The Bifurcation and Traveling Wave Existence Analysis of Spreading Virus Ebola Model

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Abstract

Ebola was originally identified in 1976 near the Ebola River in Democratic Republic of Congo. The spreading of Ebola virus is caused by contact between individuals. The spreading of Ebola virus can be defined by mathematical model to better understand this contagious disease. The model obtained shows that there is a possibility of stability changes, which is commonly known as bifurcation of the system. Furthermore, the population movement also allows for wide spreading of the disease. Because of this spreading, traveling wave existence of the system can be observed. Simulation results based on analysis show that the number of infected depends on the transmission rates of Ebola. Traveling wave simulations shows that the outbreak could occur widely in different conditions with different number of infected.

Keywords: Ebola, Bifurcation, Traveling wave

1 Introduction

The spreading of Ebola virus is caused by contact between individuals. The incubation period of this virus is 2-21 days in a single period [1]. In this case, it takes about eight hours on the virus to replicate it self. It could occur several times before symptoms in an individual appears. When the virus infects the host cell, hundreds to thousands of new virus particles are then released.
during that period before the host cell dies. The symptoms of the virus could occur within a few days after virus transmission include high fever, headache, muscle aches, abdominal pain, fatigue, diarrhea, sore throat, hiccup, rash, red and itchy eyes, blood vomiting, and bloody diarrhea. Due to the absence of drugs or vaccines on the Ebola virus then the number of fatalities reached 50% to 90% [14].

Mathematical models have been commonly used for many years to better understand contagious diseases. The formation of hypotheses on the infectious disease problems can be obtained by observing the data which obtained along with clarifying the optimal sample size in the field. Furthermore, the obtained model allows researchers to explore the effects of changing the infectivity, incubation period, and recovery period parameters of the disease along with the effect of changes in population size, and other interesting factors. It can be done in different situations based on understanding the long-term pattern of the disease. Through these observations, ideal prediction and strategy for slowing down the spreading of disease based on agreement on the model and the place where the outbreak occur can be obtained [2].

Until now, most of studies done in few years only discussed the number of one population in a particular place without considering movement occurred in the population. In fact, the world population can move beyond the scope which is commonly assumed in many researches. Therefore, the concept of the spreading should be determined so that the concept can be added to the system. For example, if the number of a population has met the amount of a particular place, then that population will spread to other places. In this case, the spreading process is called the diffusion process [11]. These spreading can also be interpreted as a dynamic movement which is conducted by individuals in one population. If individuals are moving dynamically, then the movement will depends on the traveling wave speed [7]. Therefore, the analysis of traveling wave existence must be done to know the more widely spreading of the virus [6].

In this paper, the spreading of Ebola virus model that has been modified is based on related references. This work also emphasizes the analysis of stability and Hopf bifurcation that occurs in the model. More specifically, the traveling wave of the spreading model of Ebola analyzed based on the movement that occurs in the population.

2 Preliminary

2.1 Ebola Virus Disease Model

In this observation the modelling of the spreading of Ebola will be modified. The modifications are based on the model used in Kalu, et al. [9] and de-
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developed based on [4]. The modification is made because in [4] explains that healthcare providers have a greater chance to be infected. Therefore, the rates of transmission to the general public and health workers will have different values. In [4] the natural death rates and the number of births are not related. So, we can use the assumption of [9] that there is the natural birth rate entering susceptible class. The observation of [9] shows that there is the natural death rate occurring in each compartment which is different with [4]. The death of each of the classes in this case will only caused by the disease. In last assumption, infected individuals could be treated and cured of the virus based on [9]. Because of these assumptions, the death compartment at [4] will be omitted. The modified model used is the SEIR model as developed by Kalu A.U et al. in [9]. In the model, there is an average natural birth and death (not only natural death). The natural birth will be assumed as a healthy birth and then enter the susceptible compartment [9]. In this case, deaths are only assumed as deaths due to illness suffered by infected individuals [9]. Therefore, the mathematical model of the spreading of Ebola can be described as the following differential equations

$$\frac{dS}{dt} = \pi - \beta(1-p)SLN - p(\alpha L + \alpha I)SN - \lambda S + \kappa R$$
$$\frac{dL}{dt} = \beta(1-p)SLN + p(\alpha L + \alpha I)SN - \phi L - \psi L$$
$$\frac{dI}{dt} = \phi L - \xi I - \tau I$$
$$\frac{dR}{dt} = \xi I - \kappa R - \nu R$$

where $N = S+L+I+R$. The initial condition used in the model are $S(0) = S_0$, $L(0) = L_0$ where $I(0) = I_0$ and $R(0) = R_0$.

In this model, $\beta = \rho_{\beta}c_{\beta}$ where $\rho_{\beta}$ is the rate of infection that happen to susceptible individuals. The infection happen when susceptible individual contacted with infected. $c_{\beta}$ is the contact rates of the population. So we know $\beta$ is the rates of susceptibles who being infected because of their contact with infected individuals [4].

