Estimating the Power of Lack-of-Fit Test for the Mean of Multivariate Spatial Regression

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Abstract

The purpose of this paper is to establish an estimation of a boundary crossing probability composed by a multivariate centered Gaussian processes and a vector of deterministic trends. Such probability corresponds in statistics to the power function of an asymptotic lack-of-fit test conducted based on the Kolmogorov functional of set-indexed partial (cumulative) sums process of the least squares residuals of multivariate spatial regression. Since the analytical computation of the probability is impossible, we investigate its upper and lower bounds by applying some methods relied on the multivariate Cameron-Martin translation formula on the space of high dimensional set-indexed continuous functions. Our consideration is mainly for the multivariate set-indexed Brownian pillow. The results are shown not only useful for analyzing the behavior of the test, but also worth for the abstraction and generalization of some existing results toward univariate Gaussian process studied in many literatures.

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1 Introduction

It is mentioned in the literatures of asymptotic test theory that the performance of a test can be evaluated by investigating its corresponding power function, see e.g. Serfling [16], p. 315 and Lehmann and Romano [9], p. 423–424. A recommended size $\alpha$ test should have the power larger then $\alpha$. Furthermore, an asymptotic test is said to be point-wise consistent, if the power converges in the same ways to one.

Analytical or numerical evaluation of the power of a test is frequently found complicated since it involves the computation of a non tractable probability of events. In such situation one can only give either approximation or estimation by deriving upper or lower bounds for the possible values of the function, see e.g. Bischoff and Hashorva [4] and Bischoff and et al. [5]. Monte Carlo simulation is therefore frequently conducted to approximate the finite sample behaviors of the test.

The purpose of the present paper is to give an estimation to the so-called boundary crossing probability which is closely related to the power function of a test for the mean vector in multivariate regression based on the partial boundary crossing probability which is closely related to the power function behaviors of the test.

Furthermore, an asymptotic test is said to be point-wise consistent, if the performance of a test can be evaluated by investigating its corresponding power function, see e.g. Serfling [16], p. 315 and Lehmann and Romano [9], p. 2588.
of \( \varphi_g \), defined by

\[
(pr_{W^\perp_{HZp}} \varphi_g)(A) := \varphi_g(A) - \sum_{j=1}^{m} \left( (f_j, g_i)_{L_2(P_0, G)} \right)_i^{p} \varphi_{f_j}(A), A \in \mathcal{B}(G)
\]

whereas the second component-wise projection is given as

\[
(pr^*_{W^\perp_{HZp}} Z_p)(A) := Z_p(A) - \sum_{j=1}^{m} \left( \int_G f_j(t) dZ(i)(t) \right)_i^{p} \varphi_{f_j}(A), A \in \mathcal{B}(G),
\]

provided \( \{f_1, \ldots, f_m\} \) builds an orthonormal basis of \( W \) in \( L_2(P_0, G) \cap BV(G) \).

We refer interested reader to [18, 19, 20] for more precise definition of the projection. We notice that the notation \( Z(t) \) stands for \( Z(i)(\Pi_{d=1}^{k} \left[ a_k, t_k \right]) \), cf. [22]. Hence the integral involved therein coincides path-wise with the Riemann-Stieltjes one. The integral is well defined by the reason \( Z(t) \) has continuous sample path with respect to the Euclidean distance when the index set is restricted to the Vapnich Chervonenkis Classes (VCC):

\[
\{[a, t] := \Pi_{k=1}^{p} [a_k, t_k], a_k < t_k < b_k, k = 1, \ldots, d\}
\]

of \( d \)-dimensional closed rectangles with the point \( a := (a_1, \ldots, a_d) \) as the essential point. The formula of the integration by parts for Riemann-Stieltjes integral of function with \( d \) variables was studied in Yeh [23], see also Theorem 6.4 in the Appendix.

It is worth mentioning that (1) is a generalization of the limiting power function of an asymptotic test of size \( \alpha \) for the hypothesis \( H_0 : g \in \times_{i=1}^{p} W \) against \( H_1 : g \notin \times_{i=1}^{p} W \) based on the Kolmogorov functional of the set-indexed partial sums of least squares residuals obtained from a multivariate regression model

\[
Y(t) = g(t) + \mathcal{E}(t), \quad t := (t_1, \ldots, t_d) \in G,
\]

where \( Y := (Y_i)_{i=1}^{p} \) is the vector of random observations, \( g := (g_i)_{i=1}^{p} \) is the true-unknown vector of regression function defined on \( G \) and \( \mathcal{E} := (\varepsilon_i)_{i=1}^{p} \) is the independent vector of random errors, such that \( \mathbb{E}(\mathcal{E}) = 0 \) and \( \text{Cov}(\mathcal{E}) = \Sigma \). See the asymptotic test procedures proposed in [18, 19, 20] for the technical details. The experiment was conducted under an experimental design constructed according to an algorithm proposed in Somayasa [17] and Somayasa and et al. [21]. A regularly spaced (equidistance) design which is frequently called a regular lattice is commonly used in the practice. It corresponds to the re-scaled Lebesgue measure (uniform probability measure).

Since \( \Sigma^{-1/2}(pr_{W^\perp_{HZp}} \varphi_g)(A) \) in (1) is equal to \( (pr_{W^\perp_{HZp}} \varphi_{\Sigma^{-1/2}g})(A) \), without loss of generality we consider in this work the boundary crossing probability:

\[
\Psi_{P_0}(v, \varphi_g) := P \left\{ \sup_{A \in \mathcal{B}(G)} \varphi_g(A) + \mathcal{W}_{t,P_0}(A) \geq v(A) \right\}, \quad (2)
\]
as the object of study, where $\mathcal{W}_{t,P_0} := (pr_{W \perp Z_p} Z_p)$ is a centered $p-$dimensional set-indexed Gaussian process having the covariance function, given by

$$K_{\mathcal{W}_{t,P_0}}(A_1, A_2) = \text{diag} (\gamma(A_1, A_2), \ldots, \gamma(A_1, A_2)) \in \mathcal{R}^{p \times p},$$

with $\gamma(A_1, A_2) := P_0(A_1 \cap A_2) - \sum_{j=1}^m \varphi_{f_j}(A_1) \varphi_{f_j}(A_2)$. For the case of Brownian motion and Brownian bridge on the unit interval $[0,1]$ the upper and lower bounds for (2) have been investigated in the literatures, see e.g. [4, 5] and Hasorva [7] and [8] for boundary non-crossing probabilities. As a by product to our proposed technique, we also get in this work the upper and lower bounds for the probability:

$$\mathcal{J}_{P_0}(v, \varphi) := \mathcal{P}\left\{ \inf_{A \in \mathcal{B}(G)} \varphi(A) + \mathcal{W}_{t,P_0}(A) \leq v(A) \right\},$$

which corresponds to infimum functional of the partial sums process of the residuals. This is actually another way how to define the test statistic. The inequality signs involved in (2) and (4) should be understood as a component-wise inequality in the sense for two vectors $x = (x_i)_{i=1}^p$ and $y = (y_i)_{i=1}^p$, $x \geq y (x \leq y)$ if and only if $x_i \geq y_i (x_i \leq y_i)$, for all $i = 1, \ldots, p$. By this reason (2) will be called in the sequel component-wise boundary crossing probability. For the VCC classes of subsets in $G$, the random vector $\mathcal{W}_{t,P_0}(t)$ stands for $\mathcal{W}_{t,P_0}(\Pi_{k=1}^d [a_k, t_k])$. It is clear by (3) that $\mathcal{W}_{t,P_0}(t) = 0$ a.s. for some $k$, with $t_k = a_k$, for $k = 1, \ldots, d$.

