Another Sixth-Order Iterative Method Free from Derivative for Solving Multiple Roots of a Nonlinear Equation

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Abstract

A three-step iterative method with sixth-order convergence as another modification of Newton Method was showed. We approximate them with a central difference and divided difference to avoid appearing the derivatives in the method. We show analytically that the method is of order six. We verify the theoretical result on relevant numerical problems and compare the behavior of the proposed method with some existing methods.

Keywords: central difference, free derivative method, iterative method, multiple roots

1 Introduction

Finding the roots of a nonlinear equation of the form

\[ g(x) = 0 \]
is one of foremost problem in numerical analysis. Some solutions of this problem are of the form multiple roots. In several times, there are many methods that can be used to find a multiple root of the nonlinear equation (1). Some of them requires the multiplicity of the root to be known in advance to apply the methods, and others transform the nonlinear equation (1) into a simple root problem. Schröder [6] was suggested a modified Newton’s method by adding multiplicity, \( m \), into Newton’s method for a simple root, he ends up with the following formula

\[
x_{n+1} = x_n - m \frac{g(x_n)}{g'(x_n)}, \quad \text{for given } x_0 \text{ and } m.
\]  

(2)

This method converges quadratically for multiple roots[9, p. 135]. Neta [4] proposed a new method by modifying Chebyshev’s method to have a third order iterative method for \( m \neq 3 \), that is

\[
x_{n+1} = x_n + \frac{m(m - 3)}{2} - \alpha \frac{g(x_n)}{g'(x_n)} \left( 1 + \beta \frac{g(x_n)}{g'(x_n)} \right)
\]

with \( \alpha = -\frac{m(m-3)}{2} \) and \( \beta = -\frac{m}{m-3} \). Shengguo, Xiangke and Lizhi [7] developed a fourth-order method

\[
y_n = x_n - \frac{2m}{m+2} u_n
\]

\[
x_{n+1} = x_n - \frac{1}{2} m(m - 2) \left( \frac{m}{m+2} \right)^{-m} \frac{g'(y_n)}{g'(x_n)} - \frac{m^2 g'(x_n)}{g'(y_n)} u_n
\]

where \( u_n = \frac{g(x_n)}{g'(x_n)} \). Li et al. [3] suggested a fifth-order method, using the transformation [8, p. 5]

\[
f(x) = \begin{cases} 
  \frac{g(x)}{g'(x)} & \text{if } g(x) \neq 0, \\
  0 & \text{if } g(x) = 0,
\end{cases}
\]

(3)

and

\[
y_n = x_n - \frac{f^2(x_n)}{f(x_n) + f(x_n) - f(x_n)}
\]

\[
z_n = y_n - \frac{f(x_n) f(y_n)}{f(x_n) + f(x_n) - f(x_n)}
\]

\[
x_{n+1} = z_n - \frac{f(z_n) B}{A}
\]
where
\[
A = (z_n - x_n)^2 f(x_n)(f(z_n) - f(y_n)) \\
+ (f(x_n)(f(z_n) - f(x_n)) - (z_n - x_n)(f(x_n) - f(x_n))) (z_n - y_n)^2 \\
B = (z_n - y_n)(z_n - x_n)^2 f(x_n).
\]

Qudsi et al. [5] suggested sixth order iterative method given as
\[
y_n = x_n - \frac{2f^2(x_n)}{f(x_n + f(x_n)) - f(x_n - f(x_n))},
\]
\[
z_n = y_n - \frac{2f(x_n)f(y_n)}{f(x_n + f(x_n)) - f(x_n - f(x_n))},
\]
\[
x_{n+1} = z_n - \frac{f(z_n)D}{C},
\]
where
\[
C = 2(z_n - x_n)^2 f(x_n)(f(z_n) - f(y_n)) + \left(2f(x_n)(f(z_n) - f(x_n)) \\
- (z_n - x_n)(f(x_n + f(x_n)) - f(x_n - f(x_n))\right) (z_n - y_n)^2,
\]
\[
D = 2(z_n - y_n)(z_n - x_n)^2 f(x_n).
\]

