On Degenerate Carlitz’s Type $q$-Tangent Numbers and Polynomials Associated with $p$-Adic $q$-Integral on $\mathbb{Z}_p$

C. S. Ryoo

Department of Mathematics
Hannam University, Daejeon 306-791, Korea

Abstract

In this paper, we introduce the degenerate Carlitz’s type $q$-tangent numbers and polynomials associated with the $p$-adic $q$-integral on $\mathbb{Z}_p$. We give some explicit formulas for degenerate Carlitz’s type $q$-tangent numbers and polynomials.

Mathematics Subject Classification: 11B68, 11S40, 11S80

Keywords: tangent numbers and polynomials, degenerate tangent numbers and polynomials, degenerate Carlitz’s type $q$-tangent numbers and polynomials, $p$-adic $q$-integral on $\mathbb{Z}_p$

1 Introduction

L. Carlitz constructed the degenerate Bernoulli polynomials(see [1]). Feng Qi et al.[2] introduced the partially degenerate Bernoulli polynomials of the first kind in $p$-adic field. T. Kim introduced the Barnes’ type multiple degenerate Bernoulli and Euler polynomials(see [3]), Recently, Ryoo introduced the twisted $(h,q)$-tangent numbers and tangent polynomials(see [4, 5]). In this paper, we introduce degenerate Carlitz’s type $q$-tangent numbers $\mathcal{T}_{n,q}(\lambda)$ and $q$-tangent polynomials $\mathcal{T}_{n,q}(x,\lambda)$. Let $p$ be a fixed odd prime number. Throughout this paper we use the following notations. By $\mathbb{Z}_p$, we denote the ring of
\( p \)-adic rational integers, \( \mathbb{Q}_p \) denotes the field of rational numbers, \( \mathbb{N} \) denotes the set of natural numbers, \( \mathbb{C} \) denotes the complex number field, \( \mathbb{C}_p \) denotes the completion of algebraic closure of \( \mathbb{Q}_p \), \( \mathbb{N} \) denotes the set of natural numbers and \( \mathbb{Z}_+ = \mathbb{N} \cup \{0\} \), and \( \mathbb{C} \) denotes the set of complex numbers. Let \( \nu_p \) be the normalized exponential valuation of \( \mathbb{C}_p \) with \( |p_p| = p^{-\nu_p(p)} = p^{-1} \). When one talks of \( q \)-extension, \( q \) is considered in many ways such as an indeterminate, a complex number \( q \in \mathbb{C} \), or \( p \)-adic number \( q \in \mathbb{C}_p \). If \( q \in \mathbb{C} \) one normally assume that \( |q| < 1 \). If \( q \in \mathbb{C}_p \), we normally assume that \( |q - 1|_p < p^{-1} \) so that \( q^x = \exp(x \log q) \) for \( |x|_p \leq 1 \). Throughout this paper we use the notation:
\[
[x]_q = \frac{1 - q^x}{1 - q}, \text{ cf. [1, 2, 3, 4, 5, 6] .}
\]
For
\[
g \in UD(\mathbb{Z}_p) = \{g|g : \mathbb{Z}_p \to \mathbb{C}_p \text{ is uniformly differentiable function}\},
\]
Kim[2] defined the \( p \)-adic \( q \)-integral on \( \mathbb{Z}_p \) as follows:
\[
I_{-q}(g) = \int_{\mathbb{Z}_p} g(x) d\mu_{-q}(x) = \lim_{N \to \infty} \frac{1 + q}{1 + q^{p^N}} \sum_{x=0}^{p^N-1} g(x)(-q)^x. \hspace{1cm} (1.1)
\]
From (1.1), we note that
\[
q^n I_{-q}(f_n) = (-1)^n I_{-q}(g) + [2]_q \sum_{l=0}^{n-1} (-1)^{n-1-l} q^l g(l), \hspace{1cm} (1.2)
\]
where \( g_n(x) = g(x + n) \) for \( n \in \mathbb{N} \).