The rest of the present paper is organized as follows. In Section 2 we present some preliminary results regarding the reproducing kernel Hilbert space (RKHS) as well as the basic form of the Cameron-Martin translation formula of the multivariate process $\mathcal{W}_{t,P_0}$. The derivation of the upper and lower bounds for (2) is presented in Section 3. In Section 4 the consideration is extended to the boundary crossing probability which involves additive process $\sum_{i=1}^p \mathcal{W}_{t,P_0}^{(i)}$, see Section 4. To the knowledge of the author this kind of probability has not been yet investigated in the literatures. The paper is closed with a concluding remark in Section 5.

## 2 Preliminary Results

The properties of any centered Gaussian process are entirely determined by the covariance function of the process. By generalizing the definition presented in Lifshits [10], pp.41–51 and Lifshits [11], pp. 15-16, a family of vectors

$$\left\{ m_A := (m_A^{(i)})_{i=1}^p \subset L_2^p(P_0, G) := \times_{i=1}^p L_2(P_0, G), A \in \mathcal{B}(G) \right\}$$
is said to build a model for the Gaussian process \( \mathcal{W}_{t,P_0} \), if and only if for every \( A_1, A_2 \in \mathcal{B}(G) \), it holds:

\[
K_{\mathcal{W}_{t,P_0}}(A_1, A_2) = \text{diag} \left( \langle m_{A_1}^{(1)}, m_{A_2}^{(1)} \rangle_{L_2(P_0,G)}, \ldots, \langle m_{A_1}^{(p)}, m_{A_2}^{(p)} \rangle_{L_2(P_0,G)} \right).
\]

For example, the model of \( Z_p \) is given by the family of \( p \)-dimensional vector of indicators \( \{1_A = (1_A)^{i=1}_{p} : A \in \mathcal{B}(G) \} \), because for any \( A_1, A_2 \in \mathcal{B}(G) \), we have

\[
diag \left( \langle 1_{A_1}, 1_{A_2} \rangle_{L_2(P_0,G)}, \ldots, \langle 1_{A_1}, 1_{A_2} \rangle_{L_2(P_0,G)} \right) = K_{Z_p}(A_1, A_2).
\]

Other example is the \( p \)-dimensional set-indexed Brownian pillow defined as \( Z_p^o(A) := Z_p(A) - P_0(A)Z_p(G) \), for \( A \in \mathcal{B}(G) \). This process is obtained as the limit process when under \( H_0 \) a constant model is specified. That is when \( W = \{f_1\}^{p=1}_{i=1} \), with \( f_1(t) = 1 \), for \( t \in G \). The process has the covariance function

\[
K_{Z_p^o}(A_1, A_2) = \text{diag} (\delta(A_1, A_2), \ldots, \delta(A_1, A_2)) \in \mathbb{R}^{p \times p},
\]

with \( \delta(A_1, A_2) := P_0(A_1 \cap A_2) - P_0(A_1)P_0(A_2) \). Hence, it can be shown that the family of \( p \)-dimensional vector functions

\[
\{1_A^o = (1_A^{o})^{p=1}_{i=1} := 1_A - P_0(A)1_G | A \in \mathcal{B}(G) \}
\]

can be regarded as the model of \( Z_p^o \).

In general, if the family \( \{m_A : A \in \mathcal{B}(G)\} \) constitutes a model of \( \mathcal{W}_{t,P_0} \), then by the definition of white noise integral studied e.g. in Lifshits [11], pp.14-15, we get for \( i = 1, \ldots, p \),

\[
\mathbb{E} \left( \int_{G} m_{A_1}^{(i)}(t)Z^{(i)}(dt) \int_{G} m_{A_2}^{(i)}(t)Z^{(i)}(dt) \right) = \langle m_{A_1}^{(i)}, m_{A_2}^{(i)} \rangle_{L_2(P_0,G)}.
\]

Hence, we have the expression

\[
\mathcal{W}_{t,P_0}(A) = \left( \int_{G} m_{A}^{(i)}(t)Z^{(i)}(dt) \right)^{p=1}_{i=1}, \forall A \in \mathcal{B}(G),
\]

where the integral is defined component-wise in term of white noise integral. This expression will be shown useful in the sequel.

The reproducing kernel Hilbert space (RKHS) of a \( p \)-dimensional centered Gaussian process plays important role for our result. Factorization theorem (Theorem 4.1 in [11]) can be utilized in constructing the RKHS of \( \mathcal{W}_{t,P_0} \). In particular, if there exists a family \( \{m_A \in L_2^p(P_0,G) : A \in \mathcal{B}(G)\} \), such that the covariance function of \( \mathcal{W}_{t,P_0} \) admits the representation presented in Equation (5), then the corresponding RKHS is given by

\[
\mathcal{H}_{\mathcal{W}_{t,P_0}} := \left\{ h | \exists \ell = (\ell_i)^{p=1}_{i=1}, h(A) = \int_{G} \left( m_{A}^{(i)}(t)\ell_i(t) \right)^{p=1}_{i=1} P_0(dt) \right\}.
\]

(6)
Furthermore, if the vector $\ell$ that satisfies $h(A) = \langle m_A, \ell \rangle_{L^2_2(P_0, \mathcal{G})}$ is unique, then the norm furnishing $\mathcal{H}_{W_r,P_0}$ is defined as

$$\|h\|^2_{\mathcal{H}_{W_r,P_0}} := \|\ell\|^2_{L^2_2(P_0, \mathcal{G})} = \sum_{i=1}^{p} \|\ell_i\|^2_{L^2_2(P_0, \mathcal{G})},$$

otherwise the norm is defined as

$$\|h\|^2_{\mathcal{H}_{W_r,P_0}} := \inf_{\{\ell : h(A) = \int_G (m_A(t)\ell(t)) dt, \ A \in \mathcal{B}(\mathcal{G})\}} \|\ell\|^2_{L^2_2(P_0, \mathcal{G})}.$$

The respective scalar product denoted by $\langle \cdot, \cdot \rangle_{\mathcal{H}_{W_r,P_0}}$, is defined by

$$\langle h_1, h_2 \rangle_{\mathcal{H}_{W_r,P_0}} = \langle \ell_1, \ell_2 \rangle_{L^2_2(P_0, \mathcal{G})} = \sum_{i=1}^{p} \langle \ell_i, \ell_i \rangle_{L^2_2(P_0, \mathcal{G})}, \ \forall h_1, h_2 \in \mathcal{H}_{W_r,P_0},$$

with

$$h_j(A) = \int_G (m_A(t)\ell_j(t)) dt, \ j = 1, 2.$$ 

For $i = 1, \ldots, p$, let $\mathcal{H}_{\mathcal{W}_r^{(i)},P_0}$ be the RKHS of $\mathcal{W}_r^{(i)}$. Then by the preceding result, $\mathcal{H}_{W_r,P_0} = \times_{i=1}^{p} \mathcal{H}_{\mathcal{W}_r^{(i)},P_0}$. For examples, the RKHS of $Z_p$ is given by

$$\mathcal{H}_{Z_p} := \left\{ h \mid \exists \ell \in (\ell_i)_{i=1}^{p} \in L^2_2(P_0, \mathcal{G}), \ h(A) = \int_A \ell(t)P_0(dt) \right\}.$$  

For $Z_p^0$ with $P_0(\mathcal{G}) = 1$, we have for every $A \in \mathcal{B}(\mathcal{G})$,

$$h_i(A) = \int_G 1_A(t)\ell_i(t)P_0(dt) = \int_A \ell_i(t)P_0(dt) - P_0(A)\int_G \ell_i(t)P_0(dt),$$

so $h(\mathcal{G}) = 0 = h(\emptyset)$. Hence, by the similar way as before, the RKHS of $Z_p^0$ is written as

$$\mathcal{H}_{Z_p^0} := \left\{ h \mid \exists \ell \in L^2_2(P_0, \mathcal{G}), \ h(A) = \int_A \ell(t)P_0(dt) - P_0(A)\int_G \ell(t)P_0(dt) \right\}.$$  

The following theorem gives the Cameron-Martin density formula of a shifted $p$-variate centered Gaussian process $X$, say. See also Theorem 5.1 in [11] for further reference. Bischoff and Gegg [3] established the Cameron-Martin formula for the Slepian process on $[0, 1]$, a case where $\ell$ is not unique.

