Modified Adomian-Rach Decomposition

Method for Solving Nonlinear

Time-Dependent IVPs

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Abstract

In this work, we modify the multi-stage Adomian-Rach decomposition method (shortly, ARDM) based on combining series solution and decomposition method for solving time-dependent nonlinear second-order ordinary initial value problems (IVPs) with separated nonhomogeneity. The proposed method overcomes the complexity of integral calculations in the standard Adomian decomposition method and the demerit of power series method. Approximate analytic solutions are obtained on a long domain with error estimations. Two modeling problems are investigated to demonstrate the efficiency, performance and high accuracy of the method.

Keywords: Adomian-Rach decomposition method, Series solution, Analytic solution, Error analysis, Aging mass–spring system

1 Introduction

The Adomian decomposition method was investigated and developed by George Adomian for solving nonlinear differential equations [1]. This method avoids
linearization, perturbation, discretization or any unrealistic assumptions. The method provides the solution in a rapid convergent series with easily computable components. Over the last two decades, it has modified, merged and emerged as an alternative method for solving a wide range of problems whose mathematical models involve algebraic, ordinary, partial, integral and integro-differential equations [2-10] and references therein.

In this paper, the multi-stage modification presented by Rach and Adomian [11] and reacted by Duan et al. [10] is considered and improved to get highly-accurate numeric-analytic solutions for some time-dependent nonhomogeneous second order ordinary IVPs.

2 Multi-Stage ARDM

Consider the second-order IVP in general form

\[ y''(t) = f(t, y), \quad y(0) = y_0, \quad y'(0) = y_1, \quad t \in [0, T]. \]  

where \( f \) is an analytic and separable, i.e. \( f \) can be decomposed as a product of two analytic functions \( g(t) \) and \( h(y) \). Consider the equally-spaced partition on the given domain \( 0 = t_0 < t_1 < t_2 \cdots < t_N = T \), with the step-size \( h = (b - a)/N \). The multi-stage ARDM states that the solution \( y(t) \) on the subinterval \( (t_m, t_{m+1}] \), \( m = 0, 1, \ldots, N - 1 \), is represented by the decomposition series

\[ y(t) = \sum_{n=0}^{\infty} a_n (t - t_m)^n. \]  

The nonlinear term \( h(y) \) is decomposed in terms of solution coefficients \( a_n \)'s as

\[ h(y) = \sum_{n=0}^{\infty} A_n (a_0, a_1, \ldots, a_n)(t - t_m)^n. \]

That is since,

\[ A_n = \frac{1}{n!} \left[ \frac{\partial^n}{\partial \lambda^n} \left( \sum_{k=0}^{n} u_k \lambda^k \right) \right]_{\lambda=0} = \frac{1}{n!} \left[ \frac{\partial^n}{\partial \lambda^n} \left( \sum_{k=0}^{n} a_k t^k \lambda^k \right) \right]_{\lambda=0}. \]

Consequently,

\[ A_n = t^n \frac{1}{n!} \left[ \frac{\partial^n}{\partial \lambda^n} \left( \sum_{k=0}^{n} a_k \lambda^k \right) \right]_{\lambda=0}. \]

Thus, we present an alternating proof of the Adomian-Rach theorem:

**Theorem 2.1.** Suppose \( y(t) = \sum_{n=0}^{\infty} a_n (t - t_m)^n \), a convergent series, an analytic nonlinear operator \( h(y) \) is given with

\[ h(y) = \sum_{n=0}^{\infty} A_n (u_0, \ldots, u_n), \]

where \( A_n \)'s are the Adomian polynomials defined by

...
Modified Adomian-Rach decomposition method

\[ A_n = \frac{1}{n!} \left[ \frac{\partial^n}{\partial \lambda^n} h \left( \sum_{k=0}^{n} u_k \lambda^k \right) \right]_{\lambda=0}, \ n \geq 0. \]  

The \( A_k \)'s can be defined in terms of \( a_k \)'s, i.e. \( A_k = A_k(a_0, \ldots, a_k) \), and

\[ h(y) = \sum_{n=0}^{\infty} A_n(a_0, a_1, \ldots, a_n)(t - t_m)^n. \]  

Applying Theorem 2.1 and since \( g(t) \) is analytic, the Cauchy product of infinite series implies that

\[ f(t, y) = g(t) h(y) = \left( \sum_{j=0}^{\infty} c_j (t - t_m)^j \right) \left( \sum_{i=0}^{\infty} A_i(a_0, a_1, \ldots, a_i)(t - t_m)^i \right) = \sum_{n=0}^{\infty} b_n (t - t_m)^n, \]

where, \( b_n = \sum_{k=0}^{n} c_k A_{n-k} \) and \( c_k \)'s are the coefficients of the Taylor expansion of \( g(t) \) about \( t_m \). Substituting the obtained expansions into Eq.(1) gives the following recurrence relation, in compact form, for the solution coefficients

\[ a_{n+2} = \frac{b_n}{(n+2)(n+1)}, \ n \geq 0. \]  