2.1.1 Fixed Points of Ebola Virus Spreading System

If $x^*$ is the fixed points of the system $\dot{x} = F(x)$, then $F(x^*) = 0$ [17]. Because of that, the model 1 has fixed points if it satisfies:

$$\pi - \beta(1-p)SLN - p(\alpha L + \alpha I)SN - \lambda S + \kappa R = 0$$

$$\beta(1-p)SLN + p(\alpha L + \alpha I)SN - \phi L - \psi L = 0$$
\[ \phi L - \xi I - \tau I = 0 \] (4)
\[ \xi I - \kappa R - \nu R = 0 \] (5)

Based on equations 2 to 5, two fixed points \((\frac{\pi}{\lambda}, 0, 0, 0)\) and \((S^*, L^*, I^*, R^*)\) can be obtained, with

\[ m = \left( n + \frac{\kappa \xi (\xi + \tau)}{\phi (\kappa + v)} + \frac{\kappa \xi (\xi + \tau)}{\phi (\phi + \psi)(\kappa + v) - \kappa \xi (\xi + \tau)} \right) \]
\[ n = \frac{\beta (1 - p) \pi}{\lambda N} + \frac{p \alpha_L \pi}{\lambda N} + \frac{p \alpha_I \pi}{\lambda N} - \frac{\kappa \xi (\xi + \tau)}{\phi (\kappa + v)} \] (6)

and \(L^* = \frac{n}{m}\), along with

\[ S^* = \frac{\pi \phi (\kappa + v) + (\kappa \xi (\xi + \tau) - \phi (\phi + \psi)(\kappa + v)) \frac{\pi}{m}}{\lambda \phi (\kappa + v)} \] (7)
\[ R^* = \frac{\xi (\xi + \tau) n}{\phi (\kappa + v) m} \] (8)
\[ I^* = \frac{\xi + \tau n}{\phi m} \] (9)

\((\frac{\pi}{\lambda}, 0, 0, 0)\) is the disease-free equilibrium and \((S^*, L^*, I^*, R^*)\) is endemic equilibrium with \(S^* \neq 0, L^* \neq 0, I^* \neq 0\) and \(R^* \neq 0\).

### 2.2 The Hyperbolic Fixed Point

A fixed point is called hyperbolic, if for any fixed point it has nonzero eigenvalues in its real part on the system of linear differential equations given.

**Theorem 2.1** [17]

Given a system of differential equations i.e. \(\ddot{x} = Ax\) so that the following properties obtain:

a If for every eigenvalues \(\lambda\) has a negative value on its real part, then the fixed point is asymptotically stable. Stable points, decreasing stable points, and stable foci are all asymptotically stable.

b If one of the eigenvalue \(\lambda\) has a positive value on its real part, then the starting point is unstable. Saddle points, unstable points, decreasing unstable points, and unstable foci are all unstable. A saddle point has an interesting line and refusing on the other part. However, the saddle point remains to be an unstable point.
In two dimensions if each eigenvalue is pure imaginary $\pm i\beta$, then the starting point has L-stable but asymptotically unstable.

d In two dimensions if one of the eigenvalues is 0 and the other has a negative eigenvalue, then the starting point has L-stable but is asymptotically unstable.

**Theorem 2.2** [13]
Suppose there is a smooth curve from a fixed point $(x(\mu), \mu)$ where $x(\mu_0) = x_0$ on the system $\dot{x} = f_\mu(x)$. The following conditions hold:

A1 The Jacobian of matrix $D_x f_\mu_0(x_0)$ has a pair of imaginary eigenvalues, while the other eigenvalues are negative. Now, a smooth curve from a fixed point is $(x(\mu), \mu)$ where $x(\mu_0) = x_0$. Eigenvalues $\lambda(\mu), \bar{\lambda}(\mu)$ from $J(\mu) = D_x f_\mu(x(\mu))$ will have imaginary values where $\mu = \mu_0$ and

A2 If $d(Re(\lambda(\mu_0)))/d\mu \neq 0$, then there exists simple Hopf bifurcation on the system $\dot{x} = f_\mu(x)$.

Based on the system $\dot{x} = f_\mu(x)$ we can obtain characteristic polynomial based on Jacobian $J(\mu)$ with

$$p(\lambda; \mu) = \det(\lambda I_n - J(\mu)) = p_0(\mu) + p_1(\mu)\lambda + p_2(\mu)\lambda^2 + \ldots + p_n(\mu)\lambda^n \quad (10)$$

To satisfy A1 and A2, the following inequalities must hold:

B1 $p_0(\mu_0) > 0, D_1(\mu_0) > 0, \ldots, D_{n-2}(\mu_0) > 0, D_{n-1}(\mu_0) = 0$