**Proposition 2.1 (Multivariate Cameron-Martin density)** Let $X$ be a $p$-dimensional centered Gaussian process with sample paths in a linear space $\chi$ and $h$ be a deterministic vector in $\chi$. Let $P^X$ and $P^{X+h}$ denote the probability distributions of $X$ and $X + h$, respectively, defined by

$$P^{X+h}(B) := P^X(B - h), \ \forall B \in \mathcal{B}(\chi).$$
Let $\mathcal{H}_X$ be the RKHS of $X$ with the corresponding norm $\| \cdot \|_{\mathcal{H}_X}$. Then $\mathbf{P}^{X+h}$ is absolutely continuous with respect to $\mathbf{P}^X$, if and only if $h \in \mathcal{H}_X$. Moreover, if $h \in \mathcal{H}_X$, then

$$\frac{d\mathbf{P}^{X+h}}{d\mathbf{P}^X}(u) = \exp \left\{ A(u) - \frac{1}{2} \| h \|_{\mathcal{H}_X}^2 \right\}, \text{ for } \mathbf{P}^X - \text{almost all } u \in \chi,$$

where $A$ is a linear functional such that $E(X_i A(X_i)) = h_i, \forall i = 1, \ldots, p$.

Finding the Cameron-Martin density formula is a well adopted technique proposed in the literatures for obtaining the upper and lower bounds of the boundary crossing probabilities, cf. [4, 5, 7, 8]. In this paper we use this method in deriving the bounds of (2) as well as (4) by firstly deriving the formula of $\frac{d\mathbf{P}^\phi g}{d\mathbf{P}^Z_0}(x)$, Theorem 2.1 suggests that in order to get the explicit formula of $\frac{d\mathbf{P}^\phi g}{d\mathbf{P}^Z_0}(x)$, the RKHS $\mathcal{H}_{W_{0, P_0}}$ of the Gaussian process $W_{0, P_0}$ together with its inherent norm must be determined.

### 3 Component-wise boundary

In this section and throughout this article the consideration is restricted to the case in which $W_{0, P_0}$ is given by the $p-$dimensional set-indexed Brownian pillow $Z^0_p$ introduced in Section 2. In the context of lack-of-fit test for multivariate spatial regression, $Z^0_p$ appears as the limit process of partial sums of the residuals when the assumed model is a constant model, cf. [18, 19, 20]. See also [14, 6, 22] for the univariate case. Theorem 2.1 can be used as a basis in deriving the Cameron-Martin density of $\mathbf{P}^{\phi g + Z^0_p}$ with respect to $\mathbf{P}^{Z^0_p}$ on $C(\mathcal{B}(G))$, where $C(\mathcal{B}(G))$ is the space of continuous functions with respect to a metric $d_{P_0}$ defined e.g. in [1, 15].

**Proposition 3.1** Let $\mathcal{H}_{Z^0_p BV^p_{H}}(G)$ be the set of vectors in $\mathcal{H}_{Z^0_p}$ with the associated density lies in $L^2_p(P_0, G) \cap BV^p_{H}(G)$ (see Definition 6.3 in the Appendix for the notion of $BV^p_{H}(G)$). If $\phi_g \in \mathcal{H}_{Z^0_p BV^p_{H}}(G)$, with

$$\phi_g(A) = \int_A g(t) P_0(dt) - P_0(A) \int_G g(t) P_0(dt), \text{ } A \in \mathcal{B}(G),$$

then the Cameron-Martin density of $\mathbf{P}^{\phi_g + Z^0_p}$ with respect to $\mathbf{P}^{Z^0_p}$ is given by

$$\frac{d\mathbf{P}^{\phi_g + Z^0_p}}{d\mathbf{P}^{Z^0_p}}(x) = \exp \left\{ \int_G g^\top(t) dx(t) - \frac{1}{2} \| \phi_g \|_{\mathcal{H}_{Z^0_p}}^2 \right\},$$

for $\mathbf{P}^{Z^0_p} - \text{almost all } x \in C(\mathcal{B}(G))$. The integral $\int_G g_i(t) dx^{(i)}(t)$ is defined in the sense of Wiener integral.
Proof. By Proposition 2.1, we only need to show that for every \( A \in \mathcal{B}(\mathcal{G}) \) and \( i = 1, \ldots, p \), it holds

\[
E \left( Z^o_p(i)(A) \int_g g_i(t) dZ^o(t) \right) = \int_g 1^o_A(t) g_i(t) P_0(dt).
\]

For the more general process we have by the definition of \( \mathcal{W}_{t,P_0} \) presented in Section 1 that for all \( i = 1, \ldots, p \) and \( A \in \mathcal{B}(\mathcal{G}) \),

\[
E \left( \mathcal{W}^{(i)}_{t,P_0}(A) \int_g g_i(t) d\mathcal{W}^{(i)}_{t,P_0}(t) \right) = E \left( \int_g m^o_A(t) dZ^o(t) \int_g g_i(t) d\mathcal{W}^{(i)}_{t,P_0}(t) \right)
\]

\[= \int_g m^o_A(t) g_i(t) P_0(dt) - \sum_{j=1}^m \langle m^o_A, f_j \rangle_{L_2(P_0, \mathcal{G})} \langle g_i, f_j \rangle_{L_2(P_0, \mathcal{G})},\]

provided \( \{ m_A = (m^o_A)^p_{i=1} \in L_2^p(P_0, \mathcal{G}) : A \in \mathcal{B}(\mathcal{G}) \} \) is a model of \( \mathcal{W}_{t,P_0} \). Next, by substituting \( 1^o_A \) for \( m^o_A \) and 1 for \( f_j \) for all \( j = 1, \ldots, m \), we establish the proof.

It is important to note that the white noise integral \( \int g_i(t) \mathcal{W}^{(i)}_{t,P_0}(dt) \) coincides path-wise with the Riemann-Stieltjes integral by the fact \( g_i \in BV_H(\mathcal{G}) \) and \( \mathcal{W}^{(i)}_{t,P_0} \) has continuous sample paths, cf. [23].

The following Proposition can be immediately derived by the direct application of the Cameron-Martin density formula presented in Proposition 3.1. It will be the starting point for all results exhibited in this work.

**Proposition 3.2** Under the similar conditions as those of Proposition 3.1, it holds true for the \( p \)-variate set-indexed Brownian pillow, that

\[
\Psi_{P_0}(v, \varphi_G) = 1 - \exp \left\{ -\frac{1}{2} \| \varphi_G \|^2_{\mathcal{H}Z_p^o} \right\} \times
\]

\[
E \left( \exp \left( \int_g g^\top(t) dZ^o_p(t) \right) \right) 1\{ \sup_{A \in \mathcal{B}(\mathcal{G})} Z^o_p(A) \leq v(A) \}
\]

Proof. Since \( 1 - \Psi_{P_0}(v, \varphi_G) = P\{ \sup_{A \in \mathcal{B}(\mathcal{G})} \varphi_G(A) + \mathcal{W}_{t,P_0}(A) \leq v(A) \} \), then by the Cameron-Martin formula, we get

\[
1 - \Psi_{P_0}(v, \varphi_G) = \int_{C^p(B(\mathcal{G}))} 1\{ \sup_{A \in \mathcal{B}(\mathcal{G})} x(A) \leq v(A) \} \mathcal{P}^{\varphi_G + Z^o_p}(dx)
\]

\[= \int_{C^p(B(\mathcal{G}))} 1\{ \sup_{A \in \mathcal{B}(\mathcal{G})} x(A) \leq v(A) \} \exp \left\{ \int_g g^\top(t) dx(t) - \frac{1}{2} \| \varphi_G \|^2_{\mathcal{H}Z_p^o} \right\} \mathcal{P}^Z_p(dx)
\]

\[= \exp \left\{ -\frac{1}{2} \| \varphi_G \|^2_{\mathcal{H}Z_p^o} \right\} \int_\Omega \exp \left\{ \int_g g^\top(t) dZ^o_p(t) \right\} 1\{ \sup_{A \in \mathcal{B}(\mathcal{G})} Z^o_p(A) \leq v(A) \} \mathcal{P}(d\omega)
\]

which is completing the proof.

