*, J \subset I\), satisfying (11). Note that the sequence \(v(m_j)(\mathbb{N}), j \in J\), does not \((\mathcal{F}(J))\)-converge to \(0\). Moreover, since \(J \in \mathcal{F}^*\) and \(\mathcal{F}\) is block-respecting, then \(\mathcal{F}(J)\) is block-respecting too (see also [2, 3]). By 3.1.3) used with \(m_j, j \in J\), and \(\mathcal{F}(J)\), it follows that \((\mathcal{F}(J)) \lim_{j \in J} v(m_j)(\mathbb{N}) = 0\), getting a contradiction. This proves 3.1.4).

\(\square\)

The following Vitali-Hahn-Saks-type theorem is a consequence of Theorem 3.1 and extends [5, Theorem 3.3] and [6, Theorem 4.2].

**Theorem 3.2.** Let \(\mathcal{R}\) be a super Dedekind complete and weakly \(\sigma\)-distributive \((\mathcal{L})\)-group, \(\mathcal{F}\) be a diagonal and block-respecting filter of \(\mathbb{N}\), \(\tau\) be a Fréchet-Nikodým topology on \(\Sigma\), \(m_j: \Sigma \to \mathcal{R}, j \in \mathbb{N}\), be a sequence of equibounded, \(\tau\)-continuous and \(k\)-triangular set functions. Assume that the family \(m_j(A), A \in \Sigma, j \in \mathbb{N}\), \((RDF)\)-converges to \(0\). Then for each decreasing sequence \((H_n)_n\) in \(\Sigma\) with \(\tau\-\lim H_n = \emptyset\) and for any \(\mathcal{F}\)-stationary set \(I \subset \mathbb{N}\) there is an \(\mathcal{F}\)-stationary set \(J \subset I\), with

\[
\bigwedge_n \left[ \bigvee_{j \in J} v_{L}(m_j)(H_n) \right] = 0,
\]

where \(\mathcal{L}\) is the \(\sigma\)-algebra generated by the \(H_n\)'s in \(H_1\).

**Proof:** Put \(C_n = H_n \setminus H_{n+1}\), \(n \in \mathbb{N}\). Since \(m_j\) is \(\tau\)-continuous for every \(j = 0, 1, \ldots\), then \(\bigwedge_n v_{L}(m_j)(H_n) = \bigwedge_n \bigvee_{C} v(m_j)(C)\), \(C \in \mathcal{L}\) with \(C \subset H_n\) = 0. For any \(A \in \mathcal{P}(\mathbb{N})\) and \(j = 0, 1, \ldots\), define \(v_j(A) = m_j(\bigcup_{n \in A} C_n)\). By Proposition 2.4, the \(v_j\)'s are continuous from above at \(\emptyset\). The equiboundedness of the \(v_j\)'s and \((RDF)\)-convergence of the family \(v_j(A), A \in \mathcal{P}(\mathbb{N})\), \(j \in \mathbb{N}\), to 0, follow easily from the equiboundedness of the \(m_j\)'s and \((RDF)\)-convergence of the family \(m_j(A), A \in \Sigma, j \in \mathbb{N}\), to 0, respectively. From Theorem 3.1 and (3) it follows that for every \(\mathcal{F}\)-stationary set \(I \subset \mathbb{N}\) there exists an \(\mathcal{F}\)-stationary set \(J \subset I\), satisfying (12). \(\square\)
Analogously as in Theorem 3.2 it is possible to prove the following Nikodým convergence-type theorem, extending [5, Theorem 3.4] and [6, Theorem 3.2].

**Theorem 3.3.** Let $R, \mathcal{F}, \Sigma, \mathcal{L}$ be as in Theorem 3.2, $m_j: \Sigma \to R, j \in \mathbb{N}$, be a sequence of equibounded $k$-triangular set functions, continuous from above at $\emptyset$. Suppose that the family $m_j(A), A \in \Sigma, j \in \mathbb{N}$, $(RD\mathcal{F})$-converges to 0. Then for each decreasing sequence $(H_n)_n$ in $\Sigma$ with $\bigcap_{n=1}^{\infty} H_n = \emptyset$ and for every $\mathcal{F}$-stationary set $I \subset \mathbb{N}$ there exists an $\mathcal{F}$-stationary set $J \subset I$, with $\bigwedge_n [\bigvee_{j \in J} \nu_l(m_j)(H_n)] = 0$.

**Open problems:** (a) Find some versions of limit theorems for $k$-triangular set functions, in which the limit measure is not necessarily required to be continuous from above at $\emptyset$.
(b) Prove similar results for other classes of filters and/or set functions.
(c) Prove some limit theorems with respect to other kinds of continuity and/or semivariation.

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**References**


Schur-type theorems for $k$-triangular lattice group-valued set functions


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