B2 $dD_{n-1}(\mu_0)/d\mu \neq 0$

with

$$J_n(\mu) = \begin{pmatrix}
p_1(\mu) & p_0(\mu) & \ldots & 0 \\
p_3(\mu) & p_2(\mu) & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
p_{2n-1}(\mu) & p_{2n-2}(\mu) & \ldots & p_n(\mu)
\end{pmatrix}$$

$$D_1 = \det(J_1(\mu)) = p_1(\mu)$$

$$D_2(\mu) = \det(J_2(\mu)) = \det\begin{pmatrix} p_1(\mu) & p_0(\mu) \\
p_3(\mu) & p_2(\mu) \end{pmatrix}$$

$$\ldots$$

$$D_n(\mu) = \det(J_n(\mu)) > 0$$
2.3 The Travelling Waves of Dynamic System

Given the population that described into two parts which are the traveling wave model that satisfy Definition 2.3 and Theorem 2.4, given as follows:

\[
\frac{\partial L(x,t)}{\partial t} = d(x) \nabla L(x,t) + f(x,t,L(x,t)) \tag{11}
\]

\[
\frac{\partial I(x,t)}{\partial t} = d(x) \nabla I(x,t) + g(x,t,I(x,t))
\]

**Definition 2.3** [7]

The solution of traveling wave with the speed \( c \) in model 11 is a solution which has the form 11, furthermore corresponds to the disease-free equilibrium point and the endemic equilibrium point so that if \( u(x + ct), w(x + ct) \) Then

\[
\lim_{u \to -\infty} (L, I) = (L^*, I^*) \tag{12}
\]

\[
\lim_{u \to \infty} (L, I) = (0, 0)
\]

**Theorem 2.4** [7]

If the minimum speed in traveling wave is \( c_0 \) such that for every \( c \geq c_0 \) then the nonlinear system model will have a non increasing traveling wave \( (L(x - ct), I(x - ct)) \) with the speed \( c \) which is satisfies \( \lim_{u \to \infty} (L, I) = (L^*, I^*) \) and \( \lim_{u \to -\infty} (L, I) = (0, 0) \).

3 Main Results

3.1 Stability of Ebola Virus Spreading System

The spreading of Ebola virus model 1 is a nonlinear equation system. Therefore, the linear form of the model can be obtained with \( \dot{x} = Jx \), with \( J \) is the Jacobian of the equation system. So, to analyze the stability of the system, the eigenvalues of the system must be found by performing calculations \( |J - \omega I| = 0 \) with \( \omega \) is the eigenvalue of the system. In first case at a fixed point \( (\frac{\xi}{N}, 0, 0, 0) \) the eigenvalues \( \omega_1 = -\lambda < 0, \omega_2 = -(\kappa + v) < 0 \) can be obtained, with

\[
\omega_3 = \frac{-b - \sqrt{b^2 + 4c}}{2} \tag{13}
\]

and

\[
\omega_4 = \frac{-b + \sqrt{b^2 + 4c}}{2} \tag{14}
\]

with

\[
b = (\xi + \tau) - (\beta(1 - p) + p\alpha_L)\frac{\pi}{N\lambda} - \phi - \psi \tag{15}
\]
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\[ c = (\xi + \tau)[(\beta(1 - p) + p\alpha_L) \frac{\pi}{N\lambda} - \phi - \psi] - \phi p\alpha_I \frac{\pi}{N\lambda} \] (16)

In this case, the system is stable if \( b > 0 \) and \( c < 0 \). Then

\[ \beta < \frac{(\xi + \tau + \phi + \psi) N\lambda}{\pi} - p\alpha_L \] (17)

and

\[ \beta < \frac{\phi p\alpha_I}{((\xi + \tau))} + (\phi + \psi) \frac{N\lambda}{\pi} - p\alpha_L \] (18)

When \( c < 0 \), this condition indicates that multiplication of the rates of recovery individuals, the deaths and the rates of transmission in susceptible individuals as well as on health workers although it reduced by the rates of the deaths of latent individuals and the average of latent individual entering infected compartment is smaller than multiplication of the average transmission to the medical officer when making contact with the infected individual and the average latent individual who enters the infected class. Furthermore when \( b > 0 \), this condition indicates that the sum of the recovery rates, the death of the infected individual, the average of latent individuals entering the infected compartment and the rates of the death of latent individuals because of infection is greater than the magnitude transmission insusceptible individuals. It means that the Ebola virus will be disappeared if a good control of Ebola is given in the population.

