Steady Creeping Slip Flow of Viscous Fluid through a Permeable Slit with Exponential Reabsorption

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Abstract

This paper presents the hydrodynamics of creeping flow of viscous fluid through a permeable slit with exponential reabsorption by taking into consideration the influence of the slip velocity at the walls. A system of partial differential equations (PDEs) is converted into a single PDE using the stream function. Then, taking a particular form of the stream function, exact solutions are determined. Expressions for velocity components, axial flow rate, mass flow rate, pressure distribution,

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wall shear stress, fractional reabsorption and leakage flux are obtained explicitly. Using the data of a rat’s kidney, variation of the exponential reabsorption parameter for various values of fractional reabsorption is tabulated. These values are calculated using Newton’s method with the help of computer software MAPLE 15. The effect of exponential reabsorption and slip parameters on velocity components, axial flow rate, pressure distribution, wall shear stress and leakage flux are graphically discussed. It is observed that these parameters play a vital role in altering the flow properties, which are useful in analyzing the reabsorption of glomerular filtrate through an abnormal proximal convoluted tubule.

**Keywords:** Slip condition; Exact solution; Creeping flow; Porous slit; Proximal convoluted tubule

### 1 Introduction

The study of creeping flow through a permeable duct has many applications in the field of engineering, such as flow through geothermal energy extraction, drying of food, insulation of buildings, transpiration cooling, reverse osmosis desalination, glomerular tubule ultrafiltration, proximal tubule reabsorption and artificial kidneys. Out of these many applications, flow through kidneys is perhaps the most significant. It has been reported in [1] that about 20% of the up to 50,000 people who die each year from kidney disease could benefit from artificial kidney treatment or kidney transplantation.

Kidneys are natural filtration plants found in the body of most living creatures. The main functions of kidneys are to remove waste materials from the blood and to balance body fluid by the process of reabsorption. Each kidney contains over a million tiny filtering units, known as nephrons, that are similar in structure and function. The two major parts of a kidney nephron are the renal corpuscle and the renal tubule. A renal corpuscle is the initial blood-filtering component of a nephron, and consists of two structures: a glomerulus and a Bowman’s capsule. Blood is forced from glomerular capillaries at higher pressure to allow filtration and the resulting filtrate known as glomerular filtrate enters the Bowman’s capsule. The portion of the nephron following the Bowman’s capsule is known as the renal tubule. This is where most of the materials obtained from glomerular filtrate such as water, glucose and electrolytes are reabsorbed through the tubular walls if any of these substances are at low levels in the body. Every day, 200 quarts of blood enter the nephrons for purification, and after reabsorption of valuable nutrients from the blood, about 2 quarts of waste products and extra water create urine [2]. If the kidneys of a patient no longer filters the blood and reabsorbs nutrients properly, artificial kidneys might be capable of postponing death from irreversible kidney failure for several years.
The reabsorption of the glomerular filtrate is about 80% under normal conditions but is reduced in the presence of a tubular disorder. The structure of the renal tubule is composed of epithelial cells and nutrients from the blood are reabsorbed through these cells. Several diseases like tubular proteinuria, acute pyelonephritis, allergic interstitial nephritis and tubular interstitial injury result in abnormal reabsorption, and this abnormal reabsorption has been studied by several researchers [3, 4, 5, 6]. Other conditions such as urinary tract infections, i.e., pyelonephritis and obstructive uropathy, can also cause abnormal reabsorption. In these types of diseases, microorganisms such as bacteria grow at the tubular walls and pore blockage phenomena results [7]. This pore blockage causes abnormal reabsorption, which then results in the reduction of the amount of useful fluid needed to fulfill the body’s needs. (see Figure 5)

Tubular diseases have a major impact on kidney transplants and the usage of artificial kidneys; there is a need for mathematical models to help understand the physiological process of flow through the renal tubule under both normal and diseased conditions. Macey [8, 9] studied the hydrodynamics of viscous fluid through a porous tube with linear reabsorption and exponential reabsorption at the walls. Kozinski et al. [10] completed the solutions of Macey [9], and obtained the exact expressions for velocity and pressure difference. Several researchers [11, 12, 13, 14, 15, 16, 17] later studied the flow behavior of glomerular filtrate through the renal tubule under various physiological conditions. Recently, Haroon et al. [18] studied the behavior of glomerular filtrate through a proximal convoluted tubule with uniform reabsorption at the walls. They determined the exact solution of the momentum equation and discussed the fractional reabsorption at the walls using the data from one of the rat kidneys. Siddiqui et al. [19] examined the hydrodynamics of a viscous fluid through a porous slit with linear absorption at the walls. They discovered that the adsorption of microorganisms at the tubular wall reduced the efficiency of the renal tubule. Later, Haroon et al. [20] extended the work of Siddiqui et al. [19] and investigated the flow of glomerular filtrate through a renal tubule with periodic reabsorption at the walls. They concluded that the pore blockage phenomenon is random, and perhaps the periodic nature of pore blockage may be one of the causes of the disease.