We recall that the classical Stirling numbers of the first kind \( S_1(n,k) \) and \( S_2(n,k) \) are defined by the relations(see [6])
\[
(x)_n = \sum_{k=0}^{n} S_1(n,k)x^k \text{ and } x^n = \sum_{k=0}^{n} S_2(n,k)(x)_k,
\]
respectively. Here \( (x)_n = x(x-1)\cdots(x-n+1) \) denotes the falling factorial polynomial of order \( n \). We also have
\[
\sum_{n=m}^{\infty} S_2(n,m) \frac{t^n}{n!} = \frac{(e^t - 1)^m}{m!} \text{ and } \sum_{n=m}^{\infty} S_1(n,m) \frac{t^n}{n!} = \frac{(\log(1 + t))^m}{m!}. \hspace{1cm} (1.3)
\]
The generalized falling factorial \( (x|\lambda)_n \) with increment \( \lambda \) is defined by
\[
(x|\lambda)_n = \prod_{k=0}^{n-1} (x - \lambda k) \hspace{1cm} (1.4)
\]
On degenerate Carlitz’s type $q$-tangent numbers and polynomials

for positive integer $n$, with the convention $(x|\lambda)_0 = 1$. We also need the binomial theorem: for a variable $x$,

$$(1 + \lambda t)^{x/\lambda} = \sum_{n=0}^{\infty} (x|\lambda)_n \frac{t^n}{n!}. \quad (1.5)$$

For $q \in \mathbb{C}_p$ with $|1 - q|^p \leq 1$, if we take $g(x) = e^{2xt}$ in (1.2), then we easily see that

$$I_{-q}(e^{[2x]q}t) = \int_{\mathbb{Z}_p} e^{[2x]q}t d\mu_{-q}(x) = [2]_q \sum_{m=0}^{\infty} (-1)^m q^m e^{[2m]q} t.$$ 

Let us define the tangent numbers $T_{n,q}$ and polynomials $T_{n,q}(x)$ as follows:

$$\int_{\mathbb{Z}_p} e^{[2y]q}t d\mu_{-q}(y) = \sum_{n=0}^{\infty} T_{n,q} \frac{t^n}{n!}, \quad (1.6)$$

$$\int_{\mathbb{Z}_p} e^{[x+2y]q}t d\mu_{-q}(y) = \sum_{n=0}^{\infty} T_{n,q}(x) \frac{t^n}{n!}. \quad (1.7)$$

Many mathematicians have worked in the area of the $q$-analogues of the degenerate Bernoulli numbers and polynomials, Euler numbers and polynomials, and tangent numbers and polynomials (see [1, 2, 3, 4, 5, 6]). Our aim in this paper is to define degenerate Carlitz’s type $q$-tangent polynomials $T_{n,q}(x, \lambda)$. We investigate some properties which are related to degenerate Carlitz’s type $q$-tangent numbers $T_{n,q}(\lambda)$ and polynomials $T_{n,q}(x, \lambda)$.

2 Degenerate Carlitz’s type $q$-tangent numbers and polynomials

In this section, we introduce the degenerate Carlitz’s type twisted $q$-tangent numbers and polynomials, and we obtain explicit formulas for them. We assume that $t, \lambda \in \mathbb{C}_p$ with $0 < |\lambda|^p \leq 1$ and $|t|^p < p^{-\frac{1}{p-1}}$. Then we have $|\lambda t|^p < p^{-\frac{1}{p-1}}$ and $\frac{1}{\lambda} \log(1 + \lambda t)_p = |t|^p < p^{-\frac{1}{p-1}}$. By (1.3), we have

$$\frac{[x+2y]_q}{\lambda} = e^{\frac{[x+2y]_q}{\lambda} \log(1+\lambda t)}$$

$$= \sum_{n=0}^{\infty} \left( \frac{[x+2y]_q}{\lambda} \frac{\log(1+\lambda t)}{n!} \right)^n$$

