On Hybrid Chatterjea Type Fixed Point Theorem
with $\mathcal{MT}(\lambda)$-Functions

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Abstract
In this work, we establish a new fixed point theorem for hybrid Chatterjea type mappings and $\mathcal{MT}(\lambda)$-functions which generalizes Chatterjea’s fixed point theorem. As applications, many new fixed point theorems are also given.

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1. Introduction and preliminaries

In 1972, Chatterjea established his interesting fixed point theorem (so-called the Chatterjea’s fixed point theorem [3]) as follows:

Theorem 1.1. (Chatterjea [3]) Let $(X, d)$ be a complete metric space and $T : X \to X$ be a self-mapping on $X$. Suppose that there exists $\gamma \in [0, \frac{1}{2})$ such that
\[ d(Tx, Ty) \leq \gamma(d(x, Ty) + d(y, Tx)) \]
for all $x, y \in X$. 

Then $T$ admits a unique fixed point in $X$.

It is known that Chatterjea’s fixed point theorem is different from both the famous Banach contraction principle \[1, 10, 12, 14, 15\] and Kannan’s fixed point theorem \[4, 9, 11, 13\] and has been generalized in many various different directions; for more detail, one can refer to \[2, 4, 11\] and references therein.

Let $f$ be a real-valued function defined on $\mathbb{R}$. For $c \in \mathbb{R}$, we recall that

$$\limsup_{x \rightarrow c^+} f(x) = \inf_{\varepsilon > 0} \sup_{c < x < c + \varepsilon} f(x).$$

**Definition 1.1.** \[4-9, 11\] A function $\varphi : [0, \infty) \rightarrow [0, 1)$ is said to be an $\mathcal{MT}$-function (or $\mathcal{R}$-function) if $\limsup_{s \rightarrow t^+} \varphi(s) < 1$ for all $t \in [0, \infty)$.

It is obvious that if $\varphi : [0, \infty) \rightarrow [0, 1)$ is a nondecreasing function or a nonincreasing function, then $\varphi$ is an $\mathcal{MT}$-function. So the set of $\mathcal{MT}$-functions is a rich class.

**Definition 1.2.** \[8, 9\] A real sequence $\{a_n\}_{n \in \mathbb{N}}$ is called

(i) *eventually strictly decreasing* if there exists $\ell \in \mathbb{N}$ such that $a_{n+1} < a_n$ for all $n \in \mathbb{N}$ with $n \geq \ell$;

(ii) *eventually strictly increasing* if there exists $\ell \in \mathbb{N}$ such that $a_{n+1} > a_n$ for all $n \in \mathbb{N}$ with $n \geq \ell$;

(iii) *eventually nonincreasing* if there exists $\ell \in \mathbb{N}$ such that $a_{n+1} \leq a_n$ for all $n \in \mathbb{N}$ with $n \geq \ell$;

(iv) *eventually nondecreasing* if there exists $\ell \in \mathbb{N}$ such that $a_{n+1} \geq a_n$ for all $n \in \mathbb{N}$ with $n \geq \ell$.

Very recently, Du \[8\] presented some new characterizations of $\mathcal{MT}$-functions linked with eventually nonincreasing and eventually strictly decreasing sequences as follows.

**Theorem 1.2 (see \[8, Theorem 2.1\]).** Let $\varphi : [0, \infty) \rightarrow [0, 1)$ be a function. Then the following statements are equivalent.

(a) $\varphi$ is an $\mathcal{MT}$-function.

(b) For each $t \in [0, \infty)$, there exist $r^{(1)}_t \in [0, 1)$ and $\varepsilon^{(1)}_t > 0$ such that $\varphi(s) \leq r^{(1)}_t$ for all $s \in (t, t + \varepsilon^{(1)}_t)$. 

For each \( t \in [0, \infty) \), there exist \( r_t^{(2)} \in [0, 1) \) and \( \varepsilon_t^{(2)} > 0 \) such that \( \varphi(s) \leq r_t^{(2)} \) for all \( s \in [t, t + \varepsilon_t^{(2)}] \).

For each \( t \in [0, \infty) \), there exist \( r_t^{(3)} \in [0, 1) \) and \( \varepsilon_t^{(3)} > 0 \) such that \( \varphi(s) \leq r_t^{(3)} \) for all \( s \in (t, t + \varepsilon_t^{(3)}) \).