Hopf bifurcation can occur at this fixed point if \( b = 0 \) and \( c < 0 \) such that

\[ \beta = \frac{(\xi + \tau + \phi + \psi) N\lambda}{\pi} - p\alpha_L \] (19)

The other criterion of Hopf bifurcation at this fixed point also applies because of

\[ R_e(\beta) = -\frac{1}{2}[(\xi + \tau) - (\beta(1 - p) + p\alpha_L) \frac{\pi}{N\lambda} - \phi - \psi] \] (20)

Differentiating Equation 20 gives

\[ \frac{d}{d\beta} R_e(\beta) = \frac{1}{2} (1 - p) \frac{\pi}{N\lambda} \neq 0 \] (21)

The stability at \((S^*, L^*, I^*, R^*)\) can also obtained by analyzing the eigenvalues associated with the fixed point. By using the same way as in the previous sections, \( \omega_1 = -(\kappa + \upsilon) < 0 \) can be obtained as the first eigenvalue. Solving \(|J - \omega I| = 0\) in this fixed point to get the polynomial as follows:

\[ \omega^3 + d_1 \omega^2 + d_2 \omega + d_3 = 0 \] (22)
\[ d_1 = ((\xi + \tau) - (\beta(1 - p)\frac{S^*}{N} + p\alpha_L\frac{S^*}{N} - \phi - \psi)) + \]
\[ (\beta(1 - p)\frac{L^*}{N} + p(\alpha_L L^* + \alpha_I I^*)\frac{1}{N} + \lambda) \]  

(23)

\[ d_2 = (\beta(1 - p)\frac{S^*}{N} + p\alpha_L\frac{S^*}{N} - \phi - \psi)(-(\xi + \tau)) - \phi\alpha_I S^* + \]
\[ ((\xi + \tau) - (\beta(1 - p)\frac{S^*}{N} + p\alpha_L\frac{S^*}{N} - \phi - \psi)) \]
\[ (\beta(1 - p)\frac{L^*}{N} + p(\alpha_L L^* + \alpha_I I^*)\frac{1}{N} + \lambda) + \]
\[ (\beta(1 - p)\frac{L^*}{N} + p(\alpha_L L^* + \alpha_I I^*)\frac{1}{N})((\beta(1 - p)\frac{S^*}{N} + p\alpha_L S^*) \]  

(24)

and

\[ d_3 = [(\xi + \tau)(\beta(1 - p)\frac{S^*}{N} + p\alpha_L\frac{S^*}{N} + \phi\alpha_I S^*)] \]
\[ (\beta(1 - p)\frac{L^*}{N} + p(\alpha_L L^* + \alpha_I I^*)\frac{1}{N}) + \]
\[ (\beta(1 - p)\frac{L^*}{N} + p(\alpha_L L^* + \alpha_I I^*)\frac{1}{N} + \lambda) \]
\[ (\beta(1 - p)\frac{S^*}{N} + p\alpha_L\frac{S^*}{N} - \phi - \psi)(-(\xi + \tau)) - \phi\alpha_I S^* \]  

(25)

Based on Equation 22 the stability of the system is observed by using Routh Hurwitz criterion. Therefore,

\[ H_3 = \begin{pmatrix} d_2 & d_3 & 0 \\ 1 & d_1 & d_2 \\ 0 & 0 & 1 \end{pmatrix} \]  

(26)

The system is stable if \( d_2 > 0, d_3 > 0 \) and \( (d_1d_2 - d_3) > 0 \). If \( d_1 > 0 \) then it can be obtained:

\[ (\xi + \tau) + (\beta(1 - p)\frac{L^*}{N} + p(\alpha_L L^* + \alpha_I I^*)\frac{1}{N} + \lambda) + \phi + \psi \]
\[ > (\beta(1 - p)\frac{S^*}{N} + p\alpha_L S^*) \]  

(27)

Equation 27 indicates that the sum of individual recovery rates, average death of the latent individual without diagnosis, rates of individual latent entering the infected class, average of the death infected individual and transmission rate multiplied by the endemic point \( I^* \) and \( L^* \) is greater than the average of
transmissions multiplied by $S^*$. Therefore, Ebola virus disease can be eradicated with a better control. Based on the analysis, we can conclude that the system is stable if $d_1 d_2 > d_3$.

Hopf bifurcation in $(S^*, L^*, I^*, R^*)$ can occur if $d_2 > 0$, $d_3 > 0$ and $(d_1 d_2 - d_3) = 0$. Therefore if $d_1 \geq 0$ is taken then it is obvious that $d_1 d_2 = d_3$. So the first criterion of a Hopf bifurcation can be obtained, with $d_1 d_2 \neq d_3$ for $d_1 < 0$.

The other criterion of Hopf bifurcation also applies with

$$\frac{dL^*}{d\beta} = \frac{1 - p}{\lambda N m} \pi (1 - L \frac{(\phi + \psi)(\kappa + v) - \kappa \xi (\xi + \tau)}{\phi(\kappa + v)}) \quad (28)$$

$$\frac{dS^*}{d\beta} = \frac{(\phi + \psi)(\kappa + v) - \kappa \xi (\xi + \tau)}{\phi(\kappa + v)} \frac{dL^*}{d\beta} \quad (29)$$

$$\frac{dI^*}{d\beta} = \frac{\xi + \tau}{\phi} \frac{dL^*}{d\beta} \quad (30)$$

and

$$r_1 = -\left[\frac{1 - p}{N} (S^* + \beta \frac{dS^*}{d\beta}) + \frac{p \alpha L}{N} \frac{dS^*}{d\beta}\right] \quad (31)$$

and

$$r_2 = \frac{1 - p}{N} (L^* + \beta \frac{dL^*}{d\beta}) + \frac{p \alpha L}{N} (\alpha L \frac{dL^*}{d\beta} + \alpha I \frac{dI^*}{d\beta}) \quad (32)$$