$$= \sum_{n=0}^{\infty} \left( \sum_{m=0}^{n} S_1(n, m) \lambda^{n-m}[x+2y]_q^m \right) \frac{t^n}{n!}. \quad (2.1)$$
Definition 2.1 The degenerate Carlitz’s type \( q \)-tangent numbers \( T_{n,q}(\lambda) \) and polynomials \( T_{n,q}(x, \lambda) \) are defined by means of the generating functions
\[
\int_{\mathbb{Z}_p} (1 + \lambda t) \frac{[2y]_q}{\lambda} d\mu_{-q}(y) = \sum_{n=0}^{\infty} T_{n,q}(\lambda) \frac{t^n}{n!},
\]
and
\[
\int_{\mathbb{Z}_p} (1 + \lambda t) \frac{[x + 2y]_q}{\lambda} d\mu_{-q}(y) = \sum_{n=0}^{\infty} T_{n,q}(x, \lambda) \frac{t^n}{n!},
\]
respectively.

Note that \((1 + \lambda t)^{1/\lambda}\) tends to \( e^t \) as \( \lambda \to 0 \). From (2.3) and (1.7), we note that
\[
\sum_{n=0}^{\infty} \lim_{\lambda \to 0} T_{n,q}(x, \lambda) \frac{t^n}{n!} = \int_{\mathbb{Z}_p} e^{[x+2y]_q t} d\mu_{-q}(y)
\]
\[
= \sum_{n=0}^{\infty} T_{n,q}(x) \frac{t^n}{n!}.
\]
Thus, for \( n \geq 0 \), we get
\[
\lim_{\lambda \to 0} T_{n,q}(x, \lambda) = T_{n,q}(x) \text{ and } \lim_{\lambda \to 0} T_{n,q}(\lambda) = T_{n,q}.
\]
For \( q \in \mathbb{C}_p \) with \(|q - 1|_p < 1\), we observe that
\[
T_{n,q} = [2]_q \left( \frac{1}{1-q} \right)^n \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{1}{1+q^{2l+1}}.
\]
From (1.5), we have
\[
\sum_{n=0}^{\infty} T_{n,q}(x, \lambda) \frac{t^n}{n!} = \int_{\mathbb{Z}_p} (1 + \lambda t) \frac{[x + 2y]_q}{\lambda} d\mu_{-q}(y)
\]
\[
= \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} \left( \frac{1}{\lambda} \frac{[x + 2y]_q}{n} \right) \lambda^n t^n d\mu_{-q}(y)
\]
\[
= \sum_{n=0}^{\infty} \left( \int_{\mathbb{Z}_p} \frac{[x + 2y]_q}{\lambda} \right) \lambda^n d\mu_{-q}(y) \frac{t^n}{n!},
\]
where
\[
\left( \frac{[x + 2y]_q}{\lambda} \right)_n = \left( \frac{[x + 2y]_q}{\lambda} \right) \left( \frac{[x + 2y]_q}{\lambda} - 1 \right) \cdots \left( \frac{[x + 2y]_q}{\lambda} - n + 1 \right).
\]
By comparing of the coefficients \( \frac{t^n}{n!} \) on the both sides of (2.5), we have the following theorem.
Theorem 2.2 For \( n \geq 0 \), we have
\[
\mathcal{T}_{n,q}(x, \lambda) = \int_{\mathbb{Z}_p} \left( \frac{[x + 2y]_q}{\lambda} \right)^n \lambda^n d\mu_{-q}(y).
\]

We introduce a \( q \)-analogue of the generalized falling factorial \((x|\lambda)_n\) with increment \( \lambda \). The \( q \)-generalized falling factorial \((x|\lambda)_n\) with increment \( \lambda \) is defined by
\[
([x]_q|\lambda)_n = \prod_{k=0}^{n-1} ([x]_q - \lambda k)
\tag{2.6}
\]
for positive integer \( n \), with the convention \((x|\lambda)_0 = 1\). Note that \( \lim_{q \to 1} ([x]_q|\lambda)_n = (x|\lambda)_n \).