Khattri [2] also proposed another transformation for finding multiple roots, that is
\[
y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \quad (4)
\]
\[
z_n = x_n - \frac{f(x_n)}{f'(x_n)} \left(1 + \frac{f(y_n)}{f(x_n)} \left(1 + \frac{2f(y_n)}{f(x_n)}\right)\right), \quad (5)
\]
\[
x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \left(1 + \frac{f(y_n)}{f(x_n)} \left(1 + \frac{2f(y_n)}{f(x_n)}\right)\right) \\
+ \frac{f(z_n)}{f(x_n)} \left(1 + \frac{2f(y_n)}{f(x_n)}\right). \quad (6)
\]

This form is sixth order convergent iterative method. The purpose of this paper is to construct a new iterative method free from derivative for finding multiple roots of a nonlinear equation. We present this new method and its convergence analysis in Section 2. In Section 3, we present some numerical examples showing the performance of our proposed method compared to some mention methods.
2 Proposed Method

We consider an iterative method of the form (3) and applying three step iterative method of the form (4), (5) and (6). To avoid computing $f'(x_n)$ in (4), (5) and (6), we approximate it by central difference, as follows

$$f'(x_n) = \frac{f(x_n + f(x_n)) - f(x_n - f(x_n))}{2f(x_n)}.$$  \hspace{1cm} (7)

Replacing approximation of $f'(x_n)$ given by (7), we get a new iterative method

$$y_n = x_n - t,$$  \hspace{1cm} (8)

$$z_n = x_n - t \left( 1 + \frac{f(y_n)}{f(x_n)} \left( 1 + 2 \frac{f(y_n)}{f(x_n)} \right) \right),$$  \hspace{1cm} (9)

$$x_{n+1} = x_n - t \left( 1 + \frac{f(y_n)}{f(x_n)} \left( 1 + 2 \frac{f(y_n)}{f(x_n)} \right) \right) + \frac{f(z_n)}{f(x_n)} \left( 1 + 2 \frac{f(y_n)}{f(x_n)} \right).$$  \hspace{1cm} (10)

where $t = \frac{2f^2(x_n)}{f(x_n+f(x_n))-f(x_n-f(x_n))}$. For the proposed method defined by (3), (8), (9) and (10), we have the following analysis of convergence.

**Theorem 1.** Let $x^* \in D$ be a multiple roots of multiplicity $m$ of a sufficiently differentiable function $g : D \to \mathbb{R}$ on open interval $D$. If $x_0$ is sufficiently close to $x^*$, then iterative method defined by (3), (8), (9) and (10) has a sixth-order convergence.

**Proof.** From the assumption, (1) can be write down as

$$g(x) = (x - x^*)^m h(x),$$  \hspace{1cm} (11)

where $x^*$ is a root of (1) with multiplicity $m$, and $h(x)$ is a continuous function with $h(x^*) \neq 0$. Taking the derivative of $g$ with respect to $x$, we obtain

$$g'(x) = m(x - x^*)^{m-1} h(x) + (x - x^*)^m h'(x).$$  \hspace{1cm} (12)

Substituting (11) and (12) into (3), we get

$$f(x) = \frac{g(x)}{g'(x)} = \frac{(x - x^*)h(x)}{mh(x) + (x - x^*)h'(x)}.$$  \hspace{1cm} (13)
Here we have transformed a multiple root problem into a simple root problem. Let $e_n = x_n - x^*$. Using Taylor’s expansion about $x_n = x^*$, we have respectively

$$h(x_n) = h(x^*)(1 + b_1 e_n + b_2 e_n^2 + b_3 e_n^3 + b_4 e_n^4 + b_5 e_n^5 + b_6 e_n^6 + O(e_n^7)), \quad (14)$$

and

$$h'(x_n) = h(x^*)(b_1 + 2b_2 e_n + 3b_3 e_n^2 + 4b_4 e_n^3 + 5b_5 e_n^4 + 6b_6 e_n^5 + 7b_7 e_n^6 + O(e_n^7)), \quad (15)$$

