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An Option Pricing with Nonlinear Payoff

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Abstract

In this paper we show how pricing an exotic call option with nonlinear payoff. The payoff is given by the square root of the difference between the price asset S and the strike K, in a given time T. The pricing of this option is based on the method used by Fisher Black and Myron Scholes to calculate the price of an european call option. The price obtained is expressed in terms of a non-elementary function. Finally, we compare two derivatives a basic option and the exotic option presented here.

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1 Introduction

According to the definition given in [2], [5] or [7] a european stock option (plain-vanilla option) is a contract which gives the owner the right, but not the obligation to buy or sell the stock (underlying asset) by a specified price K (exercise price) for a specified date T (maturity date), regardless of the underlying asset price S at that time. A call option gives you the right to buy a stock, while a put option gives you the right to sell it. Currently a large

variety of these derivatives are traded in many specialized markets. The best known is the CBOE (Chicago Board Options Exchange).
The complexity of these financial derivatives requires the use of mathematical tools to develop the models that allow to find the theoretical price "fair" or premium to be paid by the buyer. This is widely known as option pricing model.This is usually obtained by analytical and numerical methods based mainly on the classic Black-Scholes model [1].
If we make some modifications to a vanilla option, we get a exotic option. In [8] we can see a deep analysis of how these powerful instruments for risk management among other uses, can be valued in spite of the complexity of them. Such changes are due to many factors such as market volatility, the type of underlying asset and the different positions between who buys the contract (the buyer) and who subscribes (the writer). Consequently, one of these contracts is presented here, based on a European call option, where the linear payoff $S-K$ for the non-linear $\sqrt{S-K}$ is changed.

2 The model

Let us consider that we have a European call option on an asset, which fulfills all the hypotheses and requirements of the Black-Sholes model [1] for valuation, except that the payoff or final payment at the expiration time T is given by:

$$C(S,T) = \begin{cases} \sqrt{S-K} & if \quad S \geq K \\ \\ 0 & if \quad S < K \end{cases} \tag{1}$$

Where, $C(S,t)$ is the premium or value of the option, with price of the underlying $S \geq 0$, strike (strike price) $K > 0$ at time $t \geq 0$. With this final condition for $C(S,T)$ we have an exotic option that has the following model:

$$\left.\begin{array}{l} \frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} - rC = 0 \\ \\ S \in [0,\infty), t \in [0,T], C(0,t) = 0 \\ \\ C(S,T) = \begin{cases} \sqrt{S-K} & if \quad S \geq K \\ 0 & if \quad S < K \end{cases} \end{array}\right\} \tag{2}$$

Where r is a positive constant representing the risk-free rate and σ also positive constant represents the volatility of the underlying asset.

3 Mathematical model solution

To solve the problem given in (2), we started making the substitutions:

$$x = ln\left(\frac{S}{K}\right), \quad \tau = \frac{1}{2}\sigma^2(T-t), \quad v(x,\tau) = \frac{C(S,t)}{\sqrt{K}} \tag{3}$$

so, we have:

$$S = Ke^x, \ \ t = T - \frac{2\tau}{\sigma^2}, \ \ C(S,t) = \sqrt{K}v(x,\tau) \tag{4}$$

but now that $\tau(T) = 0$, the boundary condition in (2) becomes the initial condition:

$$v(x,0) = \begin{cases} \sqrt{e^x - 1} & if \quad x \geq 0 \\ 0 & if \quad x < 0 \end{cases} \tag{5}$$

If we also do:

$$\lambda = \frac{2r}{\sigma^2} \tag{6}$$

the problem (2) becomes:

$$\left.\begin{array}{l} \frac{\partial v}{\partial \tau} - \frac{\partial^2 v}{\partial x^2} - (\lambda - 1)\frac{\partial v}{\partial x} + \lambda v = 0 \\ \\ x \in \mathbb{R}, \ \ \tau \in \left[0, T\frac{\sigma^2}{2}\right] \\ \\ v(x,0) = \begin{cases} \sqrt{e^x - 1} & if \quad x \geq 0 \\ 0 & if \quad x < 0 \end{cases} \end{array}\right\} \tag{7}$$

Now we can substitute $v(x,\tau) = e^{\alpha x + \beta \tau}u(x,\tau)$ where α y β must be chosen appropriately to simplify the model, this leads us to do $\alpha = -\frac{\lambda-1}{2}$, and $\beta = -\frac{(\lambda+1)^2}{4}$, so the equation becomes:

$$v(x,\tau) = e^{-\frac{\lambda-1}{2}x - \frac{(\lambda+1)^2}{4}\tau}u(x,\tau) \tag{8}$$

and the initial condition of (7) becomes:

$$u(x,0) = u_0(x) = \begin{cases} \sqrt{e^{\lambda x} - e^{(\lambda-1)x}} & if \quad x \geq 0 \\ 0 & if \quad x < 0 \end{cases} \tag{9}$$