The used initial values in Eq(6) for the first subinterval \([0, t_1]\) are \( a_0 = y_0 \) and \( a_1 = y_1 \). The \( k \)th order approximate analytic solution on this subdomain is defined as

\[ \phi_1(t) = \sum_{n=0}^{k} a_n t^n. \]

For \( m = 2, 3, \ldots, N \), the \( k \)th order approximate analytic solution on the subinterval \((t_{m-1}, t_m]\) is expressed as

\[ \phi_m(t) = \sum_{n=0}^{k} a_{n,m} (t - t_{m-1})^n, \]

with

\[ a_{n+2,m} = \frac{b_{n,m}}{(n+2)(n+1)}, \ n \geq 0. \]

and used starting values

\[ a_{0,m} = \phi_{m-1}'(t_m), \ a_{1,m} = \phi_{m-1}''(t_m). \]

The exact solution on the given subdomain is obtained by
\[ y_m(t) = \lim_{k \to \infty} \sum_{n=0}^{k} a_{n,m} (t-t_{m-1})^n = \sum_{n=0}^{\infty} a_{n,m} (t-t_{m-1})^n, \]  

(11)

The \( k \)th order numeric approximations at the mesh points is generated to be

\[ \{\alpha, \phi_{n+1,m}(t_m), m = 1,2,\ldots,N \}. \]  

(12)

We define an approximate analytic solution for Eq. (1) on whole domain by the infinitely differentiable multi-rule function

\[ u_{app}(t) = \phi_m(t), \ t_{m-1} \leq t \leq t_m, \ m = 1,2,\ldots,N, \]  

(13)

and in the same way, for problems with unknown exact solution, the corresponding global absolute error is defined by [12-15]

\[ E_{abs}(t) = \{L_m(t) = |\phi_m(t) - f(t, \phi_m(t))|, \ t_{m-1} \leq t \leq t_m, \ m = 1,2,\ldots,N \}. \]  

(14)

Now, we state the convergence theorem of the assumed series solution Eq. (2) to discussed problem Eq. (1).

**Theorem 2.2.** The power series solution defined in Eq. (2) with nonzero coefficients obtained recursively in Eq. (9) converges uniformly to the solution \( y(t) \) of the IVP Eq.(1) on \( |t-t_m| < \rho \) where \( 0 < \rho < R \leq \infty \) and \( R = \lim_{n \to \infty} \left| \frac{a_{n+1,m}}{a_{n,m}} \right| \) is the radius of convergence.

**Proof:** Let

\[ \rho = \lim_{n \to \infty} \left| \frac{a_{n+1,m}}{a_{n,m}} \right| (t-t_{m-1})^{n+1} = |t-t_{m-1}| \lim_{n \to \infty} \left| \frac{a_{n+1,m}}{a_{n,m}} \right| = R \]

By the ratio test, the power series Eq.(12) converges if \( 0 \leq \rho < 1 \), or \( |t-t_{m-1}| < R \), which proves the result.

To determine \( R \) in the previous theorem, Hadamard applied the root test to the power series and proved that

\[ R = 1 / \limsup_{n \to \infty} \left| a_{n,m} \right|^\frac{1}{n}, \]

where \( R = 0 \) if the denominator diverges to \( \infty \), and \( R = \infty \) if it is \( 0 \).

**3 Applications**

To give a clear overview of the content of this work, the computer application program MATHEMATICA was used to execute the algorithms that
were used with two examples, one of which with known exact solution and the other is of physical interest known as aging spring-mass system.

**Example 3.1.** Consider the nonlinear IVP

\[ y''(t) = \frac{8y^2(t)}{1+2t}, \quad y(0) = 1, \quad y'(0) = -2, \quad t \in [0, T], \]  

(15)

whose exact solution is \( y(t) = 1/(2t+1) \). Following the procedure discussed in the previous section, we make the equally spaced partition for the interval \([0, T]\) by selecting an integer \( N \) with step-size \( h = T/N = t_m - t_{m-1} \) that guarantees the uniformly convergence of power series solution in Theorem 2.2. The time-dependent nonlinear term \( f(t, y) \) in Eq.(15) on the subdomain \( (t_{m-1}, t_m) \) may be written according to the formula in Eq.(5) in the form

\[
f(t, y) = \frac{8}{1+2t} y^2(t) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-1)^k}{(1+2t_{m-1})^{k+1}} A_n(t - t_{m-1})^n = \sum_{n=0}^{\infty} b_{n,m} (t - t_{m-1})^n,
\]

(16)

The first few terms of Adomian polynomials \( A_j \) corresponding to the nonlinear operator \( h(y) = y^2 \) in terms of series solution coefficients are as follows

\[ A_0 = h(a_0) = a_0^2, \quad A_1 = a_1 h'(a_0) = 2a_0 a_1, \quad A_2 = 2a_0 a_2 + a_0^2, \quad A_3 = 2a_0 a_3 + 2a_0 a_2, \ldots
\]

The \( a_n \)'s can be obtained by the recurrence relation in Eq.(9) with initials

\[ a_{0,m} = \phi_{m-1}(t_m), \quad a_{1,m} = \phi'_{m-1}(t_m), \quad m = 1, 2, \ldots N. \]

