Applied Mathematical Sciences, Vol. 11, 2017, no. 7, 331 - 343
HIKARI Ltd, www.m-hikari.com
https://doi.org/10.12988/ams.2017.612287

A Multi-Period Mean-Variance Analysis for Portfolio Tracking Error

Wanderlei Lima de Paulo

Post-Graduate Business Administration Program
Faculty of Campo Limpo Paulista, Campo Limpo Paulista (SP), Brazil

Yeison Andrés Zabala

Department of Telecommunications Engineering and Control
University of São Paulo, São Paulo (SP), Brazil

Oswaldo Luiz do Valle Costa

Department of Telecommunications Engineering and Control
University of São Paulo, São Paulo (SP), Brazil

Abstract

In this paper we deal with a discrete-time multi-period mean-variance portfolio tracking error problem. By tracking error we mean the difference between the value of a managed portfolio and a benchmark portfolio, obtained from a pre-specified investment strategy. The goal is to analytically derive an optimal control policy for this mean-variance tracking error problem in a multi-period set up, generalizing the uni-period case considered in Roll [7]. In particular it is shown that, as in uni-period case, the muti-period minimum tracking error variance frontier is a constant translation of the multi-period minimum variance frontier.

Mathematics Subject Classification: 93E20, 37N40, 91B28

Keywords: Mean-variance criterion, Tracking error, Multi-period

1 Introduction

A classical financial problem, known as mean-variance optimization, is the one in which it is desired to reduce risks by diversifying assets allocation. The seminal work of Markowitz [6] paved the foundation for what is nowadays known as the modern portfolio selection uni-period problem. Since then the research on the mean-variance approach to portfolio selection has increased in order to provide financial models with more realistic assumptions (see for example Ling and Tang [5], Alexander and Baptista [1], Costa and Paiva [2] and Roll [7]).

The multi-period mean-variance problem was tackled in Ni and Ng [4], where the authors provided an analytical optimal portfolio policy and an analytical expression of the mean-variance efficient frontier. Recently there has been a continuing effort in extending portfolio selection from the uni-period to the multi-period case under different formulations (see for example Yin and Zhou [8], Leippold, Trojani and Vanini [3], Xiao and Liu [9], Zhang and Gao [10]). In particular, Xiao and Liu [9] study a multi-period mean-variance portfolio selection based on a benchmark process in a discrete-time framework. The authors derive the optimal portfolio and the mean-variance efficient frontier in closed form.

Differently from Xiao and Liu [9], in this paper we present an analytical solution for three kinds of mean-variance tracking error analysis when all assets are risky and when one of the assets is riskless. These optimal solutions reduce to the analytical solution of the uni-period mean-variance problem. Furthermore, we show that the minimum tracking error variance frontier is a constant translation of the general minimum variance frontier. These results show that the multi-period mean-variance portfolio tracking error problem considered in this paper can be viewed as a generalization of the analytical work of Roll [7] which was derived within the uni-period mean-variance formulation.

2 Problem formulation

Throughout the paper we will denote by $\mathbb{R}^n$ the n-dimensional Euclidean real space and by $\mathbb{R}^{n\times m}$ the Euclidean space of all $n \times m$ real matrices. The vector formed by $1's$ in all its components will be denoted by e. The superscript "$'$" will denote the transpose of a vector or matrix. We will consider a financial market with $n+1$ risky assets on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathcal{P})$. The assets' price will be described by the random vector $\mathcal{S}(t) = (\mathcal{S}_0(t), \ldots, \mathcal{S}_n(t))'$ taking values in $\mathbb{R}^{n+1}$ with $t = 0, \ldots, T$. The filtration $\mathcal{F}_t$ is such that the random vectors $\{\mathcal{S}(k); k = 0, \ldots, t\}$ are $\mathcal{F}_t$-measurable. Set $\mathcal{R}_i(t) = S_i(t+1)/S_i(t)$, $R(t) = (\mathcal{R}_0(t) \quad \mathcal{R}(t))'$ and $\mathcal{R}(t) = (\mathcal{R}_1(t) \ldots \mathcal{R}_n(t))'$. We have that $R(t)$ can be written as $R(t) = \bar{\eta}(t) + Z(t)$, where $Z(t)$ are null mean vectors and $\bar{\eta}(t) \in \mathbb{R}^{n+1}$ represent the mean value of $R(t)$. We write $\bar{\eta}(t) = (\eta_0(t) \quad \eta(t))'$, $\eta(t) = (\eta_1(t) \dots \eta_n(t))'$ and make the following assumptions: $\{Z(t); t = 0, \dots, T-1\}$ are independent random vectors and $E(R(t)R(t)') > 0$ for each $t = 0, \dots, T-1$.