For each \( t \in [0, \infty) \), there exist \( r_t^{(4)} \in [0, 1) \) and \( \varepsilon_t^{(4)} > 0 \) such that \( \varphi(s) \leq r_t^{(4)} \) for all \( s \in [t, t + \varepsilon_t^{(4)}] \).

For any nonincreasing sequence \( \{x_n\}_{n \in \mathbb{N}} \) in \( [0, \infty) \), we have \( 0 \leq \sup_{n \in \mathbb{N}} \varphi(x_n) < 1 \).

\( \varphi \) is a function of contractive factor; that is, for any strictly decreasing sequence \( \{x_n\}_{n \in \mathbb{N}} \) in \( [0, \infty) \), we have \( 0 \leq \sup_{n \in \mathbb{N}} \varphi(x_n) < 1 \).

For any eventually nonincreasing sequence \( \{x_n\}_{n \in \mathbb{N}} \) in \( [0, \infty) \), we have \( 0 \leq \sup_{n \in \mathbb{N}} \varphi(x_n) < 1 \).

For any eventually strictly decreasing sequence \( \{x_n\}_{n \in \mathbb{N}} \) in \( [0, \infty) \), we have \( 0 \leq \sup_{n \in \mathbb{N}} \varphi(x_n) < 1 \).

In [8], Du introduced the concept of \( \mathcal{MT}(\lambda) \)-function which generalizes the concept of \( \mathcal{P} \)-function [16].

**Definition 1.3 [8, 9].** Let \( \lambda > 0 \). A function \( \mu : [0, \infty) \rightarrow [0, \lambda) \) is said to be an \( \mathcal{MT}(\lambda) \)-function if \( \limsup_{s \rightarrow t^+} \mu(s) < \lambda \) for all \( t \in [0, \infty) \).

**Remark 1.1.**

(i) Obviously, an \( \mathcal{MT} \)-function is an \( \mathcal{MT}(1) \)-function and a \( \mathcal{P} \)-function is an \( \mathcal{MT} \left( \frac{1}{2} \right) \)-function.

(ii) It is easy to see that \( \mu \) is an \( \mathcal{MT}(\lambda) \)-function if and only if \( \lambda^{-1} \mu \) is an \( \mathcal{MT} \)-function.

The following characterizations of \( \mathcal{MT}(\lambda) \)-functions is an immediate consequence of Theorem 1.2.

**Theorem 1.3 (see [8, Theorem 2.4]).** Let \( \lambda > 0 \) and let \( \mu : [0, \infty) \rightarrow [0, \lambda) \) be a function. Then the following statements are equivalent.

(1) \( \mu \) is an \( \mathcal{MT}(\lambda) \)-function.
(2) \( \lambda^{-1} \mu \) is an \( \mathcal{MT} \)-function.

(3) For each \( t \in [0, \infty) \), there exist \( \xi_t^{(1)} \in [0, \lambda) \) and \( \epsilon_t^{(1)} > 0 \) such that 
\[ \mu(s) \leq \xi_t^{(1)} \] for all \( s \in (t, t + \epsilon_t^{(1)}) \).

(4) For each \( t \in [0, \infty) \), there exist \( \xi_t^{(2)} \in [0, \lambda) \) and \( \epsilon_t^{(2)} > 0 \) such that 
\[ \mu(s) \leq \xi_t^{(2)} \] for all \( s \in [t, t + \epsilon_t^{(2)}] \).

(5) For each \( t \in [0, \infty) \), there exist \( \xi_t^{(3)} \in [0, \lambda) \) and \( \epsilon_t^{(3)} > 0 \) such that 
\[ \mu(s) \leq \xi_t^{(3)} \] for all \( s \in (t, t + \epsilon_t^{(3)}) \).

(6) For each \( t \in [0, \infty) \), there exist \( \xi_t^{(4)} \in [0, \lambda) \) and \( \epsilon_t^{(4)} > 0 \) such that 
\[ \mu(s) \leq \xi_t^{(4)} \] for all \( s \in [t, t + \epsilon_t^{(4)}] \).