where $b_k = \frac{h^{(k)}(x^*)}{k!h'(x^*)}$. Substituting (14) and (15) to (13), and using geometric series, we obtain

$$f(x_n) = c_1 e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + c_5 e_n^5 + c_6 e_n^6 + O(e_n^7) \quad (16)$$

where

$$c_1 = \frac{1}{m}, \quad c_2 = -\frac{b_1}{m^2}, \quad c_3 = \frac{1}{m^3}(b_1^2 - 2b_2m + b_1^2m),$$

$$c_4 = -\frac{1}{m^4}(b_1^3m^2 - 3b_1b_2m^2 + 2b_3m - 4b_1b_2m + 3b_3m^2 + b_1^3m^4),$$

$$c_5 = \frac{1}{m^5}(-10m^2b_1^2b_2 + 3mb_4^1 + 6m^2b_1b_3 + 4b_1b_3m^3 + 2b_2m^3 - 6mb_1^2b_2 + b_4 + 4m^2b_2^2 - 4m^3b_2^2b_2 - 3b_4m^3 + 3m^2b_4^1 + m^3b_4^1),$$

$$c_6 = \frac{1}{m^6}(-16m^3b_1b_2^2 - 8b_4b_1m^3 - 8mb_1^4b_2 - 5m^4b_1^2b_2 + 12m^2b_1b_2^2 - 21m^2b_3^1b_2 - 5b_2b_3m^4 + 4mb_5^1 + 5b_2b_1m^4 + 6m^2b_1^5 + 5b_5m^4 + 4m^3b_5^1 - 12b_2b_3m^3 + m^4b_5^1 - 18m^3b_3^3b_2 - 5b_4b_1m^4 + b_5 + 14m^3b_3^2b_3 + 9m^2b_3^2b_3 + 5b_1b_3m^4),$$

$$c_7 = \frac{1}{m^7}(-36m^3b_1b_2b_3 - 12m^2b_1^2b_4 + 12m^2b_1b_3^2 + 24m^2b_1^2b_2 + 54m^2b_1^2b_2 + 39m^4b_1b_2^2 - 28m^4b_1^2b_2 - 6m^5b_1^2b_2 - 48m^3b_1^2b_2 - 6m^5b_1^2b_2 - 10mb_1^4b_2 + b_6 + m^5b_6 + 5m^4b_6 + 10m^3b_6 + 10mb_6 + 12b_1b_2b_3m^5 - 44b_1b_2b_3m^4 - 6b_4b_3^5 - 2b_3^2m^4 + 6b_1b_2m^5 - 2b_3^2m^5 + 9b_1^2m^5 + 9b_1^2m^4 - 6b_6m^5 - 8b_2^2m^3).$$
Let $w_n = x_n + f(x_n)$ and $v_n = x_n - f(x_n)$. On computing $f(w_n)$ and $f(v_n)$ using Taylor series, we end up respectively with

$$f(w_n) = \frac{1}{m^2} (1 + m) e_n + \left( - \frac{1}{m^4} \right) \left( b_1(1 + m^2 + 3m) \right) e_n^2 + \left( \frac{1}{m^6} \right) \left( b_1^2 + (-2b_2 + 6b_1^2)m + (-6b_2 + 9b_1^2)m^2 + (5b_2^2 - 8b_2)m^3 + (-2b_2 + b_1^2)m^4 \right) e_n^3$$

$$+ \left( \frac{1}{m^8} \right) \left( b_1^3 + (9b_1^3 - 4b_1b_2)m + (27b_1^3 + 3b_3 - 25b_1b_2)m^2 + (-52b_1b_2 + 34b_1^3 + 12b_3)m^3 + (18b_3 - 48b_1b_2 + 22b_1^3)m^4 + (-19b_1b_2 + 15b_3 + 7b_1^3)m^5 + (b_1^3 - 3b_1b_2 + 3b_3)m^6 \right) e_n^4 + \cdots + O(e_n^7),$$