The set of admissible investment strategies $\mathcal{U} = \{u = (u(0), \dots, u(T-1))\}$ is such that for each $u(t) = (u_1(t), \dots, u_n(t))'$, is a $\mathcal{F}_t$-measurable random vector taking values in $\mathbb{R}^n$. We have that $u(t)$ represents the amount of the wealth allocated among the n assets. We define $u_0(t)$ as the amount of the wealth invested in the reference asset 0. Associated to each admissible investment strategy u we have the portfolio's value process $\{V^u(t); t = 0, \dots, T-1\}$, which represents the investor's wealth at time t. For notational simplicity, we will suppress the superscript ^{u} whenever no confusion may arise. We must have at each time t that

$$V(t) = u_0(t) + e'u(t). \tag{1}$$

Assuming that the initial wealth $V(0) = V_0 > 0$ and that the portfolio is self-financed, we have from the wealth process in (1) that

$$V(t+1) = \mathcal{R}_0(t)u_0(t) + \mathcal{R}(t)'u(t) = \mathcal{R}_0(t)V(t) + \mathcal{P}(t)'u(t), \tag{2}$$

where

$$\mathcal{P}(t) = \mathcal{R}(t) - \mathcal{R}_0(t)e. \tag{3}$$

Note that the amount of wealth invested in the asset 0 is determined by $V(t) - e'u(t)$.

We denote by $u_B(t) = \left(u_1^B(t), \dots, u_n^B(t)\right)'$, the amount of the wealth allocated among the n assets by a benchmark portfolio, and its value by $V_B(t)$. We assume that the benchmark portfolio is determined by the investment strategy given by

$$u_B(t) = F(t)V_B(t) + G(t) \tag{4}$$

for pre-specified n dimensional vectors $F(t)$ and $G(t)$, $t = 0, \dots, T-1$. Notice that by making $G(t) = 0$ in (4) we get the special case in which the proportion of the portfolio invested on each asset i is given by the vector $F(t) = \frac{u_B(t)}{V_B(t)}$, that is, the i^{th} element $F_i(t)$ of $F(t)$ represents the weight on the asset i, $i = 1, \dots, n$. Notice also that the investment strategy as in (4) includes the multi-period optimal mean-variance portfolio policy as presented in equation (40) of Li and Ng [4]. Similarly as in (1), (2) and (3) we get that

$$V_B(t+1) = \mathcal{R}_0(t)V_B(t) + \mathcal{P}(t)'u_B(t), \text{ with } V_B(0) = V_0. \tag{5}$$

Defining the portfolio tracking error $X(t) = V(t) - V_B(t)$ we get from (2) and

(5) that

$$X(t+1) = \mathcal{R}_0(t)X(t) + \mathcal{P}(t)'U(t), \tag{6}$$
$$X(0) = 0, \tag{7}$$
$$U(t) = u(t) - u_B(t). \tag{8}$$

Note that $X(t)$ represents the difference (or exchange) between the managed portfolio value $V(t)$ and the benchmark portfolio value $V_B(t)$, so that $U(t)$ represents the portfolio deviation (or portfolio alteration) and $u(t)$ represents the managed portfolio (here called tracking portfolio). The mean-variance tracking problems to be investigated are defined as follows:

Problem $\boldsymbol{PE(\sigma)}$:

$$\begin{aligned}
&\text{maximize } E[X(T)] \\
&\text{subject to} \\
&\qquad Var[X(T)] \leq \sigma^{\mathbf{2}}, \\
&\qquad X(t+1) = \mathcal{R}_0(t)X(t) + \mathcal{P}(t)'U(t), \\
&\qquad t = 0, \ldots, T-1.
\end{aligned}$$

Problem $\boldsymbol{PV(}\chi\boldsymbol{)}$:

$$\begin{aligned}
&\text{minimize } Var[X(T)] \\
&\text{subject to} \\
&\qquad E[X(T)] \geq \chi, \\
&\qquad X(t+1) = \mathcal{R}_0(t)X(t) + \mathcal{P}(t)'U(t), \\
&\qquad t = 0, \ldots, T-1.
\end{aligned}$$

An alternative problem would be the following, for $\rho > 0$:

Problem $\boldsymbol{PMV(\rho)}$:

$$\begin{aligned}
&\text{maximize } E[X(T)] - \rho Var[X(T)] \\
&\text{subject to} \\
&\qquad X(t+1) = \mathcal{R}_0(t)X(t) + \mathcal{P}(t)'U(t), \\
&\qquad t = 0, \ldots, T-1,
\end{aligned}$$

where ρ represents an aversion risk coefficient, so that the bigger ρ is, the more aversion to the risk the investor will be.

3 Solution of the Problems

Define $\phi(t) = E[\mathcal{P}(t)\mathcal{P}(t)']$, $\varphi_0^2(t) = E[\mathcal{R}_0(t)^2]$ and $\varphi(t) = E[\mathcal{R}_0(t)\mathcal{P}(t)]$. From (3) we have $\mathcal{P}(t) = \mathcal{R}(t) - \mathcal{R}_0(t)e = (-e \quad I)R(t)$ and since $E(R(t)R(t)') > 0$ it follows that

$$\begin{pmatrix} \varphi_0^2(t) & \varphi(t)' \\ \varphi(t) & \phi(t) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -e & I \end{pmatrix} E[R(t)R(t)'] \begin{pmatrix} 1 & -e' \\ 0 & I \end{pmatrix} > 0 \tag{9}$$

and therefore, $\phi(t) = E[\mathcal{P}(t)\mathcal{P}(t)'] > 0$. Applying the Schur's complement in (9) we obtain that $\varphi_0^2(t) - \varphi(t)'\phi(t)^{-1}\varphi(t) > 0$. Notice also that