Based on Equation 28-32, solving Equation 23-25 gives

$$\frac{dd_3}{d\beta} = (\xi + \tau)[(\phi + \psi + \frac{p \alpha I}{N} S^*) r_2 + \lambda r_1 \quad (33)]$$

$$+ \frac{\phi p \alpha I}{N} \frac{dS^*}{d\beta} \left(\beta \frac{(1 - p)}{N} L^* + \frac{p}{N} (\alpha L L^* + \alpha I^*)\right)$$

$$- \frac{\phi \alpha I}{N} \frac{dS^*}{d\beta}$$

$$\frac{dd_2}{d\beta} = (\xi + \tau + \lambda) r_1 + (\xi + \tau + \phi + \psi) r_2 \quad (34)$$

$$\frac{dd_1}{d\beta} = r_1 + r_2 \quad (35)$$

So that Equation 36 can be obtained by

$$d_2 \frac{dd_1}{d\beta} + d_1 \frac{dd_2}{d\beta} = d_2 (r_1 + r_2) + d_1 ((\xi + \tau + \lambda) r_1 + (\xi + \tau + \phi + \psi) r_2)$$

$$= [d_1 (\xi + \tau + \lambda) + d_2] r_1 + [(\xi + \tau + \phi + \psi) d_1 + d_2] r_2$$
\[
\frac{d_2}{d\beta} \frac{dd_1}{d\beta} + \frac{d_1}{d\beta} \frac{dd_2}{d\beta} - \frac{dd_3}{d\beta} = \left[ d_1(\xi + \tau + \lambda) + d_2 - (\xi + \tau)(\phi + \psi + \frac{p\alpha I}{N})S^* \right] r_1 \\
+ \left[ (\xi + \tau + \phi + \psi)d_1 + d_2 - (\xi + \tau)(\phi + \psi + \frac{p\alpha I}{N})S^* \right] r_2 \\
+ \left[ 1 - (\xi + \tau) \left( \frac{\beta(1-p)}{N}L^* + \frac{p}{N}(\alpha L^* + \alpha I^*) \right) \right] \frac{\phi p\alpha I dS^*}{N \frac{d\beta}{N}} 
\]

Equation 36 shows that \( \frac{dd_1}{d\beta} + \frac{dd_2}{d\beta} - \frac{dd_3}{d\beta} \neq 0 \) which indicates that the point is indeed the possibility of Hopf bifurcation.

### 3.2 Traveling Wave Models in The Spreading Ebola Virus Model

The analysis of travelling wave on the model of the spreading of ebola system aims to observe the wave velocities that occur in the system. So that will be obtained conclusions about the wide spreading of ebola. At the first step, the equation define the flux over the populations on the system i.e \( \mathcal{J} \) in model 1

\[
\mathcal{J} = -D \frac{\partial L}{\partial x} 
\]

with \( D \) is the coefficient of diffusivity over a population in constant value. By adding the diffusion equation to the model 1 for each compartment, such that

\[
\frac{\partial S}{\partial t} = D_s \frac{\partial^2 S}{\partial x^2} + \pi - \beta(1-p)S \frac{L}{N} - p(\alpha L + \alpha I) \frac{S}{N} - \lambda S + \kappa R \\
\frac{\partial L}{\partial t} = D_L \frac{\partial^2 L}{\partial x^2} + \beta(1-p)S \frac{L}{N} + p(\alpha L + \alpha I) \frac{S}{N} - \phi L - \psi L \\
\frac{\partial I}{\partial t} = D_I \frac{\partial^2 I}{\partial x^2} + \phi L - \xi I - \tau I \\
\frac{\partial R}{\partial t} = D_R \frac{\partial^2 R}{\partial x^2} + \xi I - \kappa R - \nu R 
\]

The outbreak of Ebola virus can occur because of the contact of susceptible with latent and infected individual [4]. Therefore analysis of the model 38 for this study is limited to compartments \( L \) and \( I \). In this case, \( L(x, t) = L(x - ct) \) and \( I(x, t) = I(x - ct) \) is a function of \( x \) and \( t \) where \( c \) is the wave velocity. If \( u = x - ct \), so

\[
-c \frac{dL}{du} = D_L \frac{d^2 L}{du^2} + \beta(1-p)S \frac{L}{N} + p(\alpha L + \alpha I) \frac{S}{N} - \phi L - \psi L \\
-c \frac{dI}{du} = D_I \frac{d^2 I}{du^2} + \phi L - \xi I + \tau I 
\]
Based on the model 39, in the next step, the wave velocity on each compartment $L$ and $I$ can be found. The linearization of model 39 near the fixed point $(\lambda, 0, 0, 0)$ gives

$$
-c\frac{dL}{du} = D_L \frac{d^2L}{du^2} + [\beta(1-p) + p\alpha_L] \frac{\pi}{N\lambda} - \phi - \psi]L + [p\alpha_I \frac{\pi}{N\lambda}]I
$$

$$
-c\frac{dI}{du} = D_I \frac{d^2I}{du^2} + \phi L - (\xi + \tau)I
$$

(40)