Now we are ready to state our main results. The proof is established by applying Proposition 3.1 and Proposition 3.2.
**Theorem 3.3** Let for $i = 1, \ldots, p$, $g_i$ be nondecreasing on $G$. Let for $k = 1, \ldots, (d - 1)$ the marginal of $g_i$ on $G_k$ be constant. Then for an even positive number $d$, we have the following inequalities

$$\Psi_{P_0}(v, \varphi_g) \geq 1 - \exp \left\{ -\frac{1}{2} \| \varphi_g \|_{H_{Z_p}}^2 + v^T(b)g(b) + \int_G v^T(t) dg(t) \right\} \times (1 - \Psi_{P_0}(v_0))$$

$$\Psi_{P_0}(v, \varphi_g) \leq 1 - \exp \left\{ -\frac{1}{2} \| \varphi_g \|_{H_{Z_p}}^2 + w^T(b)g(b) + \int_G w^T(t) dg(t) \right\} \times (2 + \Psi_{P_0}(w_0) - \Psi_{P_0}(v_0)),$$

for any boundary $w := (w_i)_{i=1}^p \in C^p(B(G))$ that satisfies $w(A) \leq \varphi_g(A) + Z_{P_0}^o(A) \leq v(A)$, for all $A \in B(G)$.

Proof. By the formula presented in Theorem 6.4 in the Appendix, if for every $i$, the marginal $g_{i_{G^k_j}}(\cdot)$ is constant on $G^k_j$ for every $k = 1, 2, \ldots, (d - 1)$ and $j = 1, \ldots, |C_{d-k}|$, then for every even $d$, it holds

$$\int_G g^T(t)dZ_{P_0}(t) = \Delta_a^b \left( g^TZ_{P_0}^o \right) + \int_G Z_{P_0}^o(t) dg(t).$$

The reader is referred to Definition 6.1 for the meaning of the operator $\Delta$. However, since $Z_{P_0}(t) = 0$, almost surely, when $t_k = a_k$, for at least one $k \in \{1, \ldots, d\}$, then the first term in the right-hand side of (7) reduces to $Z_{P_0}^o(b)g(b)$. Hence, by substituting this equality we get under the indicator $1 \left\{ Z_{P_0}^o(A) \leq v(A), \forall A \in B(D) \right\}$ the first inequality. The positive sign before the integral must be changed by a negative one when $d$ is an odd positive number.

To proof the second assertion we apply the similar strategy used in [8] by starting from the following result:

$$1 - \Psi_{P_0}(v, \varphi_g) \geq \Pr \left\{ w(A) \leq \varphi_g(A) + Z_{P_0}^o(A) \leq v(A), \forall A \in B(G) \right\}$$

$$= \exp \left\{ -\frac{1}{2} \| \varphi_g \|_{W_{v,P_0}}^2 \right\} \times \mathbb{E} \left( \exp \left\{ \int_G g^T(t)dZ_{P_0}^o(t) \right\} 1_{\{w(A) \leq Z_{P_0}^o(A) \leq v(A), \forall A \in B(G)\}} \right).$$

which is true for any boundary $w = (w_i)_{i=1}^p \in C^p(B(G))$. Hence, by Equation (7) and the inequality $w(A) \leq Z_{P_0}^o(A), \forall A \in B(G)$, we further get

$$1 - \Psi_{P_0}(v, \varphi_g) \geq \exp \left\{ -\frac{1}{2} \| \varphi_g \|_{H_{Z_p}}^2 + w^T(b)g(b) + \int_G w^T(t) dg(t) \right\} \times \Pr \left\{ w(A) \leq Z_{P_0}^o(A) \leq v(A), \forall A \in B(G) \right\}.$$
Since \( P \left\{ w(A) \leq Z_p^0(A) \leq v(A), \forall A \in B(G) \right\} = 2 + \Psi_{P_0}(w, 0) - \Psi_{P_0}(v, 0) \), we are done.

To be more visible, we fix the consideration in the following corollary for the case of \( d = 2 \). The proof is analogous to that of Theorem 3.3.

**Corollary 3.4** Let \( g_i \) be such that \( g_i \) non decreasing on the half-open rectangle \([a_2, b_2] \times [a_1, b_1]\), whereas \( g_i(b_1, y) \) and \( g_i(x, b_2) \) be respectively non increasing on the closed intervals \([a_2, b_2]\) and \([a_1, b_1]\), for \( i = 1, \ldots, p \). Let \( R_1^* \) and \( R_2^* \) be positive constants defined as

\[
R_1^* := \exp \left\{ -\frac{1}{2} \| g \|_{L^2_p(P_0, G)}^2 + g^\top(b_1, b_2)v(b_1, b_2) + \int_{[a_1, b_1]} v^\top(x, b_2)d(-g(x, b_2)) \right\} \\
\quad \quad \quad \quad \quad + \int_{[a_2, b_2]} v^\top(b_1, y)d(-g(b_1, y)) + \int_G v^\top(x, y)dg(x, y) \right\} \\
R_2^* := \exp \left\{ -\frac{1}{2} \| g \|_{L^2_p(P_0, G)}^2 + g^\top(b_1, b_2)w(b_1, b_2) + \int_{[a_1, b_1]} w^\top(x, b_2)d(-g(x, b_2)) \right\} \\
\quad \quad \quad \quad \quad + \int_{[a_2, b_2]} w^\top(b_1, y)d(-g(b_1, y)) + \int_G w^\top(x, y)dg(x, y) \right\},
\]

for any vector \( w \in C^p(B(G)) \). Then

\[
\Psi_{P_0}(v, \varphi_g) \geq 1 - R_1^*(1 - \Psi_{P_0}(v, 0)) \\
\Psi_{P_0}(v, \varphi_g) \leq 1 - R_2^*(2 + \Psi_{P_0}(w, 0) - \Psi_{P_0}(v, 0)),
\]

**Remark 3.5** By the preceding results, the variability of the power functions of testing whether or not a constant model holds true can now be inferred. Let \( c_{1-\alpha} := (c_i)_{i=1}^p \) be a vector of \((1 - \alpha)^{th}\) quantiles of \( Z_p^0 \), that is a vector of constants satisfying \( P\{Z_p^0 \leq c_{1-\alpha}\} = 1 - \alpha \). By Theorem 3.3, the values of \( \Psi_{P_0}(c_{1-\alpha}, \varphi_g) \) for any vector \( g \in H_{Z_p^0 B^p(G)} \) fulfilling the condition specified under the theorem must lie in the interval \([L, U]\), where

\[
L := 1 - \exp \left\{ -\frac{1}{2} \| \varphi_g \|_{H_{Z_p^0}}^2 + c_{1-\alpha}^\top a g(b) + c_{1-\alpha}^\top b g \right\} (1 - \alpha) \\
U := 1 - \exp \left\{ -\frac{1}{2} \| \varphi_g \|_{H_{Z_p^0}}^2 + w^\top(b)g(b) + \int_G w^\top(t)dg(t) \right\} \times \\
(2 - \alpha + \Psi_{P_0}(w, 0)), \forall w, w(A) \leq \varphi_g(A) + Z_p^0(A) \leq v(A), \forall A \in B(G).
\]

Particularly, when the true model is a constant model, the signal in Model 1 vanishes simultaneously resulting in a unit Cameron-Martin density. Hence \( \Psi_{P_0}(c_{1-\alpha}, 0) \) is reaching the lower bounds \( L = \alpha \) which coincides with the size of the test.