The aim of this present study is to investigate the behavior of glomerular filtrate through a renal tubule with exponential reabsorption by taking into consideration the influence of the slip velocity at the walls. This exponential reabsorption at the walls demonstrates that the pores are gradually becoming blocked from the opening to the end of the tubule. Because of the pores at the tubular walls, the glomerular filtrate may slip at those walls, so the validity of the slip boundary condition cannot be ignored. Beavers et al. [21] suggested that the slip velocity is proportional to the velocity gradient on the porous
boundary. Some theoretical support for the Beavers-Joseph slip condition was provided by Saffman [22]. He analyzed the flow in a channel with one porous boundary as a particular case of flow through a non-uniform medium. He derived the form of the boundary condition correct up to order $k$ as:

$$u_{\text{slip}} = -\frac{\sqrt{k}}{\gamma} \frac{\partial u}{\partial y} + O(k),$$  \hspace{1cm} (1)

where $\gamma$ is a dimensionless constant and $k$ is the specific permeability of the porous medium. Additional support for the Beavers-Joseph slip condition was provided by Taylor [23] and Richardson [24]. Recently, Siddiqui et al. [25] investigated the effects of slip on the flow of an incompressible fluid through a renal tubule. They considered the boundary condition (1) and discovered that the slip coefficient considerably influenced the flow variables.

This paper is arranged as follows: in section 2, the formulation of the problem is described by writing a mathematical equation that governs the flow of glomerular filtrate through a renal tubule. A rectangular coordinate system $(x, y, z)$ is chosen such that $x$ ranges from 0 at the Bowman’s capsule to $L$ at the loop of Henle and $y$ ranges from the axis to the tubular walls. The reabsorption velocity $v$ at the walls is exponential with axial distance and longitudinal velocity $u$ experiences a slip velocity due to smaller pores. In section 3, the exact solutions for velocity components, volume flow rate, mass flow rate, pressure distribution, wall shear stress, fractional reabsorption and leakage flux are determined. In section 4, a set of physiological data from one of the rat kidneys is used to find the theoretical values of the exponential reabsorption parameter for various values of fractional reabsorption and is presented in the form of a table. The variations of flow properties are graphically presented in section 5. Conclusions are then presented in the final section.

## 2 Description of the problem

The steady, laminar flow of an incompressible, homogeneous, Newtonian fluid of constant viscosity through a porous slit having width $2H$ is considered. A rectangular Cartesian coordinate system $(x, y)$ is chosen with the $x$-axis aligned with the center line of the slit and the $y$-axis normal to it, (Figure 2). At the point $x = 0$, the slit becomes porous with the reabsorption velocity decaying exponentially throughout the length $L$ of the slit. The fluid may slip on the surface of the porous walls and the Reynolds number is assumed to be sufficiently small so that inertial forces can be neglected. The initial volume flow rate $Q_0$ is assumed to be constant at $x = 0$. Assuming that the breadth of the slit $W$ is large enough to neglect the third component of the velocity, the flow field can be considered to be two dimensional in the form given by:

$$V = [u(x, y), v(x, y)],$$  \hspace{1cm} (2)
Steady creeping slip flow of viscous fluid through a permeable slit

where $u$ and $v$ are velocity components in the $x$ and $y$ directions. Using profile (2, the continuity equation becomes:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,$$  \hspace{1cm} (3)

and for creeping flow the momentum equation takes the form:

$$0 = -\frac{\partial p}{\partial x} + \mu \nabla^2 u,$$  \hspace{1cm} (4)

$$0 = -\frac{\partial p}{\partial y} + \mu \nabla^2 v.$$  \hspace{1cm} (5)

The boundary conditions are:

$$\frac{\partial u}{\partial y} = 0, \quad v = 0, \quad \text{at} \quad y = 0,$$  \hspace{1cm} (6)

$$u + \phi \frac{\partial u}{\partial y} = 0, \quad v = V_0 e^{-\alpha x}, \quad \text{at} \quad y = H,$$  \hspace{1cm} (7)

$$Q_0 = 2W \int_0^H u(0, y) dy.$$  \hspace{1cm} (8)

where $\phi = \frac{\sqrt{k}}{\gamma}$. In equation (7, the reabsorption velocity $v$ depends upon $\alpha$ showing that the pores at the tubular walls near the exit of the slit are becoming blocked. If $\alpha \to 0$, no pore blockage phenomenon occurs; additionally, if $\phi \to 0$, the problem of uniform reabsorption that Haroon et al. [18] studied, is recovered.