If the solution of model 40 taken by $(L, I) = (j_1 e^{\delta u}, j_2 e^{\delta u})$, then the characteristic equations can be obtained, i.e.

$$
g(\delta) = D_L j_1 \delta^2 + c j_1 \delta + [\beta(1-p) + p\alpha_L] \frac{\pi}{N\lambda} - \phi - \psi]j_1 +
[p\alpha_I \frac{\pi}{N\lambda}]j_2 = 0
$$

$$
h(\delta) = D_I j_2 \delta^2 + c j_2 \delta + \phi j_1 - (\xi + \tau)j_2 = 0
$$

(41)

Given $\frac{dg}{d\delta} = 0$ and $\frac{dh}{d\delta} = 0$ then we can get the characteristic values $\delta = -\frac{c}{2D_L} < 0$ and $\delta = -\frac{c}{2D_I} < 0$. Substitution of these characteristic values into model 41 gives

$$
\frac{j_1 c^2}{4D_L} - j_1 \frac{c^2}{2D_L} + [\beta(1-p) + p\alpha_L] \frac{\pi}{N\lambda} - \phi - \psi]j_1 +
[p\alpha_I \frac{\pi}{N\lambda}]j_2 = 0
$$

$$
\frac{j_2 c^2}{4D_I} - j_2 \frac{c^2}{2D_I} + \phi j_1 - (\xi + \tau)j_2 = 0
$$

(42)

Therefore, based on equation system of 42, there will be two minimum velocities on either compartment $L$ or $I$ i.e.

$$
c_L^* = 2\sqrt{\frac{D_L[\beta(1-p) + p\alpha_L] \frac{\pi}{N\lambda} - \phi - \psi]j_1 + [p\alpha_I \frac{\pi}{N\lambda}]j_2}{j_1}}
$$

$$
c_I^* = 2\sqrt{\frac{D_I \phi j_1 - (\xi + \tau)j_2}{j_2}}
$$

(43)

With the minimum speed on Equation 43, there will be a widely spreading of infection.

The same analysis was applied at fixed points $(S^*, L^*, I^*, R^*)$ to obtain minimum rate of traveling wave on the system. Linearization of model 39 at
fixed point \((S^*, L^*, I^*, R^*)\) would give the following model:

\[-c \frac{dL}{du} = D_L \frac{d^2L}{du^2} + [\beta(1-p) \frac{L^*}{N} + p(\alpha_L L^* + \alpha_I I^*) \frac{1}{N}] S + [\beta(1-p) S^* \frac{1}{N} + p\alpha_L S^* \frac{1}{N} - \phi - \psi] L + [\alpha_I S^*] I \]  

\[-c \frac{dI}{du} = D_I \frac{d^2I}{du^2} + \phi L - (\xi + \tau) I \]  

(44)

In this case, \([\beta(1-p) \frac{L^*}{N} + p(\alpha_L L^* + \alpha_I I^*) \frac{1}{N}] S\) is Ebola’s infection transmission on susceptible individuals because of interactions which made by latent or infected individuals with them. There are some of \(S\) which could enter the \(L\) compartment, so the transmission rate of \(S\) can be written as

\[[\beta(1-p) \frac{L^*}{N} + p(\alpha_L L^* + \alpha_I I^*) \frac{1}{N}] S = \nu L \]  

(45)

Because of Equation 45, then model 44 can be expressed as

\[-c \frac{dL}{du} = D_L \frac{d^2L}{du^2} + [\nu + \beta(1-p) \frac{S^*}{N} + p\alpha_L S^* \frac{1}{N} - \phi - \psi] L + [\alpha_I S^*] I \]  

\[-c \frac{dI}{du} = D_I \frac{d^2I}{du^2} + \phi L - (\xi + \tau) I \]  

(46)

Suppose that \((L, I) = (k_1 e^{\delta_1 u}, k_2 e^{\delta_2 u})\) is the solution of Equation 46. Then the characteristic equations can be obtained as follows

\[h(\delta) = D_L k_1 \delta^2 + c k_1 \delta + [\nu + \beta(1-p) \frac{S^*}{N} + p\alpha_L S^* \frac{1}{N} - \phi - \psi] k_1 + [\alpha_I S^*] k_2 = 0 \]  

\[g(\delta) = D_I k_2 \delta^2 + c k_2 \delta + \phi k_1 - (\xi + \tau) k_2 = 0 \]  

(47)