The following Theorem gives the upper and lower bounds for (4). The proof is based on Proposition 3.2 and the definition of the event under study.
Theorem 3.6 Under the similar condition as in Theorem 3.3, it holds
\[
\mathcal{J}_{P_0}(\mathbf{v}, \varphi_\mathbf{g}) \leq 1 - \exp \left\{ -\frac{1}{2} \| \varphi_\mathbf{g} \|^2_{\mathcal{H}_{Z^p_0}} + \mathbf{v}^\top \mathbf{b} \mathbf{g}(\mathbf{b}) + \int_{\mathcal{G}} \mathbf{v}^\top(t) d\mathbf{g}(t) \right\} \times (1 - \mathcal{J}_{P_0}(\mathbf{v}, 0))
\]
\[
\mathcal{J}_{P_0}(\mathbf{v}, \varphi_\mathbf{g}) \geq 1 - \exp \left\{ -\frac{1}{2} \| \varphi_\mathbf{g} \|^2_{\mathcal{H}_{Z^p_0}} + \mathbf{v}^\top \mathbf{b} \mathbf{g}(\mathbf{b}) + \int_{\mathcal{G}} (-\mathbf{v})^\top(t) d(-\mathbf{g})(t) \right\} \times (1 - \Psi_{P_0}(-\mathbf{v}, 0))
\]

Proof. By the Cameron-Martin density formula we get
\[
P \left\{ \inf_{A \in \mathcal{B}(\mathcal{G})} \varphi_\mathbf{g}(A) + Z^0_p(A) \geq \mathbf{v}(A) \right\} = \exp \left\{ -\frac{1}{2} \| \varphi_\mathbf{g} \|^2_{\mathcal{H}_{Z^p_0}} \right\} \times \int_{\Omega} \exp \left\{ \int_{\mathcal{G}} \mathbf{g}^\top(t) dZ^0_p(t) \right\} 1_{\inf_{A \in \mathcal{B}(\mathcal{G})} Z^0_p(A) \geq \mathbf{v}(A)} P(d\omega),
\]
so that by the same conditions as those given in Theorem 3.3 we obtain the following inequality
\[
P \left\{ \inf_{A \in \mathcal{B}(\mathcal{G})} \varphi_\mathbf{g}(A) + Z^0_p(A) \geq \mathbf{v}(A) \right\} \geq \exp \left\{ -\frac{1}{2} \| \varphi_\mathbf{g} \|^2_{\mathcal{H}_{Z^p_0}} + \mathbf{v}^\top \mathbf{b} \mathbf{g}(\mathbf{b}) + \int_{\mathcal{G}} \mathbf{v}^\top(t) d\mathbf{g}(t) \right\} \times (1 - \mathcal{J}_{P_0}(\mathbf{v}, 0)).
\]

Since \( \mathcal{J}_{P_0}(\mathbf{v}, \varphi_\mathbf{g}) = 1 - P \left\{ \inf_{A \in \mathcal{B}(\mathcal{G})} \varphi_\mathbf{g}(A) + Z^0_p(A) \geq \mathbf{v}(A) \right\} \), we then get the first inequality.

To proof the second one, we start from the fact that
\[
P \left\{ \inf_{A \in \mathcal{B}(\mathcal{G})} \varphi_\mathbf{g}(A) + Z^0_p(A) \geq \mathbf{v}(A) \right\} = P \left\{ \sup_{A \in \mathcal{B}(\mathcal{G})} -\varphi_\mathbf{g}(A) + Z^0_p(A) \leq -\mathbf{v}(A) \right\},
\]
where the equality \(-Z^0_p = Z^0_p\) in distribution has been used. By applying Proposition 3.2 and integration by parts, we lead to the following inequality
\[
P \left\{ \sup_{A \in \mathcal{B}(\mathcal{G})} -\varphi_\mathbf{g}(A) + Z^0_p(A) \leq -\mathbf{v}(A) \right\} = \exp \left\{ -\frac{1}{2} \| -\varphi_\mathbf{g} \|^2_{\mathcal{H}_{Z^p_0}} \right\} \times
\]
\[
E \left( \exp \left\{ \int_{\mathcal{G}} (-\mathbf{g})^\top(t) dZ^0_p(t) \right\} 1_{\sup_{A \in \mathcal{B}(\mathcal{G})} Z^0_p(A) \leq -\mathbf{v}(A)} \right) \leq \exp \left\{ -\frac{1}{2} \| \varphi_\mathbf{g} \|^2_{\mathcal{H}_{Z^p_0}} + (-\mathbf{v})^\top \mathbf{g}(\mathbf{b}) + \int_{\mathcal{G}} (-\mathbf{v})^\top(t) d(-\mathbf{g})(t) \right\} \times (1 - \Psi_{P_0}(-\mathbf{v}, 0)),
\]
provided \( \mathbf{g} \) is nondecreasing on \( \mathcal{G} \). Hence, completing the proof.
4 Additive set-indexed process

Other type of Kolmogorov functional of the set-indexed partial sums of multivariate residuals is defined as the supremum of the sums of the components of $\mathcal{W}_{f,P_0}$. It is frequently more effective in detecting the change of the model instead of using the component-wise comparisons. This section is devoted to the investigation of the boundary crossing probability of the additive version of $\mathcal{W}_{f,P_0}$. Let $G^p = \times_{i=1}^p G$ be the product of $p$ copies of $G$, and $B^p(G)$ be the smallest $\sigma$-algebra generated by the class of rectangles $\Gamma := \{ \times_{i=1}^p A_i | A_i \in B(G) \}$. We define the additive Gaussian process $(B^p(G)$-indexed process) denoted by $S(\mathcal{W}_{f,P_0})$ as

$$S(\mathcal{W}_{f,P_0})(\times_{i=1}^p A_i) := \sum_{i=1}^p W_{f,P_0}^{(i)}(A_i), \ \times_{i=1}^p A_i \in B^p(G),$$

where $P_0$ and $P_{0i}$ are probability measures on $B^p(G)$ and $B(G)$, respectively, such that $P_0(\times_{i=1}^p A_i) = \Pi_{i=1}^p P_{0i}(A_i)$, for $\times_{i=1}^p A_i \in B^p(G)$. Hence, by the definition, $S(\mathcal{W}_{f,P_0})$ is a centered Gaussian process with the covariance function

$$K_{S(\mathcal{W}_{f,P_0})}(\times_{i=1}^p A_i, \times_{i=1}^p B_i) = \sum_{i=1}^p \left( P_{0i}(A_i \cap B_i) - \sum_{j=1}^m \varphi_{f_j}(A_i)\varphi_{f_j}(B_i) \right)$$

and the corresponding variance

$$\text{Var}(S(\mathcal{W}_{f,P_0})(\times_{i=1}^p A_i)) = \sum_{i=1}^p \left( P_{0i}(A_i) - \sum_{j=1}^m \varphi_{f_j}^2(A_i) \right), \ \forall \times_{i=1}^p A_i \in B^p(G).$$

The product probability measure $P_0 = \Pi_{i=1}^p P_{0i}$ plays the role as the control measure of $S(\mathcal{W}_{f,P_0})$, cf. [11]. In case where $P_{0i} = P_0$ for all $i = 1, \ldots, p$, the process $S(\mathcal{W}_{f,P_0})$ is the sum of the $p$-dimensional set-indexed residual partial sums limit process studied in Section 4. Let $\cong$ denote equality in distribution and for $i = 1, \ldots, p$, let $a_i$ and $b_i$ stand for $a$ and $b$, respectively. It is clear that for $i = 1, \ldots, p$, we have

$$W_{f,P_0}^{(i)}(A_i) \cong S(\mathcal{W}_{f,P_0})(0 \times \cdots \times 0 \times A_i \times 0 \times \cdots \times 0), \ \forall A_i \in B(G).$$

In particular, by writing $S(\mathcal{W}_{f,P_0})([a_1, t_1] \times \cdots \times [a_i, t_i] \times \cdots \times [a_p, t_p])$ as $S(\mathcal{W}_{f,P_0})(t_1, \ldots, t_i, t_{i+1}, \ldots, t_p)$, for $[a_i, t_i] := \Pi_{k=1}^i [a_{ik}, t_{ik}] \subseteq G$, it holds:

$$W_{f,P_0}^{(i)}(t_i) := W_{f,P_0}^{(i)}([a_i, t_i]) \cong S(\mathcal{W}_{f,P_0})(0 \times \cdots \times 0 \times [a_i, t_i] \times 0 \times \cdots \times 0) \cong S(\mathcal{W}_{f,P_0})([a_1, a_1] \times \cdots \times [a_i, t_i] \times \cdots \times [a_p, a_p] = S(\mathcal{W}_{f,P_0})(a_1, \ldots, a_{i-1}, t_i, a_{i+1}, \ldots, a_p),$$
where \( \mathcal{W}^{(i)}_{t, P_0}(t_i) \) is distributed as \( N\left(0, P_0[a_i, t_i]\right) - \sum_{j=1}^{n_i} \int_{[a_i, t_i]} f_j(x) P_0(dx) \).