Introducing the stream function $\psi(x, y)$ of the following form:

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x},$$  \hspace{1cm} (9)

it is observed that the equation of continuity (3 is identically satisfied and equations (4 -5) reduce to the following forms:

$$\frac{\partial p}{\partial x} = \mu \nabla^2 \left( \frac{\partial \psi}{\partial y} \right),$$  \hspace{1cm} (10)

$$\frac{\partial p}{\partial y} = -\mu \nabla^2 \left( \frac{\partial \psi}{\partial x} \right).$$  \hspace{1cm} (11)

Eliminating the pressure gradient from the above equations, we obtain the following partial differential equation:

$$\nabla^4 \psi = 0,$$  \hspace{1cm} (12)
where $\nabla^4 = \nabla^2(\nabla^2)$ and $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is a Laplacian operator.

Further, the boundary conditions (6-8) in term of $\psi(x, y)$ become:

\[
\begin{align*}
\frac{\partial^2 \psi}{\partial y^2} &= 0, \quad \frac{\partial \psi}{\partial x} = 0, \quad \text{at} \quad y = 0, \quad (13) \\
\frac{\partial \psi}{\partial y} + \phi \frac{\partial^2 \psi}{\partial y^2} &= 0, \quad -\frac{\partial \psi}{\partial x} = V_0 e^{-\alpha x}, \quad \text{at} \quad y = H, \quad (14) \\
\psi(0, 0) &= 0, \quad (15)
\end{align*}
\]

and

\[
\psi(0, H) = \frac{Q_0}{2W}. \quad (16)
\]

Equation (12 along with boundary conditions (13-14) and (15-16) is a boundary value problem (BVP), describing the two dimensional slip flow of Newtonian fluid through a porous slit with exponential reabsorption at the walls. The solution of this problem will be illustrated in the following section.

3 Method of solution

To obtain exact solutions of the above two dimensional BVP, we analyze the boundary condition and choose the stream function $\psi(x, y)$ of the form:

\[
\psi = V_0 e^{-\alpha x} F(y) + K(y), \quad (17)
\]

where $F(y)$ and $K(y)$ are arbitrary functions that need to be determined. Using equation (17 in equation (12, we obtain the following equation:

\[
V_0 e^{-\alpha x} \left[ \frac{d^4 F}{dy^4} + 2\alpha^2 \frac{d^2 F}{dy^2} + \alpha^4 F \right] + \frac{d^4 K}{dy^4} = 0, \quad (18)
\]

which can take the form:

\[
V_0 e^{-\alpha x} \left[ \frac{d^4 F}{dy^4} + 2\alpha^2 \frac{d^2 F}{dy^2} + \alpha^4 F \right] = 0, \quad (19)
\]

and

\[
\frac{d^4 K}{dy^4} = 0. \quad (20)
\]

Since $V_0 e^{-\alpha x} \neq 0$, therefore,

\[
\frac{d^4 F}{dy^4} + 2\alpha^2 \frac{d^2 F}{dy^2} + \alpha^4 F = 0. \quad (21)
\]
With the help of equation (17, the boundary conditions (13-16) reduce to:

\[ F(0) = 0, \quad \frac{d^2 F(0)}{dy^2} = 0, \quad (22) \]

\[ F(H) = \frac{1}{\alpha}, \quad \frac{dF(H)}{dy} + \phi \frac{d^2 F(H)}{dy^2} = 0, \quad (23) \]

and

\[ K(0) = 0, \quad \frac{d^2 K(0)}{dy^2} = 0, \quad (24) \]

\[ K(H) = \frac{\alpha Q_0 - 2V_0W}{2\alpha W}, \quad \frac{dK(H)}{dy} + \phi \frac{d^2 K(H)}{dy^2} = 0. \quad (25) \]

The solution of equation (21 using boundary conditions (22-23) is found to be:

\[ F(y) = \frac{\alpha H \sin (\alpha H) - \cos (\alpha H) + \phi \alpha^2 H \cos (\alpha H) + 2\phi \alpha \sin (\alpha H)}{(\alpha H - \cos (\alpha H) \sin (\alpha H) + 2\alpha \phi \sin^2 \alpha H) \alpha} \sin (\alpha y) \]

\[ - \frac{(\alpha \phi \sin (\alpha H) - \cos (\alpha H))}{\alpha H - \cos (\alpha H) \sin (\alpha H) + 2\alpha \phi \sin^2 (\alpha H)} y \cos (\alpha y), \quad (26) \]

and the solution of equation (20 along with the boundary conditions (24-25) is:

\[ K(y) = \frac{\alpha Q_0 - 2V_0W}{4\alpha W H^2 (H + 3\phi)} \left[ 3H(H + 2\phi) y - y^3 \right]. \quad (27) \]

The expression for the stream function \( \psi(x, y) \) is obtained by combining equations (26 and 27 in equation (17 and writing as:

\[ \psi(x, y) = V_0 e^{-\alpha x} \left[ \frac{\alpha H \sin (\alpha H) - \cos (\alpha H) + \phi \alpha^2 H \cos (\alpha H) + 2\phi \alpha \sin (\alpha H)}{(\alpha H - \cos (\alpha H) \sin (\alpha H) + 2\alpha \phi \sin^2 \alpha H) \alpha} \sin (\alpha y) \right] \]

\[ - \frac{(\alpha \phi \sin (\alpha H) - \cos (\alpha H))}{\alpha H - \cos (\alpha H) \sin (\alpha H) + 2\alpha \phi \sin^2 (\alpha H)} y \cos (\alpha y) \]

\[ + \frac{\alpha Q_0 - 2V_0W}{4\alpha W H^2 (H + 3\phi)} \left[ 3H(H + 2\phi) y - y^3 \right], \quad (28) \]

which strongly depends upon slip parameter \( \phi \), uniform reabsorption velocity \( V_0 \) and exponential reabsorption parameter \( \alpha \).