(7) For any nonincreasing sequence \( \{x_n\}_{n \in \mathbb{N}} \) in \( [0, \infty) \), we have 
\[ 0 \leq \sup_{n \in \mathbb{N}} \mu(x_n) < \lambda. \]

(8) For any strictly decreasing sequence \( \{x_n\}_{n \in \mathbb{N}} \) in \( [0, \infty) \), we have 
\[ 0 \leq \sup_{n \in \mathbb{N}} \mu(x_n) < \lambda. \]

(9) For any eventually nonincreasing sequence \( \{x_n\}_{n \in \mathbb{N}} \) in \( [0, \infty) \), we have 
\[ 0 \leq \sup_{n \in \mathbb{N}} \mu(x_n) < \lambda. \]

(10) For any eventually strictly decreasing sequence \( \{x_n\}_{n \in \mathbb{N}} \) in \( [0, \infty) \), we have 
\[ 0 \leq \sup_{n \in \mathbb{N}} \mu(x_n) < \lambda. \]

Remark 1.2. [16, Lemma 3.1] is a special case of Theorem 1.3 for \( \lambda = \frac{1}{2} \).

In this work, we first establish a new fixed point theorem for hybrid Chatterjea type mappings and \( \mathcal{MT}(\lambda) \)-functions which generalizes and extends Chatterjea’s fixed point theorem. As applications, many new fixed point theorems are also given. All new results presented in this paper are original and quite different from the well known generalizations in the literature.

2. A hybrid Chatterjea type fixed point theorem and some new fixed point theorems

In this section, we first establish the following new fixed point theorem for hybrid Chatterjea type mappings and \( \mathcal{MT}(\lambda) \)-functions which extends and generalizes Chatterjea’s fixed point theorem.

Theorem 2.1. Let \((X, d)\) be a complete metric space and \( T : X \to X \) be a selfmapping on \( X \). Suppose that
(D) there exists an $\mathcal{MT}\left(\frac{1}{2}\right)$-function $\mu : [0, \infty) \to [0, \frac{1}{2}]$ such that
\[
\min\{d(Tx, Ty), d(x, Tx)\} \leq \mu(d(x, y))(d(x, Ty) + d(y, Tx)) \quad \text{for all } x, y \in X \text{ with } x \neq y.
\]

Then $T$ admits a fixed point in $X$.

**Proof.** Let $u \in X$ be given. If $Tu = u$, then the proof is finished. Otherwise, if $Tu \neq u$, we define a sequence $\{x_n\}_{n \in \mathbb{N}}$ by $x_1 = u$ and $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$. Since $\mu(t) < \frac{1}{2}$ for all $t \in [0, \infty)$, we can define a function $\tau : [0, \infty) \to (0, \frac{1}{2})$ by
\[
\tau(t) = \frac{1}{2}\left(\frac{1}{2} + \mu(t)\right) \quad \text{for all } t \in [0, \infty).
\]
Clearly, $0 \leq \mu(t) < \tau(t) < \frac{1}{2}$ for all $t \in [0, \infty)$. Since $x_2 = Tx_1 \neq x_1$, by condition (D), we have
\[
d(x_3, x_2) = \min\{d(Tx_2, Tx_1), d(x_2, Tx_2)\} \\
\leq \mu(d(x_2, x_1))d(x_1, x_3) \\
< \tau(d(x_2, x_1))(d(x_3, x_2) + d(x_2, x_1)).
\]

If $Tx_2 = x_2$ then $x_2 \in \mathcal{F}(T)$ (here, $\mathcal{F}(T)$ denotes the set of fixed points of $T$) and we are done. Assume $Tx_2 \neq x_2$. Then $x_3 \neq x_2$. By (D) again, we obtain
\[
d(x_4, x_3) = \min\{d(Tx_3, Tx_2), d(x_3, Tx_3)\} \\
< \tau(d(x_3, x_2))(d(x_4, x_3) + d(x_3, x_2)).
\]

Hence, by induction, if $x_{n+1} \neq x_n$ then we can get
\[
d(x_{n+2}, x_{n+1}) < \tau(d(x_{n+1}, x_n))(d(x_{n+2}, x_{n+1}) + d(x_{n+1}, x_n)) \quad \text{for all } n \in \mathbb{N}.
\]  \hfill (2.1)

Suppose there exists $k \in \mathbb{N}$ such that $d(x_{k+1}, x_k) < d(x_{k+2}, x_{k+1})$. So, by (2.1), we have
\[
d(x_{k+2}, x_{k+1}) < 2\tau(d(x_{k+1}, x_k))d(x_{k+2}, x_{k+1}) < d(x_{k+2}, x_{k+1}),
\]
a contradiction. Hence it must be
\[
d(x_{n+2}, x_{n+1}) \leq d(x_{n+1}, x_n) \quad \text{for all } n \in \mathbb{N}.
\]  \hfill (2.2)