The RKHS of the \( \mathcal{B}^p(G) \)-indexed process \( \mathcal{S}(\mathcal{W}^{(i)}_{t, P_0}) \) can be derived either by applying the analogous way as that of in the preceding section by seeking a model suitable to the structure of the covariance function, or by using Proposition 4.1 of [11] and Equation (6) based on the fact that \( \mathcal{S} \) can be regarded as a linear and continuous mapping on \( \mathcal{C}(\mathcal{B}^p(G)) \). Hence, it can be easily shown that

\[
\mathcal{H}_{\mathcal{S}(\mathcal{W}^{(i)}_{t, P_0})} = \left\{ h | h(x_{i=1}^p A_i) = \sum_{i=1}^{p} h_i(A_i), h_i \in \mathcal{H}_{\mathcal{W}^{(i)}_{t, P_0}}, \forall i = 1, \ldots, p \right\},
\]

which is furnished with the inherent norm and inner product defined by

\[
\| h \|_{\mathcal{H}_{\mathcal{S}(\mathcal{W}^{(i)}_{t, P_0})}}^2 = \inf_{h(x_{i=1}^p A_i) = \sum_{i=1}^{p} h_i(A_i), h_i \in \mathcal{W}^{(i)}_{t, P_0}} \sum_{i=1}^{p} \| h_i \|_{\mathcal{H}_{\mathcal{W}^{(i)}_{t, P_0}}}^2
\]

\[
\langle h_1, h_2 \rangle_{\mathcal{H}_{\mathcal{S}(\mathcal{W}^{(i)}_{t, P_0})}} = \sum_{i=1}^{p} \langle h_1, h_2 \rangle_{\mathcal{H}_{\mathcal{W}^{(i)}_{t, P_0}}}, \text{ with } h(x_{i=1}^p A_i) = \sum_{i=1}^{p} h_i(A_i).
\]

For the sake of brevity we write throughout the rest of the present paper the Cartesian product \( \times_{i=1}^{p} A_i \) by \( A^* \).

In the following we consider the boundary crossing probability involving additive set-indexed Brownian pillow with additive trend:

\[
\Upsilon_{P_0}(v, \mathcal{S}(\varphi_G)) := \mathcal{P}\left\{ \sup_{A^* \in \mathcal{B}^p(G)} \mathcal{S}(\varphi_G)(A^*) + \mathcal{S}(\mathcal{Z}_{P_0})(A^*) \geq v(A^*) \right\}, \tag{8}
\]

for any function \( v \in \mathcal{C}(\mathcal{B}^p(G)) \), where

\[
\mathcal{S}(\mathcal{Z}_{P_0})(A^*) = \sum_{i=1}^{p} Z_{P_0}^{(i)}(A_i) = \sum_{i=1}^{p} (Z_{P_0}^{(i)}(A_i) - P_0(A_i)Z_{P_0}^{(i)}(G))
\]

with \( Z_{P_0}^{(i)} \) is the set-indexed Brownian sheet having the control measure \( P_0 \), \( \mathcal{S}(\varphi_G)(A^*) = \sum_{i=1}^{p} \varphi_i(A_i) \), with \( g_i \in L_2(P_0, G) \), for \( i = 1, \ldots, p \). The RKHS of \( \mathcal{S}(\mathcal{Z}_{P_0}) \) is given by

\[
\mathcal{H}_{\mathcal{S}(\mathcal{Z}_{P_0})} = \left\{ h | \exists \ell_i \in L_2(P_0, G), h_i(A_i) = \int_{A_i} \ell_i(t_i) P_0(dt_i) - P_0(A_i) \int_{G} \ell_i(t_i) P_0(dt_i), i = 1, \ldots, p \right\}.
\]

Thus, every \( h \in \mathcal{H}_{\mathcal{S}(\mathcal{Z}_{P_0})} \) necessarily satisfies \( h(x_{i=1}^p G) = h(x_{i=1}^p \emptyset) = 0 \).

Below we derive the upper and lower bounds for \( \Upsilon_{P_0}(v, \mathcal{S}(\varphi_G)) \). The similar technique and assumption to \( g \) as those given in Theorem 3.3 will be applied.
Theorem 4.1 Let $S(\varphi_g) \in \mathcal{H}_S(Z_{P_0}^o)$, such that for $i = 1, \ldots, p$,

$$\varphi_g(A_i) = \int_{A_i} g_i(t_i) P_{0i}(dt_i) - P_{0i}(A_i) \int_{G} g_i(t_i) P_0(dt_i), \ A_i \in B(G),$$

with $g_i \in L_2(P_{0i}, G) \cap \text{BV}(G)$. If $g_i$ is nondecreasing on $G$ having the constant marginal on $G^k$, for $k = 1, \ldots, (d - 1)$, then for $d$ even, we have

$$\Upsilon_{P_0}(v, S(\varphi_g)) \geq 1 - \exp \left\{ -\frac{1}{2} \|S(\varphi_g)\|_{H_0(S(Z_{P_0}^o))}^2 + \sum_{i=1}^{p} v(a_1, \ldots, a_{i-1}, b_i, a_{i+1}, \ldots, a_p) g_i(b_i) + \sum_{i=1}^{p} \int_{G} v(a_1, \ldots, a_{i-1}, t_i, a_{i+1}, \ldots, a_p) dg_i(t_i) \right\} \left(1 - \Upsilon_{P_0}(v, 0)\right)$$

and

$$\Upsilon_{P_0}(v, S(\varphi_g)) \leq 1 - \exp \left\{ -\frac{1}{2} \|S(\varphi_g)\|_{H_0(S(Z_{P_0}^o))}^2 + \sum_{i=1}^{p} w(a_1, \ldots, a_{i-1}, b_i, a_{i+1}, \ldots, a_p) g_i(b_i) + \sum_{i=1}^{p} \int_{G} w(a_1, \ldots, t_i, \ldots, a_p) dg_i(t_i) \right\} \left(2 + \Upsilon_{P_0}(w, 0) - \Upsilon_{P_0}(v, 0)\right),$$

for every $w \in C(B^p(G))$, such that $w(A^*) \leq S(\varphi_g)(A^*) + S(Z_{P_0}^o)(A^*) \leq v(A^*)$, for all $A^* \in B^p(G)$.