### 3.1 Components of velocity

The velocity components are obtained using relation (9:

\[ u(x, y) = V_0 e^{-\alpha x} \left[ \frac{\alpha H \sin (\alpha H) - \cos (\alpha H) + \phi \alpha^2 H \cos (\alpha H) + 2\phi \alpha \sin (\alpha H)}{(\alpha H - \cos (\alpha H) \sin (\alpha H) + 2\alpha \phi \sin^2 \alpha H) \alpha} \cos (\alpha y) \right. \]
\[ v(x, y) = V_0 e^{-\alpha x} \left[ \frac{\alpha \sin (\alpha H) - \cos (\alpha H)}{\alpha H - \cos (\alpha H) \sin (\alpha H) + 2\alpha \phi \sin^2 (\alpha H)} \{ \cos \alpha y - \alpha y \sin (\alpha y) \} \right] \]

\[ \frac{3(\alpha Q_0 - 2V_0 W)}{4\alpha WH^2(H + 3\phi)} \]

\[ u_{max} = V_0 e^{-\alpha x} \left[ \frac{\alpha H \sin (\alpha H) - \cos (\alpha H) + \phi^2 H \cos (\alpha H) + 2\phi \alpha \sin (\alpha H)}{(\alpha H - \cos (\alpha H) \sin (\alpha H) + 2\alpha \phi \sin^2 (\alpha H)) \alpha} \sin (\alpha y) \right] \]

\[ \frac{\alpha (\alpha H \sin (\alpha H) - \cos (\alpha H))}{\alpha H - \cos (\alpha H) \sin (\alpha H) + 2\alpha \phi \sin^2 (\alpha H)} y \cos (\alpha y) \right]. \]

It is found that if \( \alpha \to 0 \) and \( \phi \to 0 \), the velocity components of Haroon et al. [18] are recovered. Equations (29 and (30 give a complete description of the fluid velocities at all the points inside the slit. From equation (29, we obtain the expression for maximum longitudinal velocity as follows:

\[ u_{max} = V_0 e^{-\alpha x} \left[ \frac{\alpha H \sin (\alpha H) - \cos (\alpha H) + \phi^2 H \cos (\alpha H) + 2\phi \alpha \sin (\alpha H)}{(\alpha H - \cos (\alpha H) \sin (\alpha H) + 2\alpha \phi \sin^2 (\alpha H)) \alpha} \sin (\alpha y) \right] \]

\[ \frac{3(\alpha Q_0 - 2V_0 W)}{4\alpha WH^2(H + 3\phi)} , \]

at \( y = 0 \) for fixed axial distance \( x \). From equation (30, we get the maximum transverse velocity at the wall, which is due to reabsorption at the walls, i.e., \( v_{max} = V_0 e^{-\alpha x} \) for fixed \( x \).

The axial volume flow rate \( Q(x) \) can be obtained by using the relation:

\[ Q(x) = 2W \int_0^H u(x, y) dy. \]

Substituting equation (29, the above equation becomes:

\[ Q(x) = \frac{2W V_0 e^{-\alpha x} + \alpha Q_0 - 2V_0 W}{\alpha} \]

(32)

which is independent of slip parameter but strongly dependent upon reabsorption velocity.

The mass flow rate \( \bar{q}(x) \) can be obtained by using the following formula:

\[ \bar{q}(x) = \rho Q(x) \]

(33)

where \( \rho \) is the density of the fluid. With the help of equation (32 in (33, we get the expression for mass flow rate inside the slit as:

\[ \bar{q}(x) = \rho \left[ Q_0 + \frac{2W V_0 (e^{-\alpha x} - 1)}{\alpha} \right]. \]

(34)
It is noted that at the end of the porous slit \((x = L)\), the expression for mass flow rate becomes:

\[
\bar{q}(L) = \rho \left[ Q_0 + \frac{2WV_0(e^{-\alpha L} - 1)}{\alpha} \right].
\]  
(35)

To find the expression for the amount of fluid mass reabsorbed through the walls of the slit, we can write:

\[
\bar{q}(0) - \bar{q}(L) = \frac{2\rho WV_0(1 - e^{-\alpha L})}{\alpha},
\]  
(36)

which depends upon \(\alpha\).