(17)

With \( h = 0.01 \), the series solution \( \phi_m(t) \) is easily obtained on the interval \([0, 20]\). The approximate analytic solution \( u_{app}(t) \) defined in Eq. (13) in comparison to exact solution is represented in Fig. 1. For some values of \( t \), the obtained absolute errors using ARDM comparing to those obtained by applying the fourth-order Runge-Kutta method (RK4) are reported in Table1. The approximations using the ARDM are more accurate than others with clear difference. This problem was solved by the standard ADM [16], our modified technique shows the high accuracy on long domain while the ADM gets progressively worse away from the origin.
Example 3.2. [17] Consider the model for a mass–spring system with an aging spring and without damping

\[ m \ y''(t) + k \ e^{-\eta t} \ y(t) = 0, \ y(0) = 1, \ y'(0) = 0, \ t \in [0,T], \]

Table 1. Absolute errors using ARDM and RK4 method for Example 3.1.

<table>
<thead>
<tr>
<th>( t_i )</th>
<th>ARDM</th>
<th>RK4 method</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.02</td>
<td>0</td>
<td>1.97729 x 10^{-6}</td>
</tr>
<tr>
<td>0.8</td>
<td>6.10623 x 10^{-16}</td>
<td>6.16126 x 10^{-8}</td>
</tr>
<tr>
<td>2.0</td>
<td>2.80331 x 10^{-15}</td>
<td>3.24818 x 10^{-7}</td>
</tr>
<tr>
<td>4.0</td>
<td>1.20876 x 10^{-14}</td>
<td>1.46264 x 10^{-6}</td>
</tr>
<tr>
<td>8.0</td>
<td>6.18880 x 10^{-14}</td>
<td>7.45799 x 10^{-6}</td>
</tr>
<tr>
<td>12.0</td>
<td>1.65902 x 10^{-13}</td>
<td>2.00294 x 10^{-5}</td>
</tr>
<tr>
<td>16.0</td>
<td>3.37803 x 10^{-13}</td>
<td>4.07898 x 10^{-5}</td>
</tr>
<tr>
<td>20.0</td>
<td>5.88980 x 10^{-13}</td>
<td>7.11342 x 10^{-5}</td>
</tr>
</tbody>
</table>

Figure 1. The exact solution (red solid line) versus the 10th-order approximate analytic solution (blue dashed) with the step-size \( h = 0.01 \) for Example 4.1.

where \( m \) is the mass, \( k \) and \( \eta \) positive constants, and \( y(t) \) the displacement of the spring from its equilibrium position. For \( \eta = 0 \) the equation reduces to that of a simple harmonic motion, and for \( \eta \neq 0 \), the solution can be expressed in terms of 0th-order Bessel's functions of the first and second kind. The 10th-order approximate analytic solution \( \phi_m(t) \) using ARDM on the subinterval \((t_{m-1}, t_m)\) can be obtained with solution coefficients satisfy the recurrence relation

\[ a_{n,m} = -\sum_{k=0}^{n} (-1)^k \frac{a_{n-k}}{k! \ e^{\eta t_m}}, \quad m = 1,2,\ldots,N, \quad n = 0,1,\ldots,10 \quad (19) \]

With \( m = k = \eta = 1 \) and the step-size \( h = 0.01 \), Table 2 reports the approximated values using our approach with corresponding absolute errors defined in Eq.(16) since no closed form exact solution is known. The highly accurate
results are obvious. Furthermore, the ARDM is simpler than RK4 method and standard ADM [16] with long domain of convergence.

Table 2. Displacement using ARDM for Example 3.2, with absolute error using formula Eq.(16)

<table>
<thead>
<tr>
<th>$t_i$</th>
<th>$y(t_i)$</th>
<th>$L_m(t_i)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.02</td>
<td>0.999801</td>
<td>2.22045×10^{-16}</td>
</tr>
<tr>
<td>0.8</td>
<td>0.759803</td>
<td>0</td>
</tr>
<tr>
<td>2.0</td>
<td>0.0167241</td>
<td>4.33681×10^{-19}</td>
</tr>
<tr>
<td>4.0</td>
<td>-1.26009</td>
<td>0</td>
</tr>
<tr>
<td>8.0</td>
<td>-3.61016</td>
<td>0</td>
</tr>
<tr>
<td>12.0</td>
<td>-5.91861</td>
<td>6.77626×10^{-21}</td>
</tr>
<tr>
<td>16.0</td>
<td>-8.22555</td>
<td>1.05879×10^{-22}</td>
</tr>
<tr>
<td>20.0</td>
<td>-10.5325</td>
<td>0</td>
</tr>
</tbody>
</table>

Conclusions

In this paper, the powerful Adomian-Rach decomposition method is developed and employed for analytic treatment of nonlinear second order IVPs with separable non-homogeneity. Two tested problems are considered to show that our technique yields a more convenient, efficient and accurate form of the solution compared to the series solution of the standard decomposition method and other existing numerical methods. It is a worthwhile to mention that the new modification can be applied to other nonlinear differential equations in mathematical physics.

References


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