By (2.6) and Theorem 2.2, we obtain the following Witt’s formula.

Corollary 2.3 For \( n \in \mathbb{Z}_+ \), we have
\[
\mathcal{T}_{n,q}(x, \lambda) = \int_{\mathbb{Z}_p} ([x + 2y]_q|\lambda)_n d\mu_{-q}(y),
\]
\[
\mathcal{T}_{n,q}(\lambda) = \int_{\mathbb{Z}_p} ([2y]_q|\lambda)_n d\mu_{-q}(y).
\]

From (1.7), (2.1), we have
\[
\int_{\mathbb{Z}_p} \left( \frac{[x + 2y]_q}{\lambda} \right)^n \lambda^n d\mu_{-q}(y)
= \sum_{l=0}^{n} S_1(n, l) \lambda^{n-l} \int_{\mathbb{Z}_p} [x + 2y]^l_q d\mu_{-q}(y)
\tag{2.7}
= \sum_{l=0}^{n} S_1(n, l) \lambda^{n-l} T_{l,q}(x).
\]

By (2.7) and Theorem 2.2, we have the following theorem.

Theorem 2.4 For \( n \in \mathbb{Z}_+ \), we have
\[
\mathcal{T}_{n,q}(x, \lambda) = \sum_{l=0}^{n} S_1(n, l) \lambda^{n-l} T_{l,q}(x).
\]

By (2.4) and Theorem 2.4, we have the following corollary.

Corollary 2.5 For \( n \in \mathbb{Z}_+ \), we have
\[
\mathcal{T}_{n,q}(x, \lambda) = \sum_{l=0}^{n} \sum_{j=0}^{l} \frac{S_1(n, l) \lambda^{n-l}}{(1 - q)^j} \binom{l}{j} (-1)^j q^{xj} \frac{1}{1 + q^{2j+1}}.
\]
By replacing $t$ by $\frac{e^{\lambda t} - 1}{\lambda}$ in (2.3), we have

$$
\int_{\mathbb{Z}_p} e^{[x+2y]q} \, d\mu_{-q}(y) = \sum_{n=0}^{\infty} T_{n,q}(x, \lambda) \left( \frac{e^{\lambda t} - 1}{\lambda} \right)^n \frac{1}{n!}
$$

$$
= \sum_{n=0}^{\infty} T_{n,q}(x, \lambda) \lambda^{-n} \sum_{m=n}^{\infty} S_2(m, n) \lambda^m \frac{t^m}{m!}
$$

$$
= \sum_{m=0}^{\infty} \left( \sum_{n=0}^{m} T_{n,q}(x, \lambda) \lambda^{m-n} S_2(m, n) \right) \frac{t^m}{m!}.
$$

Thus, by (2.8) and (1.7), we have the following theorem.

**Theorem 2.6** For $n \in \mathbb{Z}_+$, we have

$$
T_{m,q}(x) = \sum_{n=0}^{m} \lambda^{m-n} S_2(m, n) T_{n,q}(x, \lambda).
$$

By replacing $t$ by $\log(1 + \lambda t)^{1/\lambda}$ in (1.7), we have

$$
\sum_{n=0}^{\infty} T_{n,q}(x) \left( \log(1 + \lambda t)^{1/\lambda} \right)^n \frac{1}{n!} = \int_{\mathbb{Z}_p} (1 + \lambda t)^{1/\lambda} \frac{[x+2y]q}{\lambda} \, d\mu_{-q}(y)
$$

$$
= \sum_{m=0}^{\infty} T_{m,q}(x, \lambda) \frac{t^m}{m!},
$$

and

$$
\sum_{n=0}^{\infty} T_{n,q}(x) \left( \log(1 + \lambda t)^{1/\lambda} \right)^n \frac{1}{n!}
$$

$$
= \sum_{m=0}^{\infty} \left( \sum_{n=0}^{m} T_{n,q}(x) \lambda^{m-n} S_1(m, n) \right) \frac{t^m}{m!}.
$$

Thus, by (2.9) and (2.10), we have the following theorem.