According to Equation 47 with \(\frac{dg}{d\delta} = 0\) and \(\frac{dh}{d\delta} = 0\), then we get the characteristic values \(\delta = -\frac{c}{2D_I} < 0\) and \(\delta = -\frac{c}{2D_L} < 0\), such that

\[c^*_I = \frac{2}{\sqrt{k_2}} \frac{D_I (\phi k_1 - (\xi + \tau) k_2)}{} \]  

\[c^*_L = \frac{2}{\sqrt{k_1}} \frac{D_L a^*}{k_2} \]  

(48)

with \(a^* = [\nu + \beta(1-p) \frac{S^*}{N} + p\alpha_L S^* \frac{1}{N} - \phi - \psi] k_1 + [\alpha_I S^*] k_2\). \(c^*_I\) and \(c^*_L\) are the minimum rates of travelling wave that caused spreading wave on the population.
3.3 Simulation

In this section, the spreading of Ebola virus model 1 is solved numerically using fourth order Runge-Kutta. This model will be discretized up to \( n = 100000 \). On paper by Tae S.do et. al., estimation of parameter value \( \phi \) was given by \( \phi = \frac{1}{9} p_I d^{-1} \), with \( p_I \) is proportion of infected and isolated patients over infected patients without isolation or confirmation about their infection. Confirmed Ebola infections in Nigeria has a relatively high isolations, up to 90% (1 not confirmed and 19 successfully confirmed). This paper also explained that \( \xi = \frac{1}{18} p_R d^{-1} \) with \( p_R \) is recovery rate. Average recovery period in Nigeria was 18 days where \( p_R = 60\% \) in Nigeria and relatively smaller in Western Africa, up to 50\%. Death to infected patients could take an average of 8-9 days, such that value \( \tau = \frac{1}{8.5} (1 - p_R) d^{-1} \) can be taken. Incubation time and critical time could be taken at an average of 9 days and 7 days, respectively.

\[
\psi = \frac{1}{(7+9)} (1 - p_I) d^{-1} \text{ and } 1 - p_I = 10\% \text{ values could be taken in Nigeria, which has bigger values in Western Africa.}
\]

On the other hand, Tae S.Do et.al. provided value of \( \alpha_L = \alpha_I = \alpha = 1.5 \beta \) because infected paramedics. Doctors, nurses, and paramedics were infected in Western Africa. In Lagos, 12 out of 20 Ebola patients were paramedics, such that \( \alpha = 1.5 \beta \). On paper by Tae S Do et. al. \( \beta \) is transmission average when virus outbreak happened. \( \beta_I \) can be determined by using the first twelve-days outbreak data in Nigeria as follows based on [4]

\[
\beta_I (1 - p) + 1.5 \beta_I p [1 + \phi (\xi + \tau) + \lambda - \phi - psi] = 0.167
\]

with \( \lambda = 0 \) such that \( \beta_I = 0.221 d^{-1} \). \( \beta_L \) on this paper would satisfy

\[
\frac{(\tau + \xi)[\beta_L (1 - p) + 1.5 \beta_L p] + 1.5 \beta_L p \phi}{(\xi + \tau) (\phi + \psi)} = \frac{1}{2}
\]

such that \( \beta_L = 0.043 \) [4]. Zhiming Li, et. al. explained that \( \pi \) was the annual average birth rate of 1000 people. They also explained that birth rate on 2014 was 0.03602 in Guinea. In one-day period, it take the value of 

\[
\pi = \frac{0.03602}{365} 11.744.951 = 1.159
\]

Death rate of \( \lambda \) by Zhiming Li et. al. were natural annual death rate of 1000 people.

Based on this value, \( \lambda = 0.0643 \) in Guinea’s Ebola transmission can be taken. This is consistent with what Sylvie Diane Djomba Njankou had obtained. Value \( \nu = 0 \) could be taken because recovery time was less than average death rate at infections period. Based on Zhiming Li et. al. [12], effective healing time was 0.093 < \( \kappa < 0.2857 \). Then the parameters used in this paper are \( \beta = 0.221 \), \( \alpha_L = 1.5 \beta \), \( \alpha_I = 1.5 \beta \), \( \phi = 0.1 \), \( \xi = 0.6/18 \), \( \tau = 0.4/8.5 \), \( \psi = 0.9/16 \), \( p = 0.1 \), \( \pi = 1.159 \), \( \lambda = 0.0643 \), \( \kappa = 0.1 \), \( \nu = 0 \) where system was on unstable condition. The given initial conditions are \((S(0), L(0), I(0), R(0)) = (50, 1, 0, 0)\).
Figure 1: Simulation Results of The Spreading Ebola Virus Model on Unstable Condition

Simulation results as shown on Figure 1 shows that susceptible individuals are decreasing because of large transmission rate. Over time, individuals in compartment converges to some values. Figure 1 also shows that latent compartment and infected individuals are increasing over time until they converge to some values too.

The next parameter values were taken such that they satisfy Hopf bifurcation conditions. The parameters are $\beta = 7.5$, $\alpha_L = 4.5\beta$, $\alpha_I = 4.5\beta$, $\phi = 40.17/9.5$, $\xi = 10.65/28.5$, $\tau = 4.8865$, $\psi = 10.5/16.5$, $p = 0.1$, $\pi = 0.1$, $\lambda = 0.07479$, $\kappa = 0.0023$, $\nu = 0.01$.