Proof. By the transformation of variable and by using the multivariate Cameron-Martin density formula formulated in Proposition 3.1, we get

$$1 - \Upsilon_{P_0}(v, S(\varphi_g)) = \mathbb{P} \left\{ \sup_{A^* \in B^p(G)} S(\varphi_g)(A^*) + S(Z_{P_0}^o)(A^*) \leq v(A^*) \right\}$$

$$= \exp \left\{ -\frac{1}{2} \|S(\varphi_g)\|_{H_0(S(Z_{P_0}^o))}^2 \right\} \times \mathbb{E} \left( \exp \left\{ \sum_{i=1}^{p} \int_{G} g_i(t_i) dZ_{P_0}^{o(i)}(t_i) \right\} \mathbb{1}_{\left\{ \sup_{A^* \in B^p(G)} S(Z_{P_0}^o)(A^*) \leq v(A^*) \right\}} \right).$$

Since $Z_{P_0}^{o(i)}(t_i) \overset{D}{=} S(Z_{P_0}^o)(a_1, \ldots, a_{i-1}, t_i, a_{i+1}, \ldots, a_p)$, for $i = 1, \ldots, p$, then

$$\sum_{i=1}^{p} \int_{G} g_i(t_i) dZ_{P_0}^{o(i)}(t_i) \overset{D}{=} \sum_{i=1}^{p} \int_{G} g_i(t_i) dS(Z_{P_0}^o)(a_1, \ldots, a_{i-1}, t_i, a_{i+1}, \ldots, a_p).$$

Furthermore, by applying integration by parts formula for function with $d$ variables (6.4) and the fact that $S(Z_{P_0}^o)(a_1, \ldots, a_{i-1}, t_i, a_{i+1}, \ldots, a_p) = 0$ a.s.,
for \( t_i = a_i, i = 1, \ldots, p \), we obtain under \( 1 \left\{ \sup_{A^* \in B^p(G)} S(Z_{P_0}^p)(A^*) \leq v(A^*) \right\} \), that

\[
1 - \Upsilon_{P_0}(v, S(\varphi_G)) \\
\leq \exp \left\{ -\frac{1}{2} \| S(\varphi_G) \|_{H_S(Z_{P_0}^p)}^2 + \sum_{i=1}^p v(a_1, \ldots, a_{i-1}, b_i, a_{i+1}, \ldots, a_p)g_i(b_i) \right. \\
\left. + \sum_{i=1}^p \int_G v(a_1, \ldots, a_{i-1}, t_i, a_{i+1}, \ldots, a_p)d\gamma_i(t_i) \right\} (1 - \Upsilon_{P_0}(v, 0)),
\]

which is establishing the lower bound.

The proof of the second inequality (the upper bound) is based on the following inequality

\[
1 - \Upsilon_{P_0}(v, S(\varphi_G)) \geq \mathbb{P} \left\{ w(A^*) \leq S(\varphi_G)(A^*) + S(Z_{P_0}^p)(A^*) \leq v(A^*), \forall A^* \subset G^p \right\}
\]

which is true for any \( w \) and \( v \) in \( C(B^p(G)) \). By applying the similar argument as in the proof of the first inequality, we get the result. We are done.

Similar results as those presented in Theorem 3.6 is summarized in Theorem 4.1 below. It describes the upper and lower bounds for

\[
\Lambda(v, S(\varphi_G)) := \mathbb{P} \left\{ \inf_{A^* \in B^p(G)} S(\varphi_G)(A^*) + S(Z_{P_0}^p)(A^*) \leq v(A^*) \right\}.
\]

The proof is left, since it can be established immediately by combining the method of proving Theorem 3.6 and that of proving Theorem 4.1.

**Theorem 4.2** Under the similar condition as in Theorem 4.1, it holds

\[
\Lambda_{P_0}(v, S(\varphi_G)) \leq \\
1 - \exp \left\{ -\frac{1}{2} \| S(\varphi_G) \|_{H_S(Z_{P_0}^p)}^2 + \sum_{i=1}^p v(a_1, \ldots, b_i, a_{i+1}, \ldots, a_p)g_i(b_i) \right. \\
\left. + \sum_{i=1}^p \int_G v(a_1, \ldots, a_{i-1}, t_i, a_{i+1}, \ldots, a_p)d\gamma_i(t_i) \right\} (1 - \Lambda_{P_0}(v, 0)),
\]

and

\[
\Lambda_{P_0}(v, S(\varphi_G)) \geq \\
1 - \exp \left\{ -\frac{1}{2} \| S(\varphi_G) \|_{H_S(Z_{P_0}^p)}^2 + \sum_{i=1}^p v(a_1, \ldots, b_i, a_{i+1}, \ldots, a_p)g_i(b_i) \right. \\
\left. + \sum_{i=1}^p \int_G (-v)(a_1, \ldots, a_{i-1}, t_i, a_{i+1}, \ldots, a_p)d(-\gamma_i)(t_i) \right\} (1 - \Lambda_{P_0}(-v, 0)).
\]
Remark 4.3 The expression $\|S(\varphi_\Sigma)\|_{H_S(Z_p)}^2$ in Theorem 4.1 and Theorem 4.2 is computed by using the formula

$$
\|S(\varphi_\Sigma)\|_{H_S(Z_p)}^2 = \sum_{i=1}^p \|\varphi_{g_i}\|_{H_{Z_{P_0}(i)}}^2 = \sum_{i=1}^p \|g_i\|_{L_2(P_{0i}, G)}^2.
$$

As an example let us consider a lack-of-fit test for a bivariate regression on $G = [1, 2] \times [2, 3]$, for checking that a constant model holds true. Let $P_{01}$ and $P_{02}$ be probability measures with the CDF $F_{01}(x, y) = 12(1 - 1/x)(1/2 - 1/y)$, $F_{02}(x, y) = (x - 1)(y - 2)$, $(x, y) \in G$, respectively. Suppose the power is evaluated at $s = (s_1, s_2)\top$, defined by $s_1(x, y) = 1 + x + y$ and $s_2(x, y) = 2 + x - y$, for $(x, y) \in G$. By the definition, the deterministic signal is produced by the function $\varphi_{\Sigma^{-1/2}g} \in C(B^2(G))$, where $g = (g_1, g_2)\top$ is computed as follows:

$$
\begin{align*}
g_1(x, y) &= s_1(x, y) - \int_G s_1(x, y)P_{01}(dx, dy) = x + y - (2 \ln(2) + 6 \ln(3/2)), \\
g_2(x, y) &= s_2(x, y) - \int_G s_2(x, y)P_{02}(dx, dy) = 1 + x - y.
\end{align*}
$$

Suppose the covariance matrix is $\Sigma = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$. Then $\|S(\varphi_{\Sigma^{-1/2}g})\|_{H_S(Z_p)}^2$ is given by

$$
\begin{align*}
\int_{[1,2] \times [2,3]} (1.366g_1(x, y) - 0.366g_2(x, y))^2P_{01}(dx, dy) \\
+ \int_{[1,2] \times [2,3]} (-0.366g_1(x, y) + 1.366g_2(x, y))^2P_{02}(dx, dy)
\end{align*}
= \begin{align*}
\int_{[1,2] \times [2,3]} (1.366(x + y - 3.819) - 0.366(1 + x - y))^2P_{01}(dx, dy) \\
+ \int_{[1,2] \times [2,3]} (-0.366(x + y - 3.819) + 1.366(1 + x - y))^2P_{02}(dx, dy)
\end{align*}
= 0.395 + 0.338 = 0.733.
$$

Remark 4.4 Results for the multivariate set-indexed Brownian sheet can be derived straightforwardly. However in this paper we only take care on Brownian pillow since Brownian sheet corresponds to zero model which is not so important from the perspective of model check or boundary detection problem.

5 Conclusion

We have established the upper and lower bounds for the boundary crossing probability which appeared in asymptotic lack-of-fit test and boundary detection problem for multivariate spatial regression based on set-indexed partial sums of the residuals. Under mild condition the results have been derived.
mainly based on the Cameron-Martin translation formula of the set-indexed residual partial sum limit processes. Our contributions are not only important in the area of statistics, but also important for the extension and abstraction of the existing results in other areas of study such as in finance mathematics and in statistical physics.