### 3.2 Pressure distribution

To get the expression for pressure distribution, we will use equation (17 in equations (10-11) to get:

\[
\frac{\partial p}{\partial x} = \mu \left[ V_0 e^{-\alpha x} \left\{ \alpha^2 \frac{dF}{dy} + \frac{d^3F}{dy^3} \right\} + \frac{d^3K}{dy^3} \right],
\]  
(37)

\[
\frac{\partial p}{\partial y} = \mu \left[ \alpha V_0 e^{-\alpha x} \left\{ \alpha^2 F + \frac{d^2F}{dy^2} \right\} \right].
\]  
(38)

Upon integrating equation (32 with respect to \(x\), we get:

\[
p(x, y) = \mu \left[ -\frac{V_0 e^{-\alpha x}}{\alpha} \left\{ \alpha^2 \frac{dF}{dy} + \frac{d^3F}{dy^3} \right\} + x \frac{d^3K}{dy^3} \right] + R(y),
\]  
(39)

where \(R(y)\) is an unknown function that needs to be determined. By differentiating equation (39 with respect to \(y\) and comparing this with equation (38 along with using of equations (21-20), we find that:

\[
\frac{dR}{dy} = 0 \quad \Rightarrow \quad R(y) = C,
\]  
(40)

where \(C\) is an unknown constant of integration. Equation (39 can be now written as:

\[
p(x, y) = \mu \left[ -\frac{V_0 e^{-\alpha x}}{\alpha} \left\{ \alpha^2 \frac{dF}{dy} + \frac{d^3F}{dy^3} \right\} + x \frac{d^3K}{dy^3} \right] + C.
\]  
(41)

Inserting equations (26 and (27 into equation (41, we have:

\[
p(x, y) - p(0, 0) = \frac{2\mu V_0 e^{-\alpha x} \left\{ \cos(\alpha H) - \phi \alpha \sin(\alpha H) \right\} (\cos(\alpha y) - 1)}{\alpha H - \cos(\alpha H) \sin(\alpha H) + 2\alpha \phi \sin^2(\alpha H)} - \frac{3\mu(\alpha Q_0 - 2V_0 W)x}{2H\alpha(H + 3\phi)},
\]  
(42)
where \( p(0,0) \) is the value of the pressure at the entrance of the slit at \( y = 0 \). The mean pressure \( \bar{p}(x) \) is calculated by using the following relation:

\[
\bar{p}(x) = \frac{1}{H} \int_0^H [p(x,y) - p(0,0)]dy. \tag{43}
\]

With the help of equation (42, we get:

\[
\bar{p}(x) = \frac{1}{H} \frac{2 \mu V_0 \alpha e^{-\alpha x} \{ \cos (\alpha H) - \phi \alpha \sin (\alpha H) \} \sin (\alpha H) - H}{\alpha H - \cos (\alpha H) \sin (\alpha H) + 2\alpha \phi \sin^2 (\alpha H)} - \frac{3 \mu (\alpha Q_0 - 2V_0 W)x}{2H^2 W \alpha (H + 3\phi)}. \tag{44}
\]

The mean pressure drop is calculated by using the following relation:

\[
\Delta \bar{p}(L) = \bar{p}(0) - \bar{p}(L). \tag{45}
\]

After substituting equation (44 in equation (45, we get:

\[
\Delta \bar{p}(L) = \bar{p}(0) - \bar{p}(L), \tag{46}
\]

\[
\Delta \bar{p}(L) = \frac{1}{H} \frac{2 \mu V_0 \alpha \{ \cos (\alpha H) - \phi \alpha \sin (\alpha H) \} \sin (\alpha H) (1 - e^{-\alpha x})}{\alpha H - \cos (\alpha H) \sin (\alpha H) + 2\alpha \phi \sin^2 (\alpha H)} + \frac{3 \mu (\alpha Q_0 - 2V_0 W)L}{2H^2 W \alpha (H + 3\phi)}. \tag{47}
\]

### 3.3 Wall shear stress

The wall shear stress may be calculated using the following expression:

\[
\tau_w \bigg|_{y=H} = -\mu \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \bigg|_{y=H}. \tag{48}
\]

If we substitute equations (29-30) in the above equation, we obtain:

\[
\tau_w \bigg|_{y=H} = \frac{2\mu \alpha^2 V_0 e^{-\alpha x} (H + \phi + \phi^2 \sin^2 \alpha H)}{\alpha H - \cos (\alpha H) \sin (\alpha H) + 2\alpha \phi \sin^2 (\alpha H)} + \frac{3\mu (\alpha Q_0 - 2W V_0)}{2\alpha W H (H + 3\phi)}, \tag{49}
\]

which shows that wall shear stress decays from entrance to exit of the slit.
3.4 Fractional reabsorption

The fractional reabsorption ($FA$) is defined as:

$$FA = \frac{Q(0) - Q(L)}{Q(0)}.$$ (50)

By using equation (29 in the above equation, the expression for $FA$ reduces to:

$$FA = \frac{2WV_0(1 - e^{-\alpha L})}{\alpha Q_0}.$$ (51)

The above equation depends upon the exponential reabsorption parameter, the reabsorption velocity and the axial flow rate.