$$\begin{pmatrix} 1 & (\eta(t) - \eta_0(t)e)' \\ (\eta(t) - \eta_0(t)e)' & \phi(t) \end{pmatrix} = E\Big[\begin{pmatrix} 1 \\ \mathcal{P}(t) \end{pmatrix} \begin{pmatrix} 1 & \mathcal{P}(t)' \end{pmatrix} \Big] \geq 0 \tag{10}$$

and applying the Schur's complement in (10) we obtain that $(\eta(t) - \eta_0(t)e)'\phi(t)^{-1}(\eta(t) - \eta_0(t)e) \leq 1$. We set $A_2(t) = \varphi_0^2(t) - \varphi(t)'\phi(t)^{-1}\varphi(t) > 0$, $A_1(t) = \eta_0(t) - (\eta(t) - \eta_0(t)e)'\phi(t)^{-1}\varphi(t)$ and $\mathcal{B}(t) = (\eta(t) - \eta_0(t)e)'\phi(t)^{-1}(\eta(t) - \eta_0(t)e) \leq 1$. Furthermore, we introduce the following backward recursive variables: $\Gamma_2(t) = A_2(t)\Gamma_2(t+1) > 0$, with $\Gamma_2(T) = 1$, $\Gamma_1(t) = A_1(t)\Gamma_1(t+1)$, with $\Gamma_1(T) = 1$, and $\Gamma_0(t) = \Gamma_0(t+1) + \frac{\Gamma_1(t+1)^2}{\Gamma_2(t+1)}\mathcal{B}(t)$, with $\Gamma_0(T) = 0$. Finally, we also set $\mathcal{C} = \frac{1}{2}\sum_{t=0}^{T-1}\left(\frac{\Gamma_1(t+1)^2}{\Gamma_2(t+1)}\right)\mathcal{B}(t)$ and $a = \frac{\mathcal{C}}{2} - \mathcal{C}^2$.

3.1 The Case with Only Risky Assets

We present in this subsection the solution of the problems $PMV(\rho)$, $PV(\chi)$ and $PE(\sigma)$ when all assets are risky, based on the results presented in Ni and Ng [4]. In fact, the equation for the dynamic evolution of the portfolio in Ni and Ng [4] is the same as in (6), the only difference being the initial condition, which in our case is given by (7), and the control $u(t)$, which is obtained from (8). Notice that the value of the benchmark portfolio $V_B(t)$ would be available since it follows a pre-specified strategy given by (4). For problem $PE(\sigma)$ the optimal solution is given by (see Ni and Ng [4], equations (24), (25) and (26), with initial portfolio value $x_0 = 0$)

$$u(t) = -K(t)(V(t) - V_B(t)) + \frac{\sigma}{\sqrt{a}}\vartheta(t) + u_B(t) \tag{11}$$

and for problem $PV(\chi)$ the optimal solution is given by

$$u(t) = -K(t)(V(t) - V_B(t)) + \frac{\chi}{\mathcal{C}}\vartheta(t) + u_B(t). \tag{12}$$

For problem $PMV(\rho)$ we obtain that the optimal control is given by (see Ni and Ng [4], equations (22) and (23), with initial portfolio condition $x_0 = 0$)

$$u(t) = -K(t)(V(t) - V_B(t)) + \left(\frac{\mathcal{C}}{2\rho a}\right)\vartheta(t) + u_B(t), \tag{13}$$

where

$$K(t) = \phi(t)^{-1}\varphi(t) \tag{14}$$
$$\vartheta(t) = \frac{1}{2}\left(\frac{\Gamma_1(t+1)}{\Gamma_2(t+1)}\right)\phi(t)^{-1}(\eta(t) - \eta_0(t)e).$$

Note that (11), (12) and (13) represent the optimal tracking portfolio strategy related to the problems $PE(\sigma)$, $PV(\chi)$ and $PMV(\rho)$, respectively. In other words, $u(t)$ represents the amount of the wealth to be allocated among the n assets, so that $u_0(t) = V(t) - \sum_{i=1}^{n} u_i(t)$ is the amount to be invested in the reference asset.

Remark 1 *Notice that the case $\sigma = 0$ in (11) (or, equivalently, $\rho \to \infty$ in (13) which means in Problem $PE(\sigma)$ an infinity risk aversion) and the case $\chi = 0$ in (12) imply, as expected, that $V(t) = V_B(t)$ and $u(t) = u_B(t)$. Indeed in this case we have from (11) with $\sigma = 0$ or (13) with $\rho \to \infty$ or (12) with $\chi = 0$ that*

$$u(t) = -K(t)(V(t) - V_B(t)) + u_B(t). \tag{15}$$

Let us show by induction that $u(t) = u_B(t)$ and $V(t) = V_B(t)$ for all t. For $t = 0$ we have by definition that $V_B(0) = V(0) = V_0$ and thus from (15) it follows that $u(0) = u_B(0)$. Suppose now that $V(t) = V_B(t)$ and $u(t) = u_B(t)$. Then it means that $X(t) = 0$, $U(t) = 0$ and from (6) it follows that $X(t+1) = V(t+1) - V_B(t+1) = 0$. From (15) it is easy to see that $u(t+1) = u_B(t+1)$ completing the induction argument.