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6 Appendix

6.1 Nondecreasing Function with Several Variables

Definition 6.1 (Yeh[23]) Let $\psi : G := \times_{k=1}^{d} [a_k, b_k] \to \mathcal{R}$ be a real function on $G$. For $k = 1, \ldots, d$, the increment of $\psi$ on the closed interval $[a_k, b_k]$ is denoted by $\Delta_{a_k}^{b_k} \psi$, given by

$$\Delta_{a_k}^{b_k} \psi := \psi(x_1, \ldots, x_{k-1}, b_k, x_{k+1}, \ldots, x_d) - \psi(x_1, \ldots, x_{k-1}, a_k, x_{k+1}, \ldots, x_d).$$

For any $d-$vectors $v := (v_k)_{k=1}^{d}$ and $w := (w_k)_{k=1}^{d} \in G$, the increment of $\psi$ over the $d-$dimensional closed rectangle $[v, w]$ is denoted by $\Delta_{v}^{w} \psi$, defined recursively as

$$\Delta_{v}^{w} \psi := (\Delta_{v_1}^{w_1} \circ \cdots \circ \Delta_{v_{d-1}}^{w_{d-1}} \circ \Delta_{v_d}^{w_d})(\psi).$$

Definition 6.2 Let $\Gamma_k := \{[x_{k_0}, x_{k_1}], [x_{k_1}, x_{k_2}], \ldots, [x_{k_{M_k-1}}, x_{k_{M_k}}]\}$ be a collection of $M_k$ closed intervals subsets of $[a_k, b_k]$ with $a_k = x_{k_0} \leq x_{k_1} \leq \ldots \leq x_{k_{M_k}} = b_k$, for $k = 1, \ldots, d$. The Cartesian product $K := \times_{k=1}^{d} \Gamma_k$ which consists of $\Pi_{k=1}^{d} M_k$ $d-$dimensional closed rectangles is called a non-overlapping finite exact cover of $G$. Next, for $1 \leq w_k \leq M_k$, let $J_{w_1 \ldots w_d}$ be the element of $K$ defined by $J_{w_1 \ldots w_d} := \times_{k=1}^{d} [x_{k_{w_k-1}}, x_{k_{w_k}}]$. An operator $\Delta_{J_{w_1 \ldots w_d}}$ acting on a function $\psi$, defined by

$$\Delta_{J_{w_1 \ldots w_d}} \psi := (\Delta_{x_{w_1}}^{x_{w_1-1}} \circ \Delta_{x_{w_2}}^{x_{w_2-1}} \circ \cdots \circ \Delta_{x_{w_d}}^{x_{w_{d-1}}})(\psi),$$

is called the increment of $\psi$ over $J_{w_1 \ldots w_d}$. The variation of $\psi$ over the finite exact cover $K$ is defined by

$$v(\psi; K) := \sum_{w_1=1}^{M_1} \cdots \sum_{w_d=1}^{M_d} \left| \Delta_{J_{w_1 \ldots w_d}} \psi \right|.$$

Accordingly, the total variation of $\psi$ over $G$ is defined by

$$V(\psi; G) := \sup_{K \in \mathcal{J}(K)} v(\psi; K).$$
Furthermore, the function $\psi$ is said to have bounded variation in the sense of Vitaly on $G$ if there exists a real number $M > 0$ s.t. $V(\psi; G) \leq M$ for some real number $M > 0$. The class of such functions is denoted by $BVV(G)$. See also Yeh [23].

**Definition 6.3** (Yeh [23]) Let $(x_k^d)_{k=1}^d$ be a variable in $G$. For a fixed $k$, let $G^k$ be a $k$-dimensional closed rectangle constructed in the following way. We choose $d - k$ components of the variable $(x_k^d)_{k=1}^d$. For each choice from all elements of the set $C_{d-k}^d$, we set each $x_i$ with $a_i$ or $b_i$ and let the remaining $k$ variables to satisfy $a_j \leq x_j \leq b_j$. Then for each $k$ we get $2^{d-k}|C_{d-k}^d|$ $k$-dimensional closed rectangles $G^k$. For convention we denote the collection of all $2^{d-k}|C_{d-k}^d|$ of closed rectangles $G^k$ by $B^k$ and the $j$-th element of $B^k$ will be denoted by $G^k_j$, for $j = 1, \ldots, 2^{d-k}|C_{d-k}^d|$. A function $\psi$ is said to have bounded variation in the sense of Hardy on $G$, if and only if for each $k = 1, \ldots, d$ and $j = 1, \ldots, 2^{d-k}|C_{d-k}^d|$, there exists a real number $M_{jk} > 0$ s.t. $V(\psi_{G^k_j}(\cdot); G^k_j) \leq M_{jk}$, where for $k = 1, \ldots, d$ and $j = 1, \ldots, 2^{d-k}|C_{d-k}^d|$, $\psi_{G^k_j}(\cdot)$ is a function with $k$ variables obtained from the function $\psi(x_1, x_2, \ldots, x_d)$ by setting the $d - k$ selected variables with $a_j$ or $b_j$, whereas the remaining $k$ variables lies in the interval $[a_k, b_k]$. The class of such functions will be denoted by $BV_H(G)$. Furthermore, $\psi_{G^k_j}(\cdot)$ is called the marginal of $\psi$ on $G^k_j$ with a notification that for $k = d$, $\psi_{G^k_j}(\cdot)$ leads to $\psi$. The marginal function $\psi_{G^k_j}$ is called non decreasing on $G^k_j$, if and only if for every $k$-dimensional closed rectangle $J \subset G^k_j$, it holds $\Delta_J \psi_{G^k_j} \geq 0$. Conversely, $\psi_{G^k_j}$ is called non decreasing on $G^k_j$, if and only if $-\psi_{G^k_j}$ is decreasing.

### 6.2 Integration by Parts of Riemann-Stieltjes integral

For the family $B^k$ defined in Definition A.4, let $I := \bigcup_{k=0}^d B^k$, where for $k = 0$, the family $B^0$ is a collection of $2^d$ different points in $G$. As an example, for $d = 2$, we have $B^0 = \{G^0_1 = (a_1, a_2), G^0_2 = (a_1, b_2), G^0_3 = (b_1, a_2), G^0_4 = (b_1, b_2)\}$. For each $k = 1, \ldots, d$ and $j = 1, \ldots, 2^{d-k}|C_{d-k}^d|$, let $\#(G^k_j)$ be the number of $b_k$’s appearing in $G^k_j$. Next let $\varphi$ and $\psi$ be defined on $G$. If $\varphi$ is Riemann-Stieltjes integrable with respect to $\psi$ on $G^k_j \in B^k$, we denote the integral by $\int_{G^k_j} \varphi \; d^k \psi$. For $k = 0$, it is understood that $\int_{G^k_j} \varphi \; d^0 \psi$ is defined as the product of $\varphi$ and $\psi$ at that point of $G^0_j$ (cf. Yeh [23]).

**Theorem 6.4** (Integration by parts (Yeh [23])) Let $\varphi$ be Riemann-Stieltjes integrable with respect to $\psi$ on each member of $I$. Then $\psi$ is Riemann-Stieltjes integrable with respect to $\varphi$ on $G$, and we have the formula

$$
\int_G \psi(x_1, \ldots, x_d) \; d\varphi(x_1, \ldots, x_d) = \sum_{k=0}^d \sum_{j=1}^{2^{d-k}|C_{d-k}^d|} (-1)^{(d-\#(G^k_j))} \int_{G^k_j} \varphi \; d^k \psi.
$$
Example 6.5 We consider first the case where $G = [a_1, b_1] \times [a_2, b_2]$. The integration formula reads (see also M´oricz (2006))

$$
\int_G \psi(x, y) d\varphi(x, y) = \Delta_{a_2}^{b_2} \Delta_{a_1}^{b_1}(\varphi \psi) \\
+ \int_{[a_2, b_2]} \varphi(a_1, y)d\psi(a_1, y) - \int_{[a_2, b_2]} \varphi(b_1, y)d\psi(b_1, y) \\
+ \int_{[a_1, b_1]} \varphi(x, a_2)d\psi(x, a_2) - \int_{[a_1, b_1]} \varphi(x, b_2)d\psi(x, b_2) + \int_G \varphi(x, y)d\psi(x, y).
$$

References


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