3.5 Leakage flux

The leakage flux $q(x)$ is defined as:

$$q(x) = -\frac{dQ(x)}{dx}.$$ (52)

Using equation (32 in the above equation, we get:

$$q(x) = 2WV_0e^{-\alpha x},$$ (53)

which demonstrates that the leakage flux is directly proportional to the reabsorption velocity.

4 Application to the flow through the proximal convoluted tubule

To investigate the hydrodynamic contribution to the proximal convoluted tubular reabsorption, a set of data available in the literature [8, 18, 19, 20] is utilized. This data was recently used by Haroon et al. [18, 20] and Siddiqui et al. [19] to determine the theoretical values of reabsorption velocity and the linear and periodic reabsorption parameters for different values of fractional reabsorption. In a recent work, Haroon et al. [18] observed that for normal nephron function, the tubular pores remain opened and the reabsorption velocity remains uniform throughout the axial distance of the renal tubule. Theoretical values of reabsorption velocity at the wall for different values of fractional reabsorption were obtained; for example, when normal nephron function of approximately 80% of the glomerular filtrate was transferred through
the tubular epithelium, the theoretical values of reabsorption velocity had to be \(1.6 \times 10^{-6} \text{ cm/sec}\). Therefore, the end of the renal tubule \(Q(L)\) represents the filtrate entering the descending limb of Henle’s loop, and when \(\frac{Q(L)}{Q_0} = 0.2\), the expression takes the form:

\[
\frac{Q(L)}{Q_0} = \frac{2WV_0(e^{-\alpha L} - 1)}{\alpha Q_0}.
\]

(54)

In order to find the theoretical values of the exponential reabsorption parameter \(\alpha\), the data is given in [8, 18, 19, 20]:

\[
H = 10^{-3} \text{ cm}, \quad L = 1 \text{ cm}, \quad \mu = 7 \times 10^{-3} \text{ dyn sec/cm}^2, \quad Q_0 = 4 \times 10^{-7} \text{ cm}^3/\text{sec} \quad \text{and} \quad W = 10^{-1} \text{ cm}.
\]

The theoretical value of the reabsorption velocity \(V_0\) is chosen as \(V_0 = 1.6 \times 10^{-6} \text{ cm/sec}\), recently calculated by Haroon et al. [18], and the value of \(\alpha\) is obtained for various values of fractional reabsorption that can be calculated from the following nonlinear equation:

\[
FA\% - \frac{2WV_0(e^{-\alpha L} - 1)}{\alpha Q_0} = 0.
\]

(55)

To determine the values of \(\alpha\), the computer software MAPLE 15 is used with an initial guess of 0.0001. After applying five iterations to approximate the root \(\alpha\), the obtained values are listed in the following table:

<table>
<thead>
<tr>
<th>FA %</th>
<th>80%</th>
<th>70%</th>
<th>60%</th>
<th>50%</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\alpha) cm</td>
<td>1.5261</td>
<td>1.5680</td>
<td>1.6091</td>
<td>1.6494</td>
</tr>
</tbody>
</table>

It is observed from the above table that as \(\alpha\) increases, \(FA\) decreases. It can be concluded that, \(\alpha\) is a controlling parameter to regulate the pore blockage at the tubular walls.

5 Results and Discussion

The creeping slip flow of Newtonian fluid through a permeable slit with exponential reabsorption at the walls is analyzed. Exact solutions are determined for velocity components, axial flow rate, mass flow rate, pressure distribution, mean pressure drop, wall shear stress, fractional reabsorption and leakage flux.
To study the variation of flow properties, the following dimensionless quantities are introduced:

\[ x^* = \frac{x}{L}, \quad y^* = \frac{y}{L}, \quad \psi^* = \frac{\psi}{V_0 L}, \quad Q^*_0 = \frac{Q_0}{V_0 WL}, \quad \alpha^* = \alpha L, \]

\[ p^* = \frac{p}{\mu V_0 / L}, \quad \tau_w^* = \frac{\tau_w}{\mu V_0 / L}, \]