Remark 2 *We verify that, by setting $T = 1$ and for fixed χ, problem $PV(\chi)$ reduces to the uni-period mean-variance problem studied in Roll [7]. To do this, we first rewrite the uni-period model considering the notation used above (we omit the dependence on t). Set $r_i = S_i(1)/S_i(0) - 1$, $\tilde{r} = (r_0 \ \ r')'$, with $r = (r_1 \dots r_n)'$ and $\tilde{\omega} = (\omega_0 \ \ \omega')'$, with $\omega = (\omega_1 \dots \omega_n)'$, where ω_i represents the proportion of the wealth invested on asset i. Similarly set $\tilde{\omega}_B$ as the $n+1$ vector representing the proportion of the benchmark portfolio invested on asset $i = 0, 1, \dots, n$, and ω_B the n dimensional vector obtained from $\tilde{\omega}_B$ excluding the first component. Considering that $(\tilde{\omega} - \tilde{\omega}_B)'e = 0$ and $r_i = \mathcal{R}_i - 1$, we have $P = (\tilde{\omega} - \tilde{\omega}_B)\tilde{r} = (\omega - \omega_B)'(\mathcal{R} - \mathcal{R}_0 e) = (\omega - \omega_B)'\mathcal{P}$. Recalling that $\phi = E(\mathcal{P}\mathcal{P}')$ and $E(\mathcal{R}_i) = \eta_i$, the expected value and the variance of P are given by $E(P) = (\omega - \omega_B)'(\eta - \eta_0 e) = \mu$ and $\sigma_P^2 = (\omega - \omega_B)'\phi(\omega - \omega_B) - \mu^2$,* *respectively. Therefore, the uni-period model can be formulated as follows:*

$$\begin{aligned} &\textit{minimize}\ (\omega-\omega_B)'\phi(\omega-\omega_B)\\ &\textit{subject to}\ (\omega-\omega_B)'(\eta-\eta_0)=\mu. \end{aligned}$$

Considering the method of Lagrange multipliers we have

$$\begin{cases} 2\phi(\omega-\omega_B)+\lambda(\eta-\eta_0 e)=\mathbf{0},\\ (\omega-\omega_B)'(\eta-\eta_0 e)=\mu, \end{cases}$$

so that the optimal solution is given by

$$\omega^* = \frac{\mu\phi^{-1}(\eta-\eta_0 e)}{(\eta-\eta_0 e)'\phi^{-1}(\eta-\eta_0 e)}+\omega_B. \tag{16}$$

On the other hand, considering $T=1$*, we have* $\mathcal{C}=\frac{1}{2}(\eta-\eta_0 e)'\phi^{-1}(\eta-\eta_0 e)$ *and* $\vartheta=\frac{1}{2}\phi^{-1}(\eta-\eta_0 e)$*, so that the optimal solution (12), with* $V=V_B=V_0$*, is given by*

$$u=\frac{\chi\phi^{-1}(\eta-\eta_0 e)}{(\eta-\eta_0 e)'\phi^{-1}(\eta-\eta_0 e)}+\omega_B V_0. \tag{17}$$

Taking $\omega=u/V_0$ *and considering that* $\mu=E(V-V_B)/V_0=\chi/V_0$*, we have that (17) is equal to (16), showing that as expected, the optimal solution (12) obtained for problem* $PV(\chi)$*, for the case* $T=1$*, coincides with the one for the uni-period problem analyzed in Roll [7].*

We recall that $X(t)=V(t)-V_B(t)$ represents the portfolio deviation value (or excess value), whereas $V(t)$ represents the managed portfolio value (here called tracking portfolio value) and $V_B(t)$ represents the benchmark portfolio value. The expected excess value and the tracking error variance, under the optimal portfolio policy $u(t)$, are respectively given by (see Li and Ng [4], equations (55) and (56), with initial portfolio value $x_0=0$) $E[X(T)]=\mathcal{C}^2/2a\rho$ and $Var[X(T)]=\mathcal{C}^2/4a\rho^2$, with $\rho=\mathcal{C}/(2\sigma\sqrt{a})$ when $PE(\sigma)$ is solved and $\rho=\mathcal{C}^2/(2a\chi)$ when $PV(\chi)$ is solved. The excess mean-variance tracking error frontier for the problems $PE(\sigma)$, $PV(\chi)$ and $PMV(\rho)$ can be written as follows (see again Li and Ng [4], equation (27), with initial portfolio value $x_0=0$) $Var[X(T)]=aE[X(T)]^2/\mathcal{C}^2$. Then, $Var[V(T)]$ represents the variance of the terminal tracking portfolio value $V(T)$ on the tracking error frontier (represented on the plane $(Var[V(T)],\ E[V(T)])$) and $Var[V_B(T)]$ represents the variance of the terminal benchmark portfolio value $V_B(T)$. Note from (4) and (5) that $V_B(t+1)=(\mathcal{R}_0(t)+\mathcal{P}(t)'F(t))V_B(t)+\mathcal{P}(t)'G(t)$ and thus, considering the statistical independence between $(\mathcal{R}_0(t),\mathcal{P}'(t))$ and $V_B(t)$, that

$$E\big[V_B(t+1)\big]=\Psi_1(t)E\big[V_B(t+1)\big]+\beta_1(t), \tag{18}$$

with $\Psi_1(t) = \eta_0(t) + (\eta'(t) - \eta_0(t)e')F(t)$ and $\beta_1(t) = (\eta(t) - \eta_0(t)e)'G(t)$ and