Thus, we have the model that corresponds to 1D heat(or Diffusion) equation, see [4].

$$\left.\begin{array}{l} \frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} \\ \\ x \in \mathbb{R}, \quad \tau \in \left[0, T\frac{\sigma^2}{2}\right] \\ \\ u(x,0) = u_0(x) = \left\{ \begin{array}{lll} \sqrt{e^{\lambda x} - e^{(\lambda-1)x}} & if & x \geq 0 \\ 0 & if & x < 0 \end{array} \right. \end{array} \right\} \tag{10}$$

A known solution of the diffusion equation is the convolution between the initial condition and the fundamental solution:

$$u(x,\tau) = \frac{1}{2\sqrt{\pi\tau}} \int_{-\infty}^{\infty} u_0(z) e^{-\frac{(x-z)^2}{4\tau}} dz \tag{11}$$

Substituting the λ parameter, the variables x, τ in addition to considering the constraint of $u_0(x)$, which cancels out for $x < 0$ we have:

$$u(S,t) = A(t) \int_0^{\infty} \sqrt{e^{\frac{2r}{\sigma^2}z} - e^{\frac{2r-\sigma^2}{\sigma^2}z}}\, e^{-\frac{\left(ln\left(\frac{S}{K}\right)-z\right)^2}{2\sigma^2(T-t)}} dz \tag{12}$$

where $A(t)$ represents:

$$A(t) = \frac{1}{2\sqrt{\pi\tau}} = \frac{\sqrt{2}}{2\sigma\sqrt{\pi(T-t)}} \tag{13}$$

Thus,

$$v(S,t) = e^{-\frac{2r-\sigma^2}{2\sigma^2} ln\left(\frac{S}{K}\right) - \frac{1}{8}\left(\frac{2r+\sigma^2}{\sigma^2}\right)^2 \sigma^2 (T-t)} u(S,t) \tag{14}$$

Finally, from (4) and (14) the premium or the price $C(S,t)$ for the option given by the model is:

$$C(S,t) = \sqrt{K} v(S,t) \tag{15}$$

4 Numerical results

Let us discuss the numerical results we obtained by implementing of the method we developed in Sections 2 and 3 (equation 15), in Symbolic and numerical software Maple 11. For further discussion of numerical methods in partial differential equations for exotics options, see [3] and [6]. The results of a first example are shown in the table below.

Premium C(12,t)		
t	exotic call option	European vanilla call option
0.0	0.6334	0.8982
0.1	0.6135	0.8435
0.2	0.5921	0.7866
0.3	0.5689	0.7271
0.4	0.5433	0.6645
0.5	0.5148	0.5981
0.6	0.4823	0.5264
0.7	0.4438	0.4475
0.8	0.3954	0.3373
0.9	0.3262	0.2451
1.0	0.0000	0.0000

Table 1: Approximations of the option price C of an European vanilla and exotic call options with $T = 1$, $K = 12$, $S = 12$, $r = 0.03$, $\sigma = 0.15$ and $t \in [0, 1]$.

Now, let's consider a second example:

Premium C(S,0)		
S	exotic call option	European vanilla call option
10.0	0.1188	0.1307
10.5	0.2054	0.2406
11.0	0.3225	0.4025
11.5	0.4672	0.6216
12.0	0.6324	0.8982
12.5	0.8128	1.2279
13.0	0.9974	1.6033
13.5	1.1802	2.0155
14.0	1.3562	2.4559

Table 2: Approximations of the option price C of an European vanilla and exotic call options with $T = 1$, $t = 0$, $K = 12$, $r = 0.03$, $\sigma = 0.15$ and $S \in [10, 14]$.

The Table 1 shows summary results of the first example. As we can observe the price of the premium for the exotic option is less than that of the vanilla option. But as the time t approaches the time of expiration $T = 1$ this difference is diminished and even very close to the end the premium of the vanilla option is greater than that of the exotic option. This is the payoff effect given in the equation (1) which is affected by a square root.

We can also observe Table 2, which corresponds to the second example of the same phenomenon, but now given that the table shows the initial price of the premium, we see more markedly the lower premium of the exotic option with respect to the vanilla option.

5 Conclusions

Without these models, the existence of these financial products would be impossible. Great utility represents this instrument for fund managers, market speculators and general investors, who can buy a cheap option although limited coverage, but for the writer allows a lower degree of risk. However, it should be clarified that the method only indicates a theoretical price, that in the markets this is only one of the tools that must be taken into account, and that the theoretical model implies conditions that are not always fulfilled in a real environment . In solving the problem (2) we obtain the theoretical value of the exotic option and we can practically model any financial derivative by complex and solve it with methods similar to those used in this paper.

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