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On Degenerate Carlitz's Type q-Tangent Numbers and Polynomials Associated with p-Adic q-Integral on $\mathbb{Z}_p$

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Abstract

In this paper, we introduce the degenerate Carlitz's type q-tangent numbers and polynomials associated with the p-adic q-integral on $\mathbb{Z}_p$. We give some explicit formulas for degenerate Carlitz's type q-tangent numbers and polynomials.

Mathematics Subject Classification: 11B68, 11S40, 11S80

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1 Introduction

L. Carlitz constructed the degenerate Bernoulli polynomials(see [1]). Feng Qi *et al.*[2] introduced the partially degenerate Bernoull polynomials of the first kind in p-adic field. T. Kim introduced the Barnes' type multiple degenerate Bernoulli and Euler polynomials(see [3]), Recently, Ryoo introduced the twisted (h, q)-tangent numbers and tangent polynomials(see [4, 5). In this paper, we introduce degenerate Carlitz's type q-tangent numbers $\mathcal{T}_{n,q}(\lambda)$ and q-tangent polynomials $\mathcal{T}_{n,q}(x, \lambda)$. Let p be a fixed odd prime number. Throughout this paper we use the following notations. By $\mathbb{Z}_p$ we denote the ring of

p-adic rational integers, $\mathbb{Q}_p$ denotes the field of rational numbers, $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{C}$ denotes the complex number field, $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$, $\mathbb{N}$ denotes the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$, and $\mathbb{C}$ denotes the set of complex numbers. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. When one talks of q-extension, q is considered in many ways such as an indeterminate, a complex number $q \in \mathbb{C}$, or p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$ one normally assume that $|q| < 1$. If $q \in \mathbb{C}_p$, we normally assume that $|q-1|_p < p^{-\frac{1}{p-1}}$ so that $q^x = \exp(x \log q)$ for $|x|_p \leq 1$.Throughout this paper we use the notation:

$$[x]_q = \frac{1-q^x}{1-q}, \text{ cf. } [1, 2, 3, 4, 5, 6] .$$

For

$$g \in UD(\mathbb{Z}_p) = \{g | g : \mathbb{Z}_p \to \mathbb{C}_p \text{ is uniformly differentiable function}\},$$

Kim[2] defined the p-adic q-integral on $\mathbb{Z}_p$ as follows:

$$I_{-q}(g) = \int_{\mathbb{Z}_p} g(x) d\mu_{-q}(x) = \lim_{N\to\infty} \frac{1+q}{1+q^{p^N}} \sum_{x=0}^{p^N-1} g(x)(-q)^x. \tag{1.1}$$

From (1.1), we note that

$$q^n I_{-q}(f_n) = (-1)^n I_{-q}(g) + [2]_q \sum_{l=0}^{n-1} (-1)^{n-1-l} q^l g(l), \tag{1.2}$$

where $g_n(x) = g(x+n)$ for $n \in \mathbb{N}$.

We recall that the classical Stirling numbers of the first kind $S_1(n,k)$ and $S_2(n,k)$ are defined by the relations(see [6])

$$(x)_n = \sum_{k=0}^{n} S_1(n,k) x^k \text{ and } x^n = \sum_{k=0}^{n} S_2(n,k)(x)_k,$$

respectively. Here $(x)_n = x(x-1)\cdots(x-n+1)$ denotes the falling factorial polynomial of order n. We also have

$$\sum_{n=m}^{\infty} S_2(n,m) \frac{t^n}{n!} = \frac{(e^t-1)^m}{m!} \text{ and } \sum_{n=m}^{\infty} S_1(n,m) \frac{t^n}{n!} = \frac{(\log(1+t))^m}{m!}. \tag{1.3}$$