$$E\big[V_B(t+1)^2\big] = \Psi_2(t)E\big[V_B(t)^2\big] + \alpha(t)E\big[V_B(t)\big] + \beta_2(t), \tag{19}$$

with $\Psi_2(t) = \varphi_0^2(t) + F'(t)\phi(t)F(t) + 2F'(t)\varphi(t)$, $\alpha(t) = 2(\vartheta(t) + \phi(t)F(t))'G(t)$ and $\beta_2(t) = G(t)'\phi(t)G(t)$, so that the variance of the terminal benchmark portfolio value can be calculated from $Var[V_B(T)] = E[V_B(T)^2)] - E[V_B(T)]^2$ by iterating equations (18) and (19).

Considering the multi-period mean-variance formulation studied in Li and Ng [4], let $Var[\tilde{V}(T)]$ denote the variance of the terminal optimal portfolio value $\tilde{V}(T)$ on the mean-variance efficient frontier. As shown in Roll [7] for the uni-period mean-variance tracking error problem, in the following we verify that for the multi-period problem $PV(\chi)$ the difference between $Var[V(T)]$ and $Var[\tilde{V}(T)]$ is the same at every expected terminal portfolio value level.

Proposition 1 *Let $\tau = \prod_{t=0}^{T-1} A_2(t)$, $\mu = \prod_{t=0}^{T-1} A_1(t)$ and $b = \mu\mathcal{C}/a$. Taking $E[\tilde{V}(T)] = E[V(T)] = \chi + E[V_B(T)]$ and $\tilde{V}(0) = V(0) = V_0$, we have for the problem $PV(\chi)$ that*

$$\begin{aligned} Var[V(T)] - Var[\tilde{V}(T)] = &-\frac{a}{\mathcal{C}^2}\big(E[V_B(T)] - (\mu + b\mathcal{C})V_0\big)^2 \\ &+ Var[V_B(T)] - cV_0^2, \end{aligned} \tag{20}$$

with $c = (\tau - \mu^2 - ab^2)$.

Proof. Under the portfolio strategy (12) the dynamics of the tracking portfolio value (2) is written by

$$\begin{aligned} V(t+1) = &\big(\mathcal{R}_0(t) - \mathcal{P}'(t)K(t)\big)V(t) + \mathcal{P}'(t)\big(\frac{\chi}{\mathcal{C}}\vartheta(t) + G(t)\big) \\ &+ \mathcal{P}'(t)\big(K(t) + F(t)\big)V_B(t), \end{aligned} \tag{21}$$

where $K(t)$ and $\vartheta(t)$ are defined in (14). Squaring both sides of (21) we have that

$$\begin{aligned} V^2(t+1) = &\big(\mathcal{R}_0(t) - \mathcal{P}'(t)K(t)\big)^2V^2(t) + \Big(\mathcal{P}'(t)\big(\frac{\chi}{\mathcal{C}}\vartheta(t) + G(t)\big)\Big)^2 \\ &+ \big(\mathcal{P}'(t)(K(t) + F(t))\big)^2V_B^2(t) \\ &+ 2\big(\mathcal{R}_0(t) - \mathcal{P}'(t)K(t)\big)\mathcal{P}'(t)\big(\frac{\chi}{\mathcal{C}}\vartheta(t) + G(t)\big)V(t) \\ &+ 2\big(\frac{\chi}{\mathcal{C}}\vartheta(t) + G(t)\big)'\mathcal{P}(t)\mathcal{P}'(t)\big(K(t) + F(t)\big)V_B(t) \\ &+ 2\big(\mathcal{R}_0(t) - \mathcal{P}'(t)K(t)\big)\mathcal{P}'(t)\big(K(t) + F(t)\big)V(t)V_B(t). \end{aligned} \tag{22}$$

Taking the expectation on both sides of (22) and considering the statistical independence between $(\mathcal{R}_0(t), \mathcal{P}'(t))$ and $(V(t), V_B(t))$, and noticing that

$E[(\mathcal{R}_0(t) - \mathcal{P}'(t)K(t))\mathcal{P}'(t)] = 0$, we have the following recursive expression for the expected value of the squared tracking portfolio value:

$$\begin{aligned} E[V^2(t+1)] &= A_2(t)E[V^2(t)] + \big(\frac{\chi}{\mathcal{C}}\vartheta(t) + G(t)\big)'\phi(t)\big(\frac{\chi}{\mathcal{C}}\vartheta(t) + G(t)\big) \\ &\quad + \big(K(t) + F(t)\big)'\phi(t)\big(K(t) + F(t)\big)E[V_B^2(t)] \\ &\quad + 2\big(\frac{\chi}{\mathcal{C}}\vartheta(t) + G(t)\big)'\phi(t)\big(K(t) + F(t)\big)E[V_B(t)]. \end{aligned} \tag{23}$$