**Theorem 2.7** For $n \in \mathbb{Z}_+$, we have

$$
T_{m,q}(x, \lambda) = \sum_{n=0}^{m} \lambda^{m-n} S_1(m, n) T_{n,q}(x).
$$
We observe that

\[
\left(1 + \lambda t\right) \frac{[x + 2y]_q}{\lambda} \cdot \frac{q^x[2y]_q}{\lambda} = (1 + \lambda t) \frac{[x]_q}{\lambda} \left(1 + \lambda t\right) \frac{q^x[2y]_q}{\lambda} = \sum_{m=0}^{\infty} \frac{([x]_q|\lambda)_m}{m!} \frac{t^m}{m!} e^{\log(1 + \lambda t)^\ell} \sum_{l=0}^{\infty} \frac{\left(q^x[2y]_q\right)^\ell}{\ell!} \sum_{k=0}^{\infty} S(k, l) \lambda^k \frac{k!}{k!}.
\]

(2.11)

By (1.2), we have

\[
q^n \int_{\mathbb{Z}_p} \frac{[2x + 2n]_q}{\lambda} d\mu(x) + (-1)^{n-1} \int_{\mathbb{Z}_p} \frac{[2x]_q}{\lambda} d\mu(x)
\]

\[
= [2]_q \sum_{l=0}^{n-1} (-1)^{n-1-l} q^l (1 + \lambda t) \frac{[2l]_q}{\lambda} \sum_{m=0}^{\infty} \frac{\left([2l]_q\right)_m}{m!} t^m \frac{n!}{n!}.
\]

(2.12)

By comparing of the coefficients \(t^n/n!\) on the both sides of (2.12), we have the following theorem.

**Theorem 2.8** For \(n \in \mathbb{Z}_+\), we have

\[
T_{n,q}(x, \lambda) = \sum_{k=0}^{n} \sum_{l=0}^{k} \binom{n}{k} ([x]_q|\lambda)_{n-k} \lambda^{k-l} q^x T_{l,q}.
\]

By (2.2), we have the following theorem.

**Theorem 2.9** For \(m \in \mathbb{Z}_+\), we have

\[
q^n T_{m,q}(2n, \lambda) + (-1)^{n-1} T_{m,q}(\lambda) = [2]_q \sum_{l=0}^{n-1} (-1)^{n-1-l} q^l ([2l]_q|\lambda)_m.
\]
For $m \equiv 1 \pmod{2}$, we have
\[
\int_{\mathbb{Z}_p} [x + 2y]^l d\mu_q(y) = \frac{[2]^q}{[2]^{q^n}} [m]^q \sum_{a=0}^{m-1} (-1)^aq^a \int_{\mathbb{Z}_p} \left[ \frac{2a + x}{m} + 2y \right]^l d\mu_{-q^m}(y)
\]
By Theorem 2.7 and (2.13), we have the following theorem.

**Theorem 2.10** For any positive integer $m (=\text{odd})$, we have
\[
\mathcal{T}_{n,q}(x, \lambda) = \sum_{l=0}^{n} \sum_{a=0}^{m-1} \lambda^{n-l} S_1(n, l) \frac{[2]^q}{[2]^{q^n}} [m]^l [(-1)^aq^a \mathcal{T}_{l,q^m} \left( \frac{2a + x}{m} \right)].
\]
Since
\[
T_{n,q^{-1}}(2 - x) = (-1)^n q^n T_{n,q}(x),
\]
we have the following property of complement.

**Theorem 2.11** For any positive integer $m$, we have
\[
\mathcal{T}_{n,q^{-1}}(2 - x, \lambda^{-1}) = (-1)^n q^n \mathcal{T}_{n,q}(x, \lambda^{-1}).
\]

**References**


Received: August 28, 2017; Published: September 17, 2017