The generalized falling factorial $(x|\lambda)_n$ with increment λ is defined by

$$(x|\lambda)_n = \prod_{k=0}^{n-1} (x - \lambda k) \tag{1.4}$$

for positive integer n, with the convention $(x|\lambda)_0 = 1$. We also need the binomial theorem: for a variable x,

$$(1+\lambda t)^{x/\lambda} = \sum_{n=0}^{\infty} (x|\lambda)_n \frac{t^n}{n!}. \tag{1.5}$$

For $q \in \mathbb{C}_p$ with $|1-q|_p \leq 1$, if we take $g(x) = e^{2xt}$ in (1.2), then we easily see that

$$I_{-q}(e^{[2x]_q t}) = \int_{\mathbb{Z}_p} e^{[2x]_q t} d\mu_{-q}(x) = [2]_q \sum_{m=0}^{\infty} (-1)^m q^m e^{[2m]_q t}.$$

Let us define the tangent numbers $T_{n,q}$ and polynomials $T_{n,q}(x)$ as follows:

$$\int_{\mathbb{Z}_p} e^{[2y]_q t} d\mu_{-q}(y) = \sum_{n=0}^{\infty} T_{n,q} \frac{t^n}{n!}, \tag{1.6}$$

$$\int_{\mathbb{Z}_p} e^{[x+2y]_q t} d\mu_{-q}(y) = \sum_{n=0}^{\infty} T_{n,q}(x) \frac{t^n}{n!}. \tag{1.7}$$

Many mathematicians have worked in the area of the q-analogues of the degenerate Bernoulli umbers and polynomials, Euler numbers and polynomials, tangent numbers and polynomials(see [1, 2, 3, 4, 5, 6]). Our aim in this paper is to define degenerate Carlitz's type q-tangent polynomials $\mathcal{T}_{n,q}(x,\lambda)$. We investigate some properties which are related to degenerate Carlitz's type q-tangent numbers $\mathcal{T}_{n,q}(\lambda)$ and polynomials $\mathcal{T}_{n,q}(x,\lambda)$.

2 Degenerate Carlitz's type q-tangent numbers and polynomials

In this section, we introduce the degenerate Carlitz's type twisted q-tangent numbers and polynomials, and we obtain explicit formulas for them. We assume that $t, \lambda \in \mathbb{C}_p$ with $0 < |\lambda|_p \leq 1$ and $|t|_p < p^{-\frac{1}{p-1}}$. Then we have $|\lambda t|_p < p^{-\frac{1}{p-1}}$ and $|\frac{1}{\lambda}\log(1+\lambda t)_p = |t|_p < p^{-\frac{1}{p-1}}$. By (1.3), we have

$$\begin{aligned}(1+\lambda t)^{\frac{[x+2y]_q}{\lambda}} &= e^{\frac{[x+2y]_q}{\lambda}\log(1+\lambda t)} \\ &= \sum_{n=0}^{\infty} \left(\frac{[x+2y]_q}{\lambda}\right)^n \frac{(\log(1+\lambda t))^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} S_1(n,m)\lambda^{n-m}[x+2y]_q^m\right) \frac{t^n}{n!}.\end{aligned} \tag{2.1}$$

Definition 2.1 *The degenerate Carlitz's type q-tangent numbers* $\mathcal{T}_{n,q}(\lambda)$ *and polynomials* $\mathcal{T}_{n,q}(x,\lambda)$ *are defined by means of the generating functions*

$$\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{[2y]_q}{\lambda}}d\mu_{-q}(y)=\sum_{n=0}^{\infty}\mathcal{T}_{n,q}(\lambda)\frac{t^n}{n!}, \tag{2.2}$$

and

$$\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{[x+2y]_q}{\lambda}}d\mu_{-q}(y)=\sum_{n=0}^{\infty}\mathcal{T}_{n,q}(x,\lambda)\frac{t^n}{n!}. \tag{2.3}$$

respectively.

Note that $(1+\lambda t)^{1/\lambda}$ tends to e^t as $\lambda\to 0$. From (2.3) and (1.7), we note that

$$\begin{aligned}\sum_{n=0}^{\infty}\lim_{\lambda\to 0}\mathcal{T}_{n,q}(x,\lambda)\frac{t^n}{n!}&=\int_{\mathbb{Z}_p}e^{[x+2y]_q t}d\mu_{-q}(y)\\&=\sum_{n=0}^{\infty}T_{n,q}(x)\frac{t^n}{n!}.\end{aligned}$$

Thus, for $n\geq 0$, we get

$$\lim_{\lambda\to 0}\mathcal{T}_{n,q}(x,\lambda)=T_{n,q}(x)\text{ and }\lim_{\lambda\to 0}\mathcal{T}_{n,q}(\lambda)=T_{n,q}.$$

For $q\in\mathbb{C}_p$ with $|q-1|_p<1$, we observe that

$$T_{n,q}=[2]_q\left(\frac{1}{1-q}\right)^n\sum_{l=0}^{n}\binom{n}{l}(-1)^l\frac{1}{1+q^{2l+1}}. \tag{2.4}$$

From (1.5), we have

$$\begin{aligned}\sum_{n=0}^{\infty}\mathcal{T}_{n,q}(x,\lambda)\frac{t^n}{n!}&=\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{[x+2y]_q}{\lambda}}d\mu_{-q}(y)\\&=\sum_{n=0}^{\infty}\int_{\mathbb{Z}_p}\binom{\frac{1}{\lambda}[x+2y]_q}{n}\lambda^n t^n d\mu_{-q}(y)\\&=\sum_{n=0}^{\infty}\left(\int_{\mathbb{Z}_p}\left(\frac{[x+2y]_q}{\lambda}\right)_n\lambda^n d\mu_{-q}(y)\right)\frac{t^n}{n!},\end{aligned} \tag{2.5}$$

where

$$\left(\frac{[x+2y]_q}{\lambda}\right)_n=\left(\frac{[x+2y]_q}{\lambda}\right)\left(\frac{[x+2y]_q}{\lambda}-1\right)\cdots\left(\frac{[x+2y]_q}{\lambda}-n+1\right).$$

By comparing of the coefficients $\frac{t^n}{n!}$ on the both sides of (2.5), we have the following theorem.

Theorem 2.2 *For $n \geq 0$, we have*

$$\mathcal{T}_{n,q}(x,\lambda) = \int_{\mathbb{Z}_p} \left(\frac{[x+2y]_q}{\lambda} \right)_n \lambda^n d\mu_{-q}(y).$$

We introduce a q-analogue of the generalized falling factorial $(x|\lambda)_n$ with increment λ. The q-generalized falling factorial $([x]_q|\lambda)_n$ with increment λ is defined by

$$([x]_q|\lambda)_n = \prod_{k=0}^{n-1} ([x]_q - \lambda k) \tag{2.6}$$

for positive integer n, with the convention $([x]_q|\lambda)_0 = 1$. Note that $\lim_{q\to 1}([x]_q|\lambda)_n = (x|\lambda)_n$.

By (2.6) and Theorem 2.2, we obtain the following Witt's formula.

Corollary 2.3 *For $n \in \mathbb{Z}_+$, we have*

$$\mathcal{T}_{n,q}(x,\lambda) = \int_{\mathbb{Z}_p} ([x+2y]_q|\lambda)_n d\mu_{-q}(y),$$
$$\mathcal{T}_{n,q}(\lambda) = \int_{\mathbb{Z}_p} ([2y]_q|\lambda)_n d\mu_{-q}(y).$$

From (1.7), (2.1), we have

$$\begin{aligned} &\int_{\mathbb{Z}_p} \left(\frac{[x+2y]_q}{\lambda} \right)_n \lambda^n d\mu_{-q}(y) \\ &= \sum_{l=0}^{n} S_1(n,l)\lambda^{n-l} \int_{\mathbb{Z}_p} [x+2y]_q^l d\mu_{-q}(y) \\ &= \sum_{l=0}^{n} S_1(n,l)\lambda^{n-l} T_{l,q}(x). \end{aligned} \tag{2.7}$$

By (2.7) and Theorem 2.2, we have the following theorem.

Theorem 2.4 *For $n \in \mathbb{Z}_+$, we have*

$$\mathcal{T}_{n,q}(x,\lambda) = \sum_{l=0}^{n} S_1(n,l)\lambda^{n-l} T_{l,q}(x).$$

By (2.4) and Theorem 2.4, we have the following corollary.

Corollary 2.5 *For $n \in \mathbb{Z}_+$, we have*

$$\mathcal{T}_{n,q}(x,\lambda) = \sum_{l=0}^{n} \sum_{j=0}^{l} \frac{S_1(n,l)\lambda^{n-l}}{(1-q)^l} \binom{l}{j} (-1)^j q^{xj} \frac{1}{1+q^{2j+1}}.$$

By replacing t by $\dfrac{e^{\lambda t}-1}{\lambda}$ in (2.3), we have

$$\begin{aligned}\int_{\mathbb{Z}_p} e^{[x+2y]_q} d\mu_{-q}(y) &= \sum_{n=0}^{\infty} \mathcal{T}_{n,q}(x,\lambda) \left(\frac{e^{\lambda t}-1}{\lambda}\right)^n \frac{1}{n!} \\ &= \sum_{n=0}^{\infty} \mathcal{T}_{n,q}(x,\lambda)\lambda^{-n} \sum_{m=n}^{\infty} S_2(m,n)\lambda^m \frac{t^m}{m!} \\ &= \sum_{m=0}^{\infty} \left(\sum_{n=0}^{m} \mathcal{T}_{n,q}(x,\lambda)\lambda^{m-n} S_2(m,n)\right) \frac{t^m}{m!}.\end{aligned} \tag{2.8}$$

Thus, by (2.8) and (1.7), we have the following theorem.

Theorem 2.6 *For $n \in \mathbb{Z}_+$, we have*

$$T_{m,q}(x) = \sum_{n=0}^{m} \lambda^{m-n} S_2(m,n) \mathcal{T}_{n,q}(x,\lambda).$$

By replacing t by $\log(1+\lambda t)^{1/\lambda}$ in (1.7), we have

$$\begin{aligned}\sum_{n=0}^{\infty} T_{n,q}(x) \left(\log(1+\lambda t)^{1/\lambda}\right)^n \frac{1}{n!} &= \int_{\mathbb{Z}_p} (1+\lambda t)^{\frac{[x+2y]_q}{\lambda}} d\mu_{-q}(y) \\ &= \sum_{m=0}^{\infty} \mathcal{T}_{m,q}(x,\lambda) \frac{t^m}{m!},\end{aligned} \tag{2.9}$$

and

$$\begin{aligned}&\sum_{n=0}^{\infty} T_{n,q}(x) \left(\log(1+\lambda t)^{1/\lambda}\right)^n \frac{1}{n!} \\ &= \sum_{m=0}^{\infty} \left(\sum_{n=0}^{m} T_{n,q}(x)\lambda^{m-n} S_1(m,n)\right) \frac{t^m}{m!}.\end{aligned} \tag{2.10}$$

Thus, by (2.9) and (2.10), we have the following theorem.

Theorem 2.7 *For $n \in \mathbb{Z}_+$, we have*

$$\mathcal{T}_{m,q}(x,\lambda) = \sum_{n=0}^{m} \lambda^{m-n} S_1(m,n) T_{n,q}(x).$$