By (2.2), we know that the sequence $\{d(x_{n+1}, x_n)\}_{n \in \mathbb{N}}$ is nonincreasing in $[0, \infty)$. Since $\mu$ is an $\mathcal{MT}\left(\frac{1}{2}\right)$-function, by applying Theorem 1.3, we have
\[
0 \leq \sup_{n \in \mathbb{N}}\mu(d(x_{n+1}, x_n)) < \frac{1}{2}
\]
and then deduces
\[
0 < \sup_{n \in \mathbb{N}}\tau(d(x_{n+1}, x_n)) = \frac{1}{2}\left[\frac{1}{2} + \sup_{n \in \mathbb{N}}\mu(d(x_{n+1}, x_n))\right] < \frac{1}{2}.
\]
Let \( \alpha := \sup_{n \in \mathbb{N}} 2\tau(d(x_{n+1}, x_n)) \). So \( \alpha \in (0, 1) \). We claim that \( \{x_n\}_{n \in \mathbb{N}} \) is a Cauchy sequence in \( X \). Indeed, for any \( n \in \mathbb{N} \), by (2.1) and (2.2), we get
\[
d(x_{n+2}, x_{n+1}) < \tau(d(x_{n+1}, x_n))(d(x_{n+2}, x_{n+1}) + d(x_{n+1}, x_n)) \\
\leq 2\tau(d(x_{n+1}, x_n))d(x_{n+1}, x_n) \\
\leq \alpha d(x_{n+1}, x_n)
\]
and hence
\[
d(x_{n+2}, x_{n+1}) < \alpha d(x_{n+1}, x_n) < \cdots < \alpha^n d(x_2, x_1).
\]
(2.3)

Let \( \xi_n = \frac{\alpha^{n-1}}{1-\alpha} d(x_2, x_1), n \in \mathbb{N} \). For \( m, n \in \mathbb{N} \) with \( m > n \), from (2.3), we obtain
\[
d(x_m, x_n) \leq \sum_{j=n}^{m-1} d(x_{j+1}, x_j) < \xi_n.
\]
(2.4)

Since \( 0 < \alpha < 1 \), we have \( \lim_{n \to \infty} \xi_n = 0 \). So, by (2.4), we get
\[
\lim_{n \to \infty} \sup_{m > n} \{d(x_m, x_n) : m > n\} = 0,
\]
which shows that \( \{x_n\}_{n \in \mathbb{N}} \) is a Cauchy sequence in \( X \). By the completeness of \( X \), there exists \( v \in X \) such that \( x_n \to v \) as \( n \to \infty \). Now, we verify that \( v \in \mathcal{F}(T) \). Clearly, the condition (\(D\)) still holds for \( x = y \in X \). So, by (\(D\)), we have
\[
\frac{1}{2} (d(x_{n+1}, Tv) + d(v, Tv) - |d(x_{n+1}, Tv) - d(v, Tv)|) \\
= \min\{d(Tv, Tx_n), d(v, Tv)\} \\
< \frac{1}{2} (d(v, x_{n+1}) + d(x_n, Tv))
\]
for all \( n \in \mathbb{N} \).

Since \( x_n \to v \) as \( n \to \infty \), by taking the limit as \( n \to \infty \) on the last inequality, we get
\[
d(v, Tv) \leq \frac{1}{2} d(v, Tv)
\]
which implies \( d(v, Tv) = 0 \). Therefore we obtain \( v \in \mathcal{F}(T) \). The proof is completed. \( \square \)

The following new fixed point theorem is an immediate consequence of Theorem 2.1.

**Corollary 2.1.** Let \((X, d)\) be a complete metric space and \( T : X \to X \) be a selfmapping on \( X \). Suppose that there exists an MT\((\frac{1}{2})\)-function \( \mu : [0, \infty) \to [0, \frac{1}{2}] \) such that
\[
d(Tx, Ty) \leq \mu(d(x, y))(d(x, Ty) + d(y, Tx)) \quad \text{for all} \; x, y \in X \; \text{with} \; x \neq y.
\]
(2.5)
Then $T$ admits a unique fixed point in $X$.

