New Nonlinear Conditions for Approximate Sequences and New Best Proximity Point Theorems

Wei-Shih Du\textsuperscript{1} and Yuan-Liang Liu

Department of Mathematics
National Kaohsiung Normal University
Kaohsiung 82444, Taiwan

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Abstract

In this paper, we establish new convergence theorems and best proximity point theorems for approximate sequences.

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1. Introduction and preliminaries

In the last decades, the theory of fixed points has become a crucial technique in the study of nonlinear functional analysis. However, as we know, the equation $Tx = x$ (i.e. $d(Tx, x) = 0$) does not necessarily have a fixed points, where $T$ is a self-mapping or non-self-mapping defined on the metric space $(X, d)$. So, in such situations, we often turn to find the best approximation of the existence of solutions. In other words, it is quite natural to investigate an element $x \in X$ such that $d(Tx, x)$ is minimum, that is, the point $x$ and the point $Tx$ are close proximity to each other. In recent years, many researcher

\textsuperscript{1}Corresponding author
have studied the use of new nonlinear conditions to promote the best proximity point theory; see, e.g., [1-4, 7, 14-24] and references therein.

Throughout this paper, we denote by \( \mathbb{N} \) and \( \mathbb{R} \), the sets of positive integers and real numbers, respectively. Let \( A \) and \( B \) be nonempty subsets of a metric space \((X, d)\) and \( T : A \cup B \to A \cup B \) be a self-mapping. Denote
\[
\text{dist}(A, B) = \inf \{ d(x, y) : x \in A, y \in B \}.
\]
A point \( x \in A \cup B \) is called to be a best proximity point for \( T \) if \( d(x, Tx) = \text{dist}(A, B) \).

The concepts of cyclic mapping and best proximity point were introduced by Kirk, Srinavasan and Veeramani [19] in 2003.

**Definition 1.1 [19].** Let \( A \) and \( B \) be nonempty subsets of a metric space \((X, d)\). A mapping \( T : A \cup B \to A \cup B \) is called cyclic if
\[
T(A) \subset B \quad \text{and} \quad T(B) \subset A.
\]

In 2006, Eldered and Veeramani [17] established some existence results about best proximity points of cyclic contraction mappings.

**Definition 1.2 [17].** Let \( A \) and \( B \) be nonempty closed subsets of a metric space \((X, d)\). A cyclic mapping \( T : A \cup B \to A \cup B \) is called a cyclic contraction if for some \( \alpha \in (0, 1) \), the condition
\[
d(Tx, Ty) \leq \alpha d(x, y) + (1 - \alpha)\text{dist}(A, B)
\]
holds for all \( x \in A \) and \( y \in B \).

**Theorem 1.1 [17].** Let \( A \) and \( B \) be nonempty closed subsets of a metric space \((X, d)\), \( T : A \cup B \to A \cup B \) be a cyclic contraction mapping. We define \( x_1 \in A \) and \( x_{n+1} = Tx_n, n \in \mathbb{N} \). Suppose \( \{x_{2n-1}\} \) has a convergent subsequence in \( A \). Then there exists \( x \in A \) such that \( d(x, Tx) = \text{dist}(A, B) \).

Let \( f \) be a real-valued function defined on \( \mathbb{R} \). For \( c \in \mathbb{R} \), we recall that
\[
\limsup_{x \to c} f(x) = \inf_{\varepsilon > 0} \sup_{0 < |x - c| < \varepsilon} f(x)
\]
and
\[
\limsup_{x \to c^+} f(x) = \inf_{\varepsilon > 0} \sup_{c < x < c + \varepsilon} f(x).
\]

In 2016, Du introduced the concept of \( \mathcal{MT}(\lambda) \)-function [7, 9-14] as follows.
Definition 1.3. Let $\lambda > 0$. A function $\mu : [0, \infty) \to [0, \lambda)$ is said to be an $MT(\lambda)$-function [7, 9-14] if $\limsup_{s \to t^+} \mu(s) < \lambda$ for all $t \in [0, \infty)$. In particular, if $\lambda = 1$, then $\mu : [0, \infty) \to [0, 1)$ is called an $MT$-function (or $M$-function) [5-15].

In [7], Du established the following useful characterizations of $MT(\lambda)$-functions; see also [9-14].

Theorem 1.2 (see [7, Theorem 2.4]). Let $\lambda > 0$ and let $\mu : [0, \infty) \to [0, \lambda)$ be a function. Then the following statements are equivalent.