Figure 2 shows simulation results using parameters with given initial conditions ($S^*, L^*, I^*, R^*$) where there is oscillation on susceptible individuals. Population in every compartment also converges to some values. Figure 3 shows that $S$ and $L$ trajectories are surrounding the initial conditions. This figure also shows that phase portrait direction also converges to the initial conditions which is an endemic point.

Traveling wave model on model 40 is solved using finite difference method. Given initial conditions are $L(t, x_0) = L^*$ and $I(t, x_0) = I^*$ with last boundary $L(t, x_1) = L^*$, where $I(t, x_1) = I^*$. Boundary conditions used is based on Theorem 2.4, with $L = 3$. Parameters taken are $\beta = 0.221$, $\alpha_L = 1.5\beta$, $\alpha_I = 1.5\beta$, $\phi = 0.1$, $\xi = 0.6/18$, $\tau = 0.4/8.5$, $\psi = 0.9/16$, $p = 0.1$, $\pi = 1.159$, $\lambda = 0.0643$, $\kappa = 0.1$, $\nu = 0$. 
Figure 2: Simulation Results of The Spreading Ebola Virus Model on Hopf Bifurcation Condition

Figure 3: $S$ and $L$ Population Trajectories and their Phase Portrait directions
The results are shown on Figure 4 and Figure 5 which told us that infected individuals are increasing over time.

When bifurcation happened, parameters being used are $\beta = 3$, $\alpha_L = 1.5\beta$, $\alpha_I = 1.5\beta$, $\phi = 5.17/9.5$, $\xi = 10.65/18.5$, $\tau = 1.5957$, $\psi = 10.5/16.5$, $p = 0.1$, $\pi = 0.1$, $\lambda = 0.07479$, $\kappa = 0.0023$, $\nu = 0.001$ Parameters being used imply
that they satisfy Hopf bifurcation condition. The results are shown on Figure 6 and Figure 7.

Figure 6: *Travelling Wave* fixed disease-free point at Hopf Bifurcation

Figure 7: Traveling Wave around fixed disease-free point at Hopf Bifurcation

Figure 8 and Figure 9 show results of model 40 with the same parameters. In this case, the boundary conditions are $L(t_0, x_0) = L^*$, $I(t_0, x_0) = I^*$, $L(t, x_1) = L^*$, and $I(t_1, x_1) = I^*$. 
Figure 8: Traveling Wave L Around Fixed Disease-Free Point at Unstable Condition

Figure 9: Traveling Wave L Around Fixed Disease-Free Point at Unstable Condition
Figure 4 through Figure 9 show existence of waves of increasing and decreasing of L and I values over time. We can conclude that there exists traveling wave with a form of different waves where waves rate depends on various parameter values used in simulations. The simulations of traveling wave show that different conditions give graphs with different wave shapes. In unstable conditions, the simulations show that the number of infected and latent are less than the initial conditions. So there is a possibility of wider outbreaks with less number of infected. The simulations on Hopf bifurcation condition show that infected and latent numbers are greater than the initial conditions. The oscillations which occurred in Figure 6 indicates a change in the number of infected and latent individuals. But in this condition, widespread with more quantities can occur. In the last simulation with different boundary conditions and same parameter as before, we can conclude that in Hopf bifurcation conditions more number of infected and latent individual can be obtained even though it also depends on the rates of transmission of Ebola.

4 Conclusion

Based on simulations on previous sections, we can conclude as follows:

1. Observation on two fixed points gave \((\frac{\pi}{\lambda}, 0, 0, 0)\) where this point is disease-free point and \((S^*, L^*, I^*, R^*)\) is endemic fixed point.

2. Observations on traveling wave obtained \(c_{\text{min}}\) on fixed point limited only for I and L compartments, such that

\[
c = 2\sqrt{\frac{D_I(\phi k_1 - (\xi + \tau)k_2)}{k_2}}
\]

\[
c = 2\sqrt{\frac{D_L\alpha^*}{k_1}}
\]
on endemic fixed point and

\[
c = 2\sqrt{\frac{D_I\phi j_1 - (\xi + \tau)j_2}{j_2}}
\]

\[
c = 2\sqrt{\frac{D_L[\beta(1 - p) + p\alpha_L] \frac{\pi}{N_k} - \phi - \psi]j_1 + [p\alpha_L \frac{\pi}{N_k}]j_2}{j_1}}
\]
on disease-free point for each I and L compartment, respectively.
3. Based on simulations, the higher transmission rate of Ebola virus then population of infected individuals were very likely to be increased as well. This is caused by interactions between susceptible individuals with latent and infected individuals. Because of movement freedom for each people, causing a wider transmission wave. This is proved by simulations i.e. there exists waves on some places with no indications of virus infections or even outbreak before.

References


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