We have the following identities:

$$(K(t) + F(t))'\phi(t)(K(t) + F(t)) = -A_2(t) + \Psi_2(t), \tag{24}$$

$$2\vartheta'(t)\phi(t)\big(K(t) + F(t)\big) = \frac{\Gamma_1(t+1)}{\Gamma_2(t+1)}(-A_1(t) + \Psi_1(t)), \tag{25}$$

$$G'(t)\phi(t)\big(K(t) + F(t)\big) = (\varphi'(t) + F'(t)\phi(t))G(t), \tag{26}$$

$$2G'(t)\phi(t)\vartheta(t) = \frac{\Gamma_1(t+1)}{\Gamma_2(t+1)}(\eta(t) - \eta_0(t)e)'G(t), \tag{27}$$

$$\vartheta'(t)\phi(t)\vartheta(t) = \frac{1}{4}\Big(\frac{\Gamma_1(t+1)}{\Gamma_2(t+1)}\Big)^2\mathcal{B}(t). \tag{28}$$

Set $Z(t) = E(V^2(t)) - E(V_B^2(t))$. By combining (24)-(28) and (18)-(19) into (23) we get that

$$\begin{aligned} Z(t+1) &= A_2(t)Z(t) + \Big(\frac{\chi}{\mathcal{C}}\Big)\Big(\frac{\Gamma_1(t+1)}{\Gamma_2(t+1)}\Big)\Big(-A_1(t)E(V_B(t)) + E(V_B(t+1))\Big) \\ &\quad + \Big(\frac{\chi}{2\mathcal{C}}\Big)^2\Big(\frac{\Gamma_1(t+1)}{\Gamma_2(t+1)}\Big)^2\mathcal{B}(t). \end{aligned} \tag{29}$$

Solving the recursive equation (29) and recalling that $V(0) = V_B(0) = V_0$ (and thus $Z(0) = 0$) and $\Gamma_2(t+1) = \prod_{k=t+1}^{T-1} A_2(k)$, $\Gamma_1(t+1) = \prod_{k=t+1}^{T-1} A_1(k)$ we get that

$$\begin{aligned} Z(T) &= \sum_{t=0}^{T-1}\Gamma_2(t+1)\Big\{\Big(\frac{\chi}{\mathcal{C}}\Big)\Big(\frac{\Gamma_1(t+1)}{\Gamma_2(t+1)}\Big)\Big[-A_1(t)E(V_B(t)) + E(V_B(t+1))\Big] \\ &\quad + \Big(\frac{\chi}{2\mathcal{C}}\Big)^2\Big(\frac{\Gamma_1(t+1)}{\Gamma_2(t+1)}\Big)^2\mathcal{B}(t)\Big\} = \Big(\frac{\chi}{\mathcal{C}}\Big)\Big(E(V_B(T)) - \mu V_0\Big) + \frac{\chi^2}{2\mathcal{C}}. \end{aligned} \tag{30}$$

Thus from (30) the expected value of the squared terminal tracking portfolio value is given by $E[V^2(T)] = \frac{\chi^2}{2\mathcal{C}} + \frac{\chi}{\mathcal{C}}\big(E[V_B(T)] - \mu V_0\big) + E[V_B^2(T)]$. Since

$E[V(T)] = \chi + E[V_B(T)]$, we have that

$$Var[V(T)] = \frac{a\chi^2}{\mathcal{C}^2} + \frac{\chi}{\mathcal{C}}\big((1-2\mathcal{C})E[V_B(T)] - \mu V_0\big) + Var[V_B(T)]. \tag{31}$$

Recall that $\tilde{V}(T)$ denote the terminal optimal portfolio value and $Var[\tilde{V}(T)]$ its respective variance. We have that (see Li and Ng [4], equations (56) and (26), with $\nu = \mathcal{C}$ and $x_0 = \tilde{V}(0)$)

$$Var[\tilde{V}(T)] = \frac{a}{\mathcal{C}^2}\big(\epsilon - (\mu + b\mathcal{C})\tilde{V}(0)\big)^2 + (\tau - \mu^2 - ab^2)\tilde{V}^2(0), \tag{32}$$

where ϵ is the pre-selected level of the expected terminal optimal portfolio value (e.g. $\epsilon = E[\tilde{V}(T)]$). Taking $\epsilon = E[V(T)] = \chi + E[V_B(T)]$ and $\tilde{V}(0) = V_0$, we have that (31) and (32) yields (20), completing the proof. ∎

Remark 3 *If the benchmark portfolio obtained from $u_B(t)$ is efficient, e.g. $E[V_B(T)] = E[\tilde{V}(T)]$ and $Var[V_B(T)] = Var[\tilde{V}(T)]$, then from equation (32), with $\epsilon = E(V_B(T))$ and $\tilde{V}(T) = V_0$, the variance of the terminal benchmark portfolio value is given by $Var[V_B(T)] = \frac{a}{\mathcal{C}^2}\big(E[V_B(T)] - (\mu + b\mathcal{C})V_0\big)^2 + (\tau - \mu^2 - ab^2)V_0^2$. Therefore, from (20) we have $Var[V(T)] - Var[V_B(T)] = 0$, as described in Roll [7] for the uni-period mean-variance tracking error problem.*