1. $\mu$ is an $MT(\lambda)$-function.
2. $\lambda^{-1} \mu$ is an $MT$-function.
3. For each $t \in [0, \infty)$, there exists $\xi_t^{(1)} \in [0, \lambda)$ and $\epsilon_t^{(1)} > 0$ such that $\mu(s) \leq \xi_t^{(1)}$ for all $s \in (t, t + \epsilon_t^{(1)})$.
4. For each $t \in [0, \infty)$, there exists $\xi_t^{(2)} \in [0, \lambda)$ and $\epsilon_t^{(2)} > 0$ such that $\mu(s) \leq \xi_t^{(2)}$ for all $s \in [t, t + \epsilon_t^{(2)}]$.
5. For each $t \in [0, \infty)$, there exists $\xi_t^{(3)} \in [0, \lambda)$ and $\epsilon_t^{(3)} > 0$ such that $\mu(s) \leq \xi_t^{(3)}$ for all $s \in (t, t + \epsilon_t^{(3)})$.
6. For each $t \in [0, \infty)$, there exists $\xi_t^{(4)} \in [0, \lambda)$ and $\epsilon_t^{(4)} > 0$ such that $\mu(s) \leq \xi_t^{(4)}$ for all $s \in [t, t + \epsilon_t^{(4)}]$.
7. For any nonincreasing sequence $\{x_n\}_{n \in \mathbb{N}}$ in $[0, \infty)$, we have $0 \leq \sup_{n \in \mathbb{N}} \mu(x_n) < \lambda$.
8. For any strictly decreasing sequence $\{x_n\}_{n \in \mathbb{N}}$ in $[0, \infty)$, we have $0 \leq \sup_{n \in \mathbb{N}} \mu(x_n) < \lambda$.
9. For any eventually nonincreasing sequence $\{x_n\}_{n \in \mathbb{N}}$ (i.e. there exists $\ell \in \mathbb{N}$ such that $x_{n+1} \leq x_n$ for all $n \in \mathbb{N}$ with $n \geq \ell$) in $[0, \infty)$, we have $0 \leq \sup_{n \in \mathbb{N}} \mu(x_n) < \lambda$.
10. For any eventually strictly decreasing sequence $\{x_n\}_{n \in \mathbb{N}}$ (i.e. there exists $\ell \in \mathbb{N}$ such that $x_{n+1} < x_n$ for all $n \in \mathbb{N}$ with $n \geq \ell$) in $[0, \infty)$, we have $0 \leq \sup_{n \in \mathbb{N}} \mu(x_n) < \lambda$. 
Let us recall the concept of approximate sequence.

**Definition 1.4** [14]. Let $A$ and $B$ be nonempty subsets of a metric space $(X, d)$ and $\tau : [0, \infty) \to [0, 1)$ be a function. A sequence $\{x_n\}_{n \in \mathbb{N}}$ in $A \cup B$ is said to be *approximate with respect to* $\tau$, if the following conditions are satisfied:

(a) one of the following conditions holds:

(i) $\{x_{2n-1}\}_{n \in \mathbb{N}} \subset A$ and $\{x_{2n}\}_{n \in \mathbb{N}} \subset B$;

(ii) $\{x_{2n}\}_{n \in \mathbb{N}} \subset A$ and $\{x_{2n-1}\}_{n \in \mathbb{N}} \subset B$,

(b) $d(x_{n+1}, x_{n+2}) \leq \tau(d(x_n, x_{n+1}))d(x_n, x_{n+1}) + (1-\tau(d(x_n, x_{n+1})))\text{dist}(A, B)$ for all $n \in \mathbb{N}$.

The following convergence theorem for approximate sequences was essentially proved by using Theorem 1.2 in [14].

**Theorem 1.3** [14]. Let $A$ and $B$ be nonempty subsets of a metric space $(X, d)$ and $\varphi : [0, \infty) \to [0, 1)$ be an $\mathcal{MT}$-function. If $\{x_n\}_{n \in \mathbb{N}} \subset A \cup B$ is an approximate sequence with respect to $\varphi$, then $\lim_{n \to \infty} d(x_n, x_{n+1}) = \inf_{n \in \mathbb{N}} d(x_n, x_{n+1}) = \text{dist}(A, B)$.

In this paper, we establish new convergence theorems and best proximity point theorems for approximate sequences.

2. Some new existence theorems

In this section, we first give the following new nonlinear condition for approximate sequences.

**Theorem 2.1.** Let $A$ and $B$ be nonempty subsets of a metric space $(X, d)$ and $T : A \cup B \to A \cup B$ be a cyclic mapping. Suppose that

(DL) there exists an $\mathcal{MT}$-function $\varphi : [0, \infty) \to [0, 1)$ such that

$$d(Tx, Ty) \leq \varphi(d(x, y)) \max \left\{ \frac{1}{4} [d(x, Ty) + 2d(Tx, Ty) + d(y, Tx)], \right.$$  

$$\frac{1}{8} [d(x, Ty) + 3d(x, Tx) + 3d(y, Ty) + d(y, Tx)] \right\}$$  

$$+ [1 - \varphi(d(x, y))] \text{dist}(A, B)$$

for all $x \in A$ and $y \in B$.  


Then there exists an approximate sequence in \( A \cup B \) with respect to \( \varphi \).

**Proof.** Because \( T \) is cyclic, we have \( T(A) \subset B \) and \( T(B) \subset A \). Let \( x_1 \in A \) be given. Define \( x_{n+1} = Tx_n \) for all \( n \in \mathbb{N} \). Then \( \{x_{2n-1}\}_{n \in \mathbb{N}} \subset A \) and \( \{x_{2n}\}_{n \in \mathbb{N}} \subset B \). In order to finish the proof it is sufficient to show that the sequence \( \{x_n\}_{n \in \mathbb{N}} \) satisfies the following:

- \( d(x_{n+1}, x_{n+2}) \leq d(x_n, x_{n+1}) \) for all \( n \in \mathbb{N} \),
- \( d(x_{n+1}, x_{n+2}) \leq \varphi(d(x_n, x_{n+1}))d(x_n, x_{n+1}) + [1 - \varphi(d(x_n, x_{n+1}))] \text{dist}(A, B) \)
  for all \( n \in \mathbb{N} \).

Suppose that there exists \( j \in \mathbb{N} \) such that \( \varphi(d(x_j, x_{j+1})) = 0 \), then, by (DL), we obtain

\[
    d(x_{j+1}, x_{j+2}) = d(Tx_j, Tx_{j+1}) \leq \text{dist}(A, B) \leq d(x_j, x_{j+1}). \tag{2.1}
\]

By (2.1), we have

\[
    d(x_{j+1}, x_{j+2}) \leq \varphi(d(x_j, x_{j+1}))d(x_j, x_{j+1}) + [1 - \varphi(d(x_j, x_{j+1}))] \text{dist}(A, B)
\]

and our claim for the case of \( \varphi(d(x_j, x_{j+1})) = 0 \) is proved. For this reason we henceforth will assume that \( \varphi(d(x_n, x_{n+1})) > 0 \) for all \( n \in \mathbb{N} \). By (DL), we have

\[
    d(x_2, x_3) \leq \varphi(d(x_1, x_2)) \max \left\{ \frac{1}{4}d(x_1, x_3) + 2d(x_2, x_3) + d(x_2, x_2), \frac{1}{8}[d(x_1, x_3) + 3d(x_1, x_2) + 3d(x_2, x_3) + d(x_2, x_2)] \right\}
      + [1 - \varphi(d(x_1, x_2))] \text{dist}(A, B)
    \leq \varphi(d(x_1, x_2)) \max \left\{ \frac{1}{4}d(x_1, x_2) + \frac{3}{4}d(x_2, x_3), \frac{1}{2}d(x_1, x_2) + \frac{1}{2}d(x_2, x_3) \right\}
      + [1 - \varphi(d(x_1, x_2))] \text{dist}(A, B). \tag{2.2}
\]

Assume that

\[
    \frac{1}{4}d(x_1, x_2) + \frac{3}{4}d(x_2, x_3) > \frac{1}{2}d(x_1, x_2) + \frac{1}{2}d(x_2, x_3). \tag{2.3}
\]

Then (2.3) implies

\[
    d(x_2, x_3) > d(x_1, x_2). \tag{2.4}
\]

By taking into account (2.2), (2.3) and (2.4), we get

\[
    d(x_2, x_3) \leq \frac{1}{4} \varphi(d(x_1, x_2))[d(x_1, x_2) + 3d(x_2, x_3)] + [1 - \varphi(d(x_1, x_2))] \text{dist}(A, B).
\]


\[
    \text{and of course, } d(x_2, x_3) = d(x_2, x_3).
\]
a contradiction. So we must have
\[ \frac{1}{4} d(x_1, x_2) + \frac{3}{4} d(x_2, x_3) \leq \frac{1}{2} d(x_1, x_2) + \frac{1}{2} d(x_2, x_3) \]  
(2.5)
and hence
\[ d(x_2, x_3) \leq d(x_1, x_2). \]  
(2.6)
Taking into account (2.2), (2.5) and (2.6), we obtain
\[ d(x_2, x_3) \leq \varphi(d(x_1, x_2)) d(x_1, x_2) + [1 - \varphi(d(x_1, x_2))] \text{dist}(A, B). \]  
Similarly, from (DL) again, we have
\[ d(x_4, x_3) \leq \varphi(d(x_3, x_2)) \max \left\{ \frac{1}{4} [d(x_3, x_3) + 2d(x_3, x_4) + d(x_2, x_4)] \right\} \]  
\[ + (1 - \varphi(d(x_3, x_2))) \text{dist}(A, B) \]  
\[ \leq \varphi(d(x_2, x_3)) \max \left\{ \frac{1}{4} d(x_2, x_3) + \frac{3}{4} d(x_3, x_4), \frac{1}{2} d(x_2, x_3) + \frac{1}{2} d(x_3, x_4) \right\} \]  
\[ + [1 - \varphi(d(x_2, x_3))] \text{dist}(A, B). \]  
(2.7)
If
\[ \frac{1}{4} d(x_2, x_3) + \frac{3}{4} d(x_3, x_4) > \frac{1}{2} d(x_2, x_3) + \frac{1}{2} d(x_3, x_4), \]  
then
\[ d(x_3, x_4) > d(x_2, x_3) \]  
and we get from (2.7) that
\[ d(x_3, x_4) \leq \frac{1}{4} \varphi(d(x_2, x_3)) [d(x_2, x_3) + 3d(x_3, x_4)] + [1 - \varphi(d(x_2, x_3))] \text{dist}(A, B). \]  
\[ < \varphi(d(x_2, x_3)) d(x_3, x_4) + (1 - \varphi(d(x_2, x_3))) d(x_3, x_4) \]  
\[ = d(x_3, x_4), \]  
which leads a contradiction. So it must be
\[ \frac{1}{4} d(x_2, x_3) + \frac{3}{4} d(x_3, x_4) \leq \frac{1}{2} d(x_2, x_3) + \frac{1}{2} d(x_3, x_4), \]  
(2.8)
and hence
\[ d(x_3, x_4) \leq d(x_2, x_3) \]  
(2.9)
By (2.7), (2.8) and (2.9), we have
\[ d(x_3, x_4) \leq \varphi(d(x_2, x_3)) d(x_2, x_3) + [1 - \varphi(d(x_2, x_3))] \text{dist}(A, B). \]
Therefore, by induction, we prove that the sequence \( \{x_n\}_{n \in \mathbb{N}} \) satisfies the following:
New nonlinear conditions for approximate sequences

(a) \( d(x_{n+1}, x_{n+2}) \leq d(x_n, x_{n+1}) \) for all \( n \in \mathbb{N} \),

(b) \( d(x_{n+1}, x_{n+2}) \leq \varphi(d(x_n, x_{n+1}))d(x_n, x_{n+1}) + [1 - \varphi(d(x_n, x_{n+1}))] \text{dist}(A, B) \)

for all \( n \in \mathbb{N} \).

Therefore, we prove that \( \{x_n\}_{n \in \mathbb{N}} \) is an approximate sequence in \( A \cup B \) with respect to \( \varphi \). \( \square \)

Applying Theorems 2.1 and 1.3, we obtain the following new convergence theorem immediately.

**Theorem 2.2.** Let \( A \) and \( B \) be nonempty subsets of a metric space \( (X, d) \) and \( T : A \cup B \to A \cup B \) be a cyclic mapping. Suppose that the condition (DL) as in Theorem 2.1 holds. Then there exists a sequence \( \{x_n\}_{n \in \mathbb{N}} \subset A \cup B \) such that

(a) \( d(x_{n+1}, x_{n+2}) \leq \varphi(d(x_n, x_{n+1}))d(x_n, x_{n+1}) + [1 - \varphi(d(x_n, x_{n+1}))] \text{dist}(A, B) \)

for all \( n \in \mathbb{N} \);

(b) \( \lim_{n \to \infty} d(x_n, x_{n+1}) = \inf_{n \in \mathbb{N}} d(x_n, x_{n+1}) = \text{dist}(A, B) \).

**Corollary 2.1.** Let \( A \) and \( B \) be nonempty subsets of a metric space \( (X, d) \) and \( T : A \cup B \to A \cup B \) be a cyclic mapping. Suppose that

(DL1) there exists an \( \mathcal{MT} \)-function \( \varphi : [0, \infty) \to [0, 1) \) such that

\[
d(Tx, Ty) \leq \frac{1}{4} \varphi(d(x, y))[d(x, Ty) + 2d(Tx, Ty) + d(y, Tx)]
+ [1 - \varphi(d(x, y))\text{dist}(A, B)]
\]

for all \( x \in A \) and \( y \in B \).

Then there exists a sequence \( \{x_n\}_{n \in \mathbb{N}} \subset A \cup B \) such that

(a) \( d(x_{n+1}, x_{n+2}) \leq \varphi(d(x_n, x_{n+1}))d(x_n, x_{n+1}) + [1 - \varphi(d(x_n, x_{n+1}))] \text{dist}(A, B) \)

for all \( n \in \mathbb{N} \);

(b) \( \lim_{n \to \infty} d(x_n, x_{n+1}) = \inf_{n \in \mathbb{N}} d(x_n, x_{n+1}) = \text{dist}(A, B) \).

**Corollary 2.2.** Let \( A \) and \( B \) be nonempty subsets of a metric space \( (X, d) \) and \( T : A \cup B \to A \cup B \) be a cyclic mapping. Suppose that
(DL2) there exists an MT-function \( \varphi : [0, \infty) \to [0, 1) \) such that

\[
d(Tx, Ty) \leq \frac{1}{8} \varphi(d(x, y)) [d(x, Ty) + 3d(x, Tx) + 3d(y, Ty) + d(y, Tx)] \\
+ [1 - \varphi(d(x, y))] \text{dist}(A, B)
\]

for all \( x \in A \) and \( y \in B \).

Then there exists a sequence \( \{x_n\}_{n \in \mathbb{N}} \subset A \cup B \) such that

\[
\begin{align*}
(a) & \quad d(x_{n+1}, x_{n+2}) \leq \varphi(d(x_n, x_{n+1}))d(x_n, x_{n+1}) + [1 - \varphi(d(x_n, x_{n+1}))] \text{dist}(A, B) \\
& \quad \text{for all } n \in \mathbb{N}; \\
(b) & \quad \lim_{n \to \infty} d(x_n, x_{n+1}) = \inf_{n \in \mathbb{N}} d(x_n, x_{n+1}) = \text{dist}(A, B).
\end{align*}
\]

Following a similar argument as the proof of [14,Theorem 2.5] , we give an existence theorem for best proximity points.

**Theorem 2.3.** In Theorem 2.2, if we further assume

\( (H) \ d(Tx, Ty) \leq d(x, y) \) for any \( x \in A \) and \( y \in B \).

Then the following statements hold.

\[
\begin{align*}
(a) & \quad \text{If } \{x_{2n-1}\}_{n \in \mathbb{N}} \text{ has a convergent subsequence in } A, \text{ then there exists } u \in A \\
& \quad \text{such that } d(u, Tu) = \text{dist}(A, B). \\
(b) & \quad \text{If } \{x_{2n}\}_{n \in \mathbb{N}} \text{ has a convergent subsequence in } B, \text{ then there exists } v \in B \\
& \quad \text{such that } d(v, Tv) = \text{dist}(A, B).
\end{align*}
\]

**Remark 2.1.**

\[
\begin{align*}
(a) & \quad \text{Theorems 2.1, 2.2 and 2.3 and Corollaries 2.1 and 2.2 are also true if the function } \varphi \text{ is nonincreasing or nondecreasing.} \\
(b) & \quad \text{In Theorem 2.3, if } "\text{Theorem 2.2}" \text{ is replaced with } "\text{Corollary 2.1}\" \text{ or } "\text{Corollary 2.2}, \text{ then we can also obtain new best proximity point theo-} \\
& \quad \text{rems.} \\
(c) & \quad \text{As a direct application of Theorem 2.2, we can establish easily some new fixed point theorems.}
\end{align*}
\]
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