**Proof.** Applying Theorem 2.1, $F(T) \neq \emptyset$. We claim that $F(T)$ is a singleton set. Suppose there exist $u, v \in F(T)$ with $u \neq v$. By (2.5), we obtain $d(u,v) > 0$ and

$$d(u,v) = d(Tu,Tv)$$

$$\leq \mu(d(u,v))(d(u,Tv) + d(v,Tu))$$

$$= \mu(d(u,v))(d(u,v) + d(v,u))$$

$$< d(u,v)$$

a contradiction. Therefore $F(T)$ is a singleton set and $T$ has a unique fixed point in $X$. \hfill \Box

**Remark 2.1.** Chatterjea’s fixed point theorem [3] is a special case of Theorem 2.1 and Corollary 2.1.

**Corollary 2.2.** Let $(X,d)$ be a complete metric space and $T : X \to X$ be a selfmapping on $X$. Suppose that there exists an $\mathcal{MT} \left( \frac{1}{2} \right)$-function $\mu : [0,\infty) \to [0,\frac{1}{2})$ such that

$$d(x,Tx) \leq \mu(d(x,y))(d(x,Ty) + d(y,Tx))$$

for all $x, y \in X$ with $x \neq y$.

Then $T$ admits a fixed point in $X$.

**Corollary 2.3.** Let $(X,d)$ be a complete metric space and $T : X \to X$ be a selfmapping on $X$. Suppose that there exists $\gamma \in [0,\frac{1}{2})$ such that

$$d(x,Tx) \leq \gamma(d(x,Ty) + d(y,Tx))$$

for all $x, y \in X$ with $x \neq y$.

Then $T$ admits a fixed point in $X$.

Applying Theorem 2.1, we can derive the following new fixed point theorems immediately.

**Theorem 2.2.** Let $(X,d)$ be a complete metric space and $T : X \to X$ be a selfmapping on $X$. Suppose that there exists an $\mathcal{MT} \left( \frac{1}{2} \right)$-function $\mu : [0,\infty) \to [0,\frac{1}{2})$ such that

$$d(Tx,Ty) + d(x,Tx) \leq 2\mu(d(x,y))(d(x,Ty) + d(y,Tx))$$

for all $x, y \in X$ with $x \neq y$. \hfill (2.6)

Then $T$ admits a fixed point in $X$.

**Theorem 2.3.** Let $(X,d)$ be a complete metric space and $T : X \to X$ be a selfmapping on $X$. Suppose that there exists an $\mathcal{MT} \left( \frac{1}{2} \right)$-function $\mu : [0,\infty) \to [0,\frac{1}{2})$ such that

$$\sqrt{d(Tx,Ty)d(x,Tx)} \leq \mu(d(x,y))(d(x,Ty) + d(y,Tx))$$

for all $x, y \in X$ with $x \neq y$. \hfill (2.7)
Then $T$ admits a fixed point in $X$.

In fact, we can establish a wide generalization of Theorem 2.2 as follows.

**Theorem 2.4.** Let $(X,d)$ be a complete metric space and $T : X \to X$ be a self-mapping on $X$. Suppose that there exists an $\mathcal{MT} \left( \frac{1}{2} \right)$-function $\mu : [0, \infty) \to [0, \frac{1}{2})$ such that

$$\frac{\alpha d(Tx,Ty) + \beta d(x,Tx)}{\alpha + \beta} \leq \mu(d(x,y))(d(x,Ty)+d(y,Tx))$$

for all $x, y \in X$ with $x \neq y$, \hspace{1cm} (2.8)

where $\alpha$ and $\beta$ are nonnegative real numbers with $\alpha + \beta > 0$. Then $T$ admits a fixed point in $X$.

**Remark 2.2.**

(i) The conclusions of Theorems 2.1-2.4 still hold if

- "min\{d(Tx,Ty), d(x,Tx)\}" is replaced with "min\{d(Tx,Ty), d(y,Ty)\}" in condition (D) of Theorem 2.1;
- "d(Tx,Ty) + d(x,Tx)" is replaced with "d(Tx,Ty) + d(y,Ty)" in (2.6) of Theorem 2.2;
- "\sqrt{d(Tx,Ty)d(x,Tx)}" is replaced with "\sqrt{d(Tx,Ty)d(y,Ty)}" in (2.7) of Theorem 2.3;
- \(\frac{\alpha d(Tx,Ty)+\beta d(x,Tx)}{\alpha+\beta}\) is replaced with \(\frac{\alpha d(Tx,Ty)+\beta d(y,Ty)}{\alpha+\beta}\) in (2.8) of Theorem 2.4.

(ii) If we take $\alpha = \beta = 1$ in Theorem 2.4, then we obtain Theorem 2.2.

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**References**


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