and the variations have been explained in figures (3-30), after skipping *. In figures (3-12), a comparison of the exponential reabsorption \((v = V_0 e^{-\alpha x})\) with the uniform reabsorption \((v = V_0)\) is provided. Fig. (3) demonstrates that the longitudinal velocity profiles \(u\) at the entrance \((x = 0.1)\) of the slit for both cases are parabolic and approximately the same. In Figs. (4-5), it is noted that the velocity profile of \(u\) for the exponential reabsorption is higher in magnitude than the uniform reabsorption at the middle position of the center of the slit and the exit of the slit. Using these three figures, we observe that the \(u\) at the entrance is higher than at the middle position and the exit of the slit. Figs. (6-8) show the comparison of reabsorption velocity \(v\) for different positions of the slit. Using these figures, it is noted that the profile of the exponential reabsorption as compared to uniform reabsorption is low at all the positions due to the pore blockage phenomena. It is also noticed that for exponential reabsorption, \(v\) is higher at the entrance than at the middle position and exit of the slit. In Fig. (9), the volume flow rate \(Q(x)\) is depicted for the exponential reabsorption and the uniform reabsorption from entrance to exit, which decreases from entrance to exit of the slit. The volume flow rate due to exponential reabsorption is higher than due to uniform reabsorption because of abnormal reabsorption of fluid across the walls of the slit. Fig. (10) shows the pressure difference from entrance to exit of the slit. It is observed that the magnitude of the pressure difference at the center line \(p(x, 0) - p(0, 0)\) of the exponential reabsorption is less than the uniform reabsorption at the exit of the slit. The shear stress \(\tau_w\) at the walls is shown in Fig. (11). Due to pore blockage, the profile of \(\tau_w\) for the exponential reabsorption has large magnitude when compared with the uniform reabsorption. The comparison of wall leakage can be seen in Fig. (12).

The effects of the exponential reabsorption parameter \(\alpha\) on \(u\) at different positions of the slit are shown in Figs. (13-15). From Fig. (13), it is observed that with increasing \(\alpha\), \(u\) increases near the walls and decreases at the center of the slit. At the middle position of the slit as shown in Fig. (14), it is observed that as \(\alpha\) increases, \(u\) increases. Similarly, at the exit of the slit, \(u\) has a parabolic profile for all values of \(\alpha\) and increases with increasing \(\alpha\) (see Fig. (15)). The pore blockage phenomenon can be controlled by varying \(\alpha\) and the effect of \(\alpha\) on reabsorption velocity \(v\) is shown in Figs. (16-18). It is also observed that \(v\) decreases with increasing \(\alpha\) at different positions of the slit. The effect of \(\alpha\) on \(Q(x)\) can be seen in Fig. (19). With increasing
α, Q(x) increases and reaches highest magnitude at the exit of the slit. In Fig. (20), the effect of α on \( p(x,0) - p(0,0) \) is seen. It is observed that \( p(x,0) - p(0,0) \) decreases downstream for \( \alpha = 1 \) and \( \alpha = 2 \); otherwise, as \( \alpha \) increases, \( p(x,0) - p(0,0) \) also increases. For \( \alpha = 3 \), \( p(x,0) - p(0,0) \) decreases downstream after the middle position of the slit. The effect of \( \alpha \) on wall shear stress is shown in Fig. (21). Maximum shear stress at the walls is recorded at the entrance of the slit and the magnitude increases as \( \alpha \) increases. Fig. (22) depicts the effect of \( \alpha \) on leakage flux.

The effect of slip parameter \( \phi \) on \( u \) at different positions of the slit is shown in Figs. (23-25). It is observed that with increasing \( \phi \), \( u \) increases near the walls due to the permeability of the walls. Due to this permeability, \( \phi \) also affects the reabsorption velocity \( v \). (see Fig. (26-28). It is observed that \( \phi \) has some effect on \( v \) between the center and the walls of the slit. The effect of \( \phi \) on \( p(x,0) - p(0,0) \) and \( \tau_w \) are shown in Figs. (29-30). It is observed that \( p(x,0) - p(0,0) \) increases and wall shear stress decreases with increasing \( \phi \).

6 Conclusion

The problem of creeping slip flow of Newtonian fluid through a porous slit with exponential reabsorption was considered. Exact solutions were obtained for the expressions of the components of velocity, volume flow rate, mass flow rate, pressure difference, pressure drop, wall shear stress, fractional reabsorption and leakage flux. From this work, we made the following observations:

1. Due to pore blockage, the profile of longitudinal velocity for exponential reabsorption at the exit is higher than the profile of uniform reabsorption, but it is lower for transverse velocity.

2. Blockage phenomenon can be controlled by varying the values of \( \alpha \), i.e., with decreasing \( \alpha \) the longitudinal velocity increases while the transverse velocity decreases.

3. Volume flow rate, pressure difference, wall shear stress and leakage flux decrease downstream.

4. 80% fractional reabsorption can be achieved if \( \alpha \) is set at 1.5261 cm.

5. Slip parameter \( \phi \) controls the longitudinal velocity at the slit walls, since it increases with increasing \( \phi \).

6. \( \phi \) does not affect the profile of transverse velocity.

7. Pressure difference and wall shear stress are affected with slip parameter \( \phi \).
It should be mentioned that this study is theoretical in nature and significant additional experimental and physiological work is needed to have complete insight in to the transport of glomerular filtrate through the renal tubule in a normal and abnormal kidney nephron. This present study would have significant impact from a biomedical point of view, as there is little information on this topic.

References


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Figure 1: Tubular walls without/with pore blockage phenomenon.

Figure 2: Geometry of the problem with exponential reabsorption at the walls.