(17)

and

$$f(v_n) = \frac{1}{m^2} (m - 1) e_n + \left( - \frac{1}{m^4} \right) \left( b_1(-3m + m^2 + 1) \right) e_n^2 + \left( \frac{1}{m^6} \right) \left( -b_1^2 + (4b_1^2 + 2b_2)m + (-6b_2 - 3b_1^2)m^2 + (-3b_1^2 + 8b_2)m^3 + (-2b_2 + b_1^2)m^4 \right) e_n^3$$

$$+ \left( - \frac{1}{m^8} \right) \left( b_1^3 + (-5b_1^3 - 4b_1b_2)m + (19b_1b_2 + 3b_3 + 5b_1^3)m^2 + (-12b_3 - 28b_1b_2 + 6b_1^3)m^3 + (12b_1b_2 - 8b_1^3 + 18b_3)m^4 + (-3b_1^3 - 15b_3 + 11b_1b_2)m^5 + (b_1^3 - 3b_1b_2 + 3b_3)m^6 \right) e_n^4 + \cdots + O(e_n^7).$$

(18)

Substituting (17), (18) and (16) into $y_n$ in (8) and simplifying, we obtain

$$y_n = x^* + \left( - \frac{b_1}{m} \right) e_n^2 + \left( \frac{1}{m^4} \right) \left( b_1^2 + m(b_1^2 - 2b_2) + m^3(2b_1^2 - 4b_2) \right) e_n^3$$

$$+ \left( - \frac{1}{m^6} \right) \left( 3b_3^2 + m(7b_3^2 - 14b_1b_2) + m^2(4b_3^2 - 12b_1b_2 + 12b_3) + 9m^4b_3 + m^3(2b_1b_2 - b_1^3) + m^4(3b_3^2 - 9b_1b_2) \right) e_n^4 + \cdots + O(e_n^7).$$

(19)

Expanding $f(y_n)$ about $y_n = x^*$ to get

$$f(y_n) = \left( - \frac{b_1}{m^2} \right) e_n^2 + \left( \frac{1}{m^5} \right) \left( 2m^3b_1^2 - 4m^3b_2 + b_1^2 - 2mb_2 + mb_1^2 \right) e_n^3$$

$$+ \left( - \frac{1}{m^6} \right) \left( 3b_3^2 + (7b_3^2 - 14b_1b_2)m + (-12b_1b_2 + 5b_1^3 + 12b_3)m^2 + (2b_1b_2 - b_1^3)m^3 + (9b_3 + 3b_3^2 - 9b_1b_2)m^4 \right) e_n^4 + \cdots + O(e_n^7).$$

(20)
and dividing by \( f(x_n) \), we get

\[
\frac{f(y_n)}{f(x_n)} = \left( -\frac{b_1}{m} \right) e_n + \left( \frac{1}{m^4} (b_1^2 + (b_1^2 - 2b_2)m - b_1^2 m^2 + (2b_2^2 - 4b_2)m^3) \right) e_n^2 \\
+ \cdots + O(e_n^8).
\] (21)

Substituting (16)–(21) into \( z_n \) in (9), we get after simplifying,

\[
z_n = \left( \frac{1}{m^5} (b_1(b_1^2 + (b_1^2 - 2b_2)m - 4b_1^2m^2) - (2b_2 + b_1^2)m^3) \right) e_n^4 \\
+ \left( -\frac{1}{m^8} (b_1^4 + (2b_1^4 - 2b_2)m - (7b_1^4 + 4b_2^2 - 4b_1^2b_2)m^2 \\
- (b_1^4 + 2b_2^2b_2)m^3 + (12b_2^2 + 19b_1^4 - 24b_1^2b_2 + 12b_1b_3)m^4 \\
+ (50b_1^2b_2 - 25b_1^4)m^5 + (6b_1b_3 - 14b_1^2b_2 + 8b_2^2 + 4b_1^4)m^6) \right) e_n^4 \\
+ \cdots + O(e_n^8).
\]

Using tranformation (3) we get

\[
f(z_n) = \left( \frac{1}{m^6} (b_1(b_1^2 + (b_1^2 - 2b_2)m - 4b_1^2m^2) - (2b_2 + b_1^2)m^3) \right) e_n^4 \\
+ \cdots + O(e_n^8).
\] (22)

