Complex Fuzzy Subgroups

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Abstract

A complex fuzzy subgroup is a fuzzy subgroup whose membership function takes values in the unit circle in the complex plane. In this paper we defined the complex fuzzy subgroup and investigate some of its characteristics.

Keywords: $\pi$–fuzzy set, $\pi$–fuzzy group, homogeneous complex fuzzy set, complex fuzzy subgroup, normal complex fuzzy subgroup

1 Introduction

In 1965, Zadeh [8] introduced the concept of fuzzy sets. A fuzzy set $A$ in $U$ is defined by a membership function $\mu_A : U \rightarrow [0,1]$ where $U$ is nonempty set, