On an Integrable System and its Relation with the Schwarz KdV Equation:
Exact Traveling Wave Solutions

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We consider a special integrable system, and we show its relation with the Schwarz KdV equation (SKdV). We use the improved tanh-coth method for solving the first system and the Exp. function method for obtain exact solutions of the SKdV equation.

Abstract

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1 Introduction

We will use the improved tanh-coth method [1] [2] to obtain exact traveling solutions of the system

\[
\begin{align*}
  u_t &= u_{xxx} - w_x w_{xx} - \frac{1}{2}(u_x^2 + w_x^2)u_x, \\
  w_t &= w_x u_{xx} - \frac{1}{2}(u_x^2 + w_x^2)w_x,
\end{align*}
\]

and the Exp. function method [3] to obtain exact traveling wave solutions for the Schwarz KdV equation (SKdV) [2]
$u_t = u_{xxx} - \frac{3}{2} \frac{u_x^2}{u_x}$,  \hspace{1cm} (2)

where $u, w$ are unknowns functions depending of the variables $x, y$ and $t$. The previous two equations have relevance today due to Eq.(2) is very related with Eq. (1), and the Eq. (1) is related with the following double Liouville integrable system

$$\begin{cases}
u_{xy} = we^u \\ w_{xy} = e^u.
\end{cases}$$  \hspace{1cm} (3)

Indeed, from (1) and taking into account (3) we have

$$(u_t)_y = we^uV, \quad (w_t)_y = e^uV,$$

where

$$V = u_{xx} - \frac{1}{2}(u_x^2 + w_x^2).$$  \hspace{1cm} (4)

Then the system (1) can be written in the form

$$\begin{cases}
(u_{xy})_t = (e^u)_t = (e^uV)_x \\
(w_{xy})_t = (we^u)_t = (we^uV)_x.
\end{cases}$$  \hspace{1cm} (5)

On the other hand, extending the system (1) to following compatible system

$$\begin{cases}
u_t = u_{xxx} - w_xw_{xx} - \frac{1}{2}(u_x^2 + w_x^2)u_x, \\
w_t = w_xu_{xx} - \frac{1}{2}(u_x^2 + w_x^2)w_x, \\
v_x = e^u, \\
v_t = e^uV,
\end{cases}$$  \hspace{1cm} (6)

we can see that $v_tw_x - v_xw_t = 0$, therefore, we can consider $w$ as function of $v$: $w(v)$, then

$$w'_x = w'(v)v_x.$$  \hspace{1cm} (7)

From third equation of (6) we have $u = \ln v_x$, so that $u_x = \frac{v_x}{v_x}$, and $u_{xx} = -\frac{v_x^2}{v_x^2} + \frac{u_{xxx}}{v_x}$. The system (6) reduces

$$\begin{cases}
u_t = v_{xxx} - \frac{3}{2} \frac{v_x^2}{v_x} - \frac{1}{2} v_x w_x^2, \\
w_t = \frac{v_{xxx}w_x}{v_x} - \frac{3}{2} \frac{v_x^2 w_x}{v_x^2} - \frac{1}{2} w_x^3.
\end{cases}$$  \hspace{1cm} (8)

Now, taking into account (7), this last system reduces to equation
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\[ v_t = v_{xxx} - \frac{3}{2} \frac{v_x^2}{v_x} - \frac{1}{2} (w'(r)^2)v_x^3. \]  

(9)

Finally, choosing an adequate punctual transformation \(v(u)\) we can obtain the Schwarz KdV equation (2).

2 Traveling wave solutions for Eq.(1)

Considering the following transformation

\[
\begin{align*}
    u(x,y,t) &= u(\xi) \\
    w(x,y,t) &= w(\xi) \\
    \xi &= x + y + \lambda t,
\end{align*}
\]

(10)

then, the system (1) can be written in the form

\[
\begin{align*}
    2\lambda u'(\xi) - 2u''(\xi) + 2w'(\xi)w''(\xi) + u'(\xi)^3 + w'(\xi)^2u'(\xi) &= 0, \\
    2\lambda w'(\xi) - 2w'(\xi)u''(\xi) + u'(\xi)w'(\xi) + w'(\xi)^3 &= 0,
\end{align*}
\]

(11)

where ' denote the ordinary derivative with respect to \(\xi\). The idea of the improved tanh-coth method [1] [2] consists on the search solutions for Eq.(1) in the form

\[
\begin{align*}
    u(\xi) &= \sum_{i=0}^{M} a_i \phi(\xi)^i + \sum_{i=M+1}^{2M} a_i \phi(\xi)^{M-i}, \\
    w(\xi) &= \sum_{i=0}^{N} b_i \phi(\xi)^i + \sum_{i=N+1}^{2N} a_i \phi(\xi)^{N-i},
\end{align*}
\]

(12)

where \(M, N\) are a positive integer and \(\phi = \phi(\xi)\) is solution of the Riccati equation

\[
\phi'(\xi) = \alpha + \beta \phi(\xi) + \gamma \phi(\xi)^2.
\]

(13)

The \(a_i, i = 1, 2, \ldots, 2M, b_i, i = 1, 2, \ldots, 2N, \alpha, \beta, \gamma\) are constants to be determined later. For sake of simplicity, we take \(M = N = 2\). So that (12) reduces to

\[
\begin{align*}
    u(\xi) &= a_0(t) + a_1 \phi(\xi) + a_2(\phi(\xi))^2 + a_3(\phi(\xi))^{-1} + a_4(\phi(\xi))^{-2}, \\ u(\xi) &= b_0 + b_1 \phi(\xi) + b_2(\phi(\xi))^2 + b_3(\phi(\xi))^{-1} + b_4(\phi(\xi))^{-2}.
\end{align*}
\]

(14)

Substituting (14) into (11), using (13) and equating to zero the coefficients of all powers of \(\phi(\xi)\), we get a set of algebraic equations for the unknowns \(a_0, a_1, a_2, a_3, a_4, b_0, b_1, b_2, b_3, b_4, \alpha, \beta, \gamma\) and \(\lambda\). We solve the system with aid the Mathematica. We consider only the following no trivial solution:
\[\begin{align*}
\alpha &= \beta = 0, \quad a_1 = a_2 = a_4 = 0, \quad b_1 = b_2 = b_4 = 0, \\
\lambda &= -\frac{1}{2} \gamma^2 (a_3^2 + b_3^2). \quad (15)
\end{align*}\]

For the previous values, the solution of the Riccati equation (13) is given by [4]
\[
\phi(\xi) = -\frac{1}{\gamma \xi}. \quad (16)
\]

Therefore, taking into account (14) and (10), we have the following solutions to system (1)
\[\begin{align*}
u(x, y, t) &= a_0 - a_3 (\gamma \xi), \\
w(x, y, t) &= b_0 - b_3 (\gamma \xi), \quad (17)
\end{align*}\]

where \(a_0, a_3, b_0, b_3\) and \(\gamma\) are arbitrary constants, \(\xi = x + y - \frac{1}{2} \gamma^2 (a_3^2 + b_3^2) t\).

### 3 Traveling wave solutions for Eq.(2)

We use the Exp. function method [3] for obtain a solution to (2). First, using the transformation \(\xi = x + y + \lambda t\), the Eq. (2) take the form
\[
2\lambda u'(\xi)^2 - 2u'(\xi)u''(\xi) + 3u''(\xi)^2 = 0. \quad (18)
\]

We seek a solution for (18) using the expression
\[
u(\xi) = \frac{a_0 + a_1 e^{k \xi} + a_{-1} e^{-k \xi}}{b_0 + b_1 e^{k \xi} + b_{-1} e^{-k \xi}}. \quad (19)
\]

As in the previous section, substituting (19) into (18) and after simplifications, we obtain a system in the variables \(a_0, a_1, a_{-1}, b_0, b_1, b_{-1}, \lambda, k\). Solving it with aid of Mathematica, we have the following solution
\[\begin{align*}
b_0 &= -\sqrt{(a_0^2 - 4a_1 a_{-1})(a_1 b_{-1} - a_{-1} b_1)^2 + a_0 (a_1 b_{-1} + a_{-1} b_1)} - \frac{a_0 (a_1 b_{-1} + a_{-1} b_1)^2 + a_0 (a_1 b_{-1} + a_{-1} b_1)}{2 a_0 a_{-1}} \\
\lambda &= -\frac{k^2}{2}. \quad (20)
\end{align*}\]

Finally, by (19) and (20) one solution to (2) can be written as
\[
u(\xi) = \frac{a_0 + a_1 e^{k \xi} + a_{-1} e^{-k \xi}}{-\sqrt{(a_0^2 - 4a_1 a_{-1})(a_1 b_{-1} - a_{-1} b_1)^2 + a_0 (a_1 b_{-1} + a_{-1} b_1)} + b_1 e^{k \xi} + b_{-1} e^{-k \xi}}, \quad (21)
\]

with \(\xi = x + y - \frac{k^2}{2} t\) and \(a_0, a_1, a_{-1}, b_1, b_{-1}, k\) arbitrary constants.
4 Conclusions

The improved tanh-coth method and the Exp. function method have been used in a satisfactory way for obtain traveling wave solutions to models (1) and (2). In particular, the system (1) is not easy to handled, however, we can see that the technique used to solve it is a very tool to solve many complicate nonlinear equations. We have showed the relation between the two equations solved by the mentioned computational methods and the double integrable Liouville system given by (3).

References


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