Based on (10), (22) divided by (16) to get

\[
\frac{f(z_n)}{f(x_n)} = \left( \frac{1}{m^5} (b_1(b_1^2 + (b_1^2 - 2b_2)m - 4b_1^2m^2) - (2b_2 + b_1^2)m^3) \right) e_n^3 \\
+ \left( -\frac{1}{m^8} (b_1^4 + (2b_1^4 - 2b_2)m - (7b_1^4 + 4b_2^2 - 4b_1^2b_2)m^2 \\
- (b_1^4 + 2b_2^2b_2)m^3 + (12b_2^2 + 19b_1^4 - 24b_1^2b_2 + 12b_1b_3)m^4 \\
+ (50b_1^2b_2 - 25b_1^4)m^5 + (6b_1b_3 - 14b_1^2b_2 + 8b_2^2 + 4b_1^4)m^6) \right) e_n^4 \\
+ \cdots + O(e_n^8).
\] (23)

Substituting (16)–(23) into (10), noting \( e_n = x_n - x^* \), we get

\[
e_{n+1} = \frac{1}{m^9} (-b_1(b_1^4 - 4b_1^2b_2 + 2b_1^4)m + 4b_2^2 - 4b_1^2b_2 - 8b_1^4)m^2 + (14b_1^2b_2 \\
- 7b_1^4)m^3 - (8b_1^2b_2 + 22b_1^4 + 8b_2^2)m^4 - (9b_1^4 + 18b_1^2b_2)m^5 + (b_1^4 \\
+ 4b_2^2 - 4b_1^2b_2)m^6) e_n^6 + \cdots + O(e_n^8)
\]

From the definition of the order of convergence [1, p. 56], this method has a sixth-order convergence, then the theorem proved \( \square \).
3 Numerical Experiments

In this section, we give some numerical results to demonstrate the efficiency of the proposed method. We compare this method (MITS) with Li’s method (MLi) defined by [3], Khattri’s method (MK) defined by [2] and Qudsi’s method (MQud) defined by [5]. All computation were done using MAPLE. We use the following functions which also have been consider in [3]

1. \( f_1(x) = (8xe^{-x^2} - 2x - 3)^8 \) with \( m = 8 \), and \( x^* = -1.7903531791589544 \).
2. \( f_2(x) = (e^{-x^2+x+3} - x + 2)^9 \) with \( m = 9 \), and \( x^* = 2.4905398276083051 \).
3. \( f_3(x) = \frac{(2x\cos(x)+x^2-3)^{10}}{x^4+1} \) with \( m = 10 \), and \( x^* = 2.9806452794385368 \).
4. \( f_4(x) = (e^{-x} + 2\sin(x))^4 \) with \( m = 4 \), and \( x^* = 3.1627488709263654 \).
5. \( f_5(x) = ((x - 3)e^x)^5 \) with \( m = 5 \), and \( x^* = 3.00000000000000000 \).

Table 1: Comparison the number of iteration and error of the discussed iterative methods

<table>
<thead>
<tr>
<th>( g(x) )</th>
<th>( x_0 )</th>
<th>( n )</th>
<th>( \text{error} )</th>
<th>( COC )</th>
<th>( n )</th>
<th>( \text{error} )</th>
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</table>

We set a tolerance, \( Toll = 1.0 \times 10^{-69} \), and we use \( |x_{n+1} - x_n| < Toll \) and \( |f(x_{n+1})| < Toll \) as stopping criteria.

Table 1 shows the number of iteration, error of each method and the Computational Order of Convergence (COC). In Tabel 1, we denote \( n \) as the number of maximum iteration, \( \text{err} \) as computational error of the iterative methods, \( NC \) as the method does not convergence and \( Fa \) as the method is fail to obtain the root. Based on numerical computation in Table 1,
Another sixth-order iterative method for solving multiple roots

References


Received: July 11, 2017; Published: August 23, 2017