We observe that

$$\begin{aligned}
&(1+\lambda t)^{\frac{[x+2y]_q}{\lambda}} \\
&= (1+\lambda t)^{\frac{[x]_q}{\lambda}} (1+\lambda t)^{\frac{q^x[2y]_q}{\lambda}} \\
&= \sum_{m=0}^{\infty} ([x]_q|\lambda)_m \frac{t^m}{m!} e^{\log(1+\lambda t)^{\frac{q^x[2y]_q}{\lambda}}} \\
&= \sum_{m=0}^{\infty} ([x]_q|\lambda)_m \frac{t^m}{m!} \sum_{l=0}^{\infty} \left(\frac{q^x[2y]_q}{\lambda}\right)^l \frac{\log(1+\lambda t)^l}{l!} \\
&= \sum_{m=0}^{\infty} ([x]_q|\lambda)_m \frac{t^m}{m!} \sum_{l=0}^{\infty} \left(\frac{q^x[2y]_q}{\lambda}\right)^l \sum_{k=l}^{\infty} S_1(k,l)\lambda^k \frac{t^k}{k!} \\
&= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} ([x]_q|\lambda)_{n-k} \sum_{l=0}^{k} \lambda^{k-l} q^{xl} [2y]_q^l S_1(n,m) \right) \frac{t^n}{n!}.
\end{aligned} \tag{2.11}$$

By (2.2), we have the following theorem.

Theorem 2.8 *For $n \in \mathbb{Z}_+$, we have*

$$\mathcal{T}_{n,q}(x,\lambda) = \sum_{k=0}^{n} \sum_{l=0}^{k} \binom{n}{k} ([x]_q|\lambda)_{n-k} \lambda^{k-l} q^{xl} S_1(n,m) T_{l,q}.$$

By (1.2), we have

$$\begin{aligned}
&q^n \int_{\mathbb{Z}_p} (1+\lambda t)^{\frac{[2x+2n]_q}{\lambda}} d\mu_{-q}(x) + (-1)^{n-1} \int_{\mathbb{Z}_p} (1+\lambda t)^{\frac{[2x]_q}{\lambda}} d\mu_{-q}(x) \\
&= [2]_q \sum_{l=0}^{n-1} (-1)^{n-1-l} q^l (1+\lambda t)^{\frac{[2l]_q}{\lambda}} \\
&= \sum_{m=0}^{\infty} \left([2]_q \sum_{l=0}^{n-1} (-1)^{n-1-l} q^l ([2l]_q|\lambda)_m \right) \frac{t^m}{m!}.
\end{aligned} \tag{2.12}$$

By comparing of the coefficients $\frac{t^n}{n!}$ on the both sides of (2.12), we have the following theorem.

Theorem 2.9 *For $m \in \mathbb{Z}_+$, we have*

$$q^n \mathcal{T}_{m,q}(2n,\lambda) + (-1)^{n-1} \mathcal{T}_{m,q}(\lambda) = [2]_q \sum_{l=0}^{n-1} (-1)^{n-1-l} q^l ([2l]_q|\lambda)_m.$$

For $m \equiv 1 \pmod 2$, we have

$$\begin{aligned}&\int_{\mathbb{Z}_p} [x+2y]_q^l d\mu_{-q}(y)\\ &= \frac{[2]_q}{[2]_{q^m}}[m]_q^l \sum_{a=0}^{m-1}(-1)^a q^a \int_{\mathbb{Z}_p}\left[\frac{2a+x}{m}+2y\right]_{q^m}^l d\mu_{-q^m}(y)\end{aligned} \tag{2.13}$$

By Theorem 2.7 and (2.13), we have the following theorem.

Theorem 2.10 *For any positive integer m(=odd), we have*

$$\mathcal{T}_{n,q}(x,\lambda) = \sum_{l=0}^{n}\sum_{a=0}^{m-1}\lambda^{n-l}S_1(n,l)\frac{[2]_q}{[2]_{q^m}}[m]_q^l(-1)^a q^a T_{l,q^m}\left(\frac{2a+x}{m}\right).$$

Since

$$T_{n,q^{-1}}(2-x) = (-1)^n q^n T_{n,q}(x),$$

we have the following property of complement.

Theorem 2.11 *For any positive integer m, we have*

$$\mathcal{T}_{n,q^{-1}}(2-x,\lambda^{-1}) = (-1)^n q^n \mathcal{T}_{n,q}(x,\lambda^{-1}).$$

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