3.2 The Special Case with One Riskless Asset

Let us investigate now the special case in which one of the assets is riskless (that is, it has no volatility). We assume the asset $i = 0$ as the riskless one. In this case $\mathcal{R}_0(t) = \eta_0(t)$ and

$$A_1(t) = \eta_0(t)(1 - \mathcal{B}(t)), A_2(t) = \eta_0(t)^2(1 - \mathcal{B}(t)), \frac{\Gamma_1(t+1)}{\Gamma_2(t+1)} = \prod_{k=t+1}^{T-1} \frac{1}{\eta_0(k)},$$

$$\Gamma_1(0) = \prod_{t=0}^{T-1} \eta_0(t)(1 - \mathcal{B}(t)),\ \Gamma_2(0) = \prod_{t=0}^{T-1} \eta_0(t)^2(1 - \mathcal{B}(t)),$$

$$a = \frac{1}{4}\prod_{t=0}^{T-1}(1 - \mathcal{B}(t))\left(1 - \prod_{t=0}^{T-1}(1 - \mathcal{B}(t))\right),\ b = 2\prod_{t=0}^{T-1}\eta_0(t),\ c = 0,$$

$$\vartheta(t) = \frac{1}{2}\left(\prod_{k=t+1}^{T-1}\frac{1}{\eta_0(k)}\right)\phi(t)^{-1}(\eta(t) - \eta_0(t)e),$$

$$\mathcal{C} = \frac{1}{2}\left(1 - \prod_{t=0}^{T-1}(1 - \mathcal{B}(t))\right),\ (bv_0 + \frac{\mathcal{C}}{2\rho a}) = 2\prod_{t=0}^{T-1}\eta_0(t)v_0 + \frac{1}{\rho\prod_{t=0}^{T-1}(1 - \mathcal{B}(t))}.$$

For problem $PE(\sigma)$ the optimal solution is given by (see Li and Ng [4], equations (73), (74) and (75), with initial portfolio value $x_0 = 0$)

$$u(t) = -\eta_0(t)\phi(t)^{-1}(\eta(t) - \eta_0(t)e)(V(t) - V_B(t)) + \frac{2\sigma\vartheta(t)}{\sqrt{[1 - \prod_{t=0}^{T-1}(1 - \mathcal{B}(t))]\prod_{t=0}^{T-1}(1 - \mathcal{B}(t))}} + u_B(t) \tag{33}$$

and for problem $PV(\chi)$ the optimal solution is given by

$$u(t) = -\eta_0(t)\phi(t)^{-1}(\eta(t) - \eta_0(t)e)(V(t) - V_B(t)) + \frac{2\chi\vartheta(t)}{1 - \prod_{t=0}^{T-1}(1 - \mathcal{B}(t))} + u_B(t). \tag{34}$$

For problem $PMV(\rho)$ we obtain that the optimal control is given by (see Li and Ng [4], equations (69) and (70), with initial portfolio condition $x_0 = 0$)

$$u(t) = -\eta_0(t)\phi(t)^{-1}(\eta(t) - \eta_0(t)e)(V(t) - V_B(t)) + \frac{\vartheta(t)}{\rho\prod_{t=0}^{T-1}(1 - \mathcal{B}(t))} + u_B(t). \tag{35}$$

As in Remark 1 the cases $\rho \to \infty$ in (35), $\sigma = 0$ in (33), and $\chi = 0$ in (34) imply, as expected, that $V(t) = V_B(t)$ and $u(t) = u_B(t)$.

The expected excess value and the variance of the excess terminal portfolio tracking error under the optimal portfolio policy $u(t)$ are respectively given by (see Li and Ng [4], equations (71) and (72), with initial portfolio value $x_0 = 0$)

$$E(X(T)) = \frac{1 - \prod_{t=0}^{T-1}(1 - \mathcal{B}(t))}{2\rho\prod_{t=0}^{T-1}(1 - \mathcal{B}(t))} \quad \text{and} \quad Var(X(T)) = \frac{1 - \prod_{t=0}^{T-1}(1 - \mathcal{B}(t))}{4\rho^2\prod_{t=0}^{T-1}(1 - \mathcal{B}(t))},$$

where $\rho = \frac{1}{2\sigma}\sqrt{\frac{1-\prod_{t=0}^{T-1}(1-\mathcal{B}(t))}{\prod_{t=0}^{T-1}(1-\mathcal{B}(t))}}$, when $PE(\sigma)$ is solved, and $\rho = \frac{1-\prod_{t=0}^{T-1}(1-\mathcal{B}(t))}{2\chi\prod_{t=0}^{T-1}(1-\mathcal{B}(t))}$, when $PV(\sigma)$ is solved. Finally, the excess mean-variance tracking error frontier for the problems $PE(\sigma)$, $PV(\chi)$ and $PMV(\rho)$ is given by (see again Li and Ng [4], equation (76), with initial portfolio value $x_0 = 0$)

$$Var(X(T)) = \frac{\prod_{t=0}^{T-1}(1 - \mathcal{B}(t))}{1 - \prod_{t=0}^{T-1}(1 - \mathcal{B}(t))}\left(E(X(T)\right)^2.$$