Figure 3: Longitudinal velocity profile $u$ for exponential reabsorption and uniform reabsorption at the entrance of the slit, when $Q_0 = 3$, $V_0 = 1$ and $\alpha = 1$
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Figure 4: Longitudinal velocity profile $u$ for exponential reabsorption and uniform reabsorption at the middle position of the slit, when $Q_0 = 3$, $V_0 = 1$ and $\alpha = 1$

Figure 5: Longitudinal velocity profile $u$ for exponential reabsorption and uniform reabsorption at the exit of the slit, when $Q_0 = 3$, $V_0 = 1$ and $\alpha = 1$
Figure 6: Transverse velocity profile $v$ for exponential reabsorption and uniform reabsorption at the exit of the slit, when $Q_0 = 3$, $V_0 = 1$ and $\alpha = 1$

Figure 7: Transverse velocity profile $v$ for exponential reabsorption and uniform reabsorption at the middle position of the slit, when $Q_0 = 3$, $V_0 = 1$ and $\alpha = 1$
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Figure 8: Transverse velocity profile $v$ for exponential reabsorption and uniform reabsorption at the exit of the slit, when $Q_0 = 3$, $V_0 = 1$ and $\alpha = 1$

Figure 9: Axial flow rate $Q(x)$ for exponential reabsorption and uniform reabsorption inside the slit, when $Q_0 = 3$, $V_0 = 1$ and $\alpha = 1$
Figure 10: Pressure difference $p(x, 0) - p(0, 0)$ for exponential reabsorption and uniform reabsorption inside the slit, when $Q_0 = 3$, $V_0 = 1$ and $\alpha = 1$

Figure 11: Wall shear stress $\tau_w$ for exponential reabsorption and uniform reabsorption inside the slit, when $Q_0 = 3$, $V_0 = 1$ and $\alpha = 1$
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Figure 12: Leakage flux $q(x)$ for exponential reabsorption and uniform reabsorption inside the slit, when $Q_0 = 3$, $V_0 = 1$ and $\alpha = 1$

Figure 13: Effect of $\alpha$ on $u$ at the entrance of the slit, when $Q_0 = 3$, $\phi = 0$ and $V_0 = 1$
Figure 14: Effect of $\alpha$ on $u$ at the middle position of the slit, when $Q_0 = 3$, $\phi = 0$ and $V_0 = 1$

Figure 15: Effect of $\alpha$ on $u$ at the exit of the slit, when $Q_0 = 3$, $\phi = 0$ and $V_0 = 1$

Figure 16: Effect of $\alpha$ on $v$ at the entrance of the slit, when $Q_0 = 3$, $\phi = 0$ and $V_0 = 1$

Figure 17: Effect of $\alpha$ on $v$ at the middle position of the slit, when $Q_0 = 3$, $\phi = 0$ and $V_0 = 1$
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Figure 18: Effect of $\alpha$ on $v$ at the exit of the slit, when $Q_0 = 3$, $\phi = 0$ and $V_0 = 1$

Figure 19: Effect of $\alpha$ on $Q(x)$ inside the slit, when $Q_0 = 3$, $\phi = 0$ and $V_0 = 1$

Figure 20: Effect of $\alpha$ on $p(x,0) - p(0,0)$ inside the slit, when $Q_0 = 3$, $\phi = 0$ and $V_0 = 1$

Figure 21: Effect of $\alpha$ on $\tau_w$ inside the slit, when $Q_0 = 3$, $\phi = 0$ and $V_0 = 1$
Figure 22: Effect of $\alpha$ on $q(x)$ inside the slit, when $Q_0 = 3$, $\phi = 0$ and $V_0 = 1$

Figure 23: Effect of $\phi$ on $u$ at the entrance of the slit, when $Q_0 = 3$, $\alpha = 1$ and $V_0 = 1$

Figure 24: Effect of $\phi$ on $u$ at the middle position of the slit, when $Q_0 = 3$, $\alpha = 1$ and $V_0 = 1$

Figure 25: Effect of $\phi$ on $u$ at the exit of the slit, when $Q_0 = 3$, $\alpha = 1$ and $V_0 = 1$
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Figure 26: Effect of \( \phi \) on \( v \) at the entrance of the slit, when \( Q_0 = 3, \alpha = 1 \) and \( V_0 = 1 \)

Figure 27: Effect of \( \phi \) on \( v \) at the middle position of the slit, when \( Q_0 = 3, \alpha = 1 \) and \( V_0 = 1 \)

Figure 28: Effect of \( \phi \) on \( v \) at the exit of the slit, when \( Q_0 = 3, \alpha = 1 \) and \( V_0 = 1 \)

Figure 29: Effect of \( \phi \) on \( p(x,0) - p(0,0) \) inside the slit, when \( Q_0 = 3, \alpha = 1 \) and \( V_0 = 1 \)
Figure 30: Effect of $\phi$ on $\tau_w$ inside the slit, when $Q_0 = 3$, $\alpha = 1$ and $V_0 = 1$