Remark 4 *As in Remark (2), in the following we verify that, by setting $T = 1$ and for fixed χ, the solution (34) for problem $PV(\chi)$ reduces to the uni-period mean-variance problem with one riskless asset (with expected return rate r_f). Considering $r = (r_1 \ldots r_n)'$ and $\omega = (\omega_1 \ldots \omega_n)'$, with $r_i = S_i(1)/S_i(0) - 1$*

and $r_i = \mathcal{R}_i - 1$, *we have* $P = (\omega - \omega_B)'r = (\omega - \omega_B)'(\mathcal{R} - e)$ *with* $E(P) = (\omega - \omega_B)'(\eta - e)$. *Setting* $\Sigma = E(\mathcal{R}\mathcal{R}')$ *and* $r_f = \eta_0 - 1$, *the uni-period model can be formulated as follows:*

$$\begin{aligned} &\textit{minimize} \;\; (\omega - \omega_B)'\Sigma(\omega - \omega_B) \\ &\textit{subject to} \;\; (1 - \omega' e)(\eta_0 - 1) + (\omega - \omega_B)'(\eta - e) = \mu. \end{aligned}$$

In this case, the optimality conditions are written as

$$\begin{cases} 2\Sigma(\omega - \omega_B) + \lambda(\eta - \eta_0 e) = \mathbf{0}, \\ \omega'(\eta - \eta_0 e) = 1 + \mu - \eta_0 + \omega_B'(\eta - e), \end{cases}$$

so that the optimal solution is given by

$$\omega^* = \frac{\mu\Sigma^{-1}(\eta - \eta_0 e)}{(\eta - \eta_0 e)'\Sigma^{-1}(\eta - \eta_0 e)} + \omega_B. \tag{36}$$

On the other hand, considering $T = 1$, *we have* $\vartheta = \frac{1}{2}\phi^{-1}(\eta - \eta_0 e)$, *so that the optimal solution (34), with* $V = V^B = V_0$, *is given by*

$$u = \frac{\chi\phi^{-1}(\eta - \eta_0 e)}{(\eta - \eta_0 e)'\phi^{-1}(\eta - \eta_0 e)} + \omega_B V_0. \tag{37}$$

Since $\sigma_{0i} = 0$, *with* $i = 0, \ldots, n$, *we have* $\phi = E(\mathcal{P}\mathcal{P}') = E(\mathcal{R}\mathcal{R}') = \Sigma$. *Taking* $\omega = u/V_0$ *and considering that* $\mu = E(V - V_B)/V_0 = \chi/V_0$, *we have that (37) is equal to (36), showing that, as expected, the solution for the multi-period problem* $PV(\chi)$ *with one riskless asset when we set* $T = 1$ *is the same as the one for the uni-period problem.*

4 Conclusion

In this paper we extended the work of Roll [7] by studying a discrete-time multi-period mean-variance tracking error portfolio selection problem. An optimal investment strategy for this mean-variance problem was analytically derived in a closed form. As a result, an explicit expression for the efficient frontier was identified. Moreover, it was shown that the muti-period minimum tracking error variance frontier is a constant translation of the multi-period minimum variance efficient frontier. Our results coincide with those in Roll [7] for the uni-period mean-variance formulation.

Acknowledgements. The third author was supported in part by CNPq, grant 304866/03-2, and Fapesp/BG, grant 2014/50279-4.

References

[1] G. J. Alexander and A. M. Baptista, Active portfolio management with benchmarking: a frontier based on alpha, *Journal of Banking & Finance,* **34** (2010), 2185 - 2197. https://doi.org/10.1016/j.jbankfin.2010.02.005

[2] O. L. V. Costa and A. C. Paiva, Robust portfolio selection using linear matrix inequalities, *Journal of Economic Dynamics and Control,* **26** (2002), 889 - 909. https://doi.org/10.1016/S0165-1889(00)00086-5

[3] M. Leippold, F. Trojani and P. Vanini, A geometric approach to multiperiod mean variance optimization of assets and liabilities, *Journal of Economic Dynamic Control,* **28** (2004), 1079 - 1113. https://doi.org/10.1016/S0165-1889(03)00067-8

[4] D. Li and W. L. Ng, Optimal dynamic portfolio selection: Multi-period mean-variance formulation, *Mathematical Finance,* **10** (2000), 387 - 406. https://doi.org/10.1111/1467-9965.00100

[5] A. Ling and L. Tang, A Numerical Study for Robust Active Portfolio Management with Worst-Case Downside Risk Measure, *Mathematical Problems in Engineering,* **2014** (2014), 1 - 13. https://doi.org/10.1155/2014/912389

[6] H. Markowitz, Portfolio selection, *Journal of Finance,* **7** (1952), 77 - 91. https://doi.org/10.1111/j.1540-6261.1952.tb01525.x

[7] R. Roll, A mean/variance analysis of tracking error, *Journal of Portfolio Management,* **18** (1992), 13 - 22. https://doi.org/10.3905/jpm.1992.701922

[8] G. Yin and X. Y. Zhou, Markowitz's mean-variance portfolio selection with regime switching: From discrete-time models to their continuous-time limits, *IEEE Transactions on Automatic Control,* **49** (2004), 349 - 360. https://doi.org/10.1109/TAC.2004.824479

[9] Q. Xiao and L. Liu, Multi-period mean-variance portfolio selection with a benchmark process, *Int. Journal of Contemp. Math. Sciences,* **7** (2012), 1727 - 1734.

[10] Q. Zhang and Y. Gao, Portfolio selection based on a benchmark process with dynamic value-at-risk constraints, *Journal of Computational and Applied Mathematics,* **313** (2017), 440 - 447. https://doi.org/10.1016/j.cam.2016.10.001

Received: January 7, 2017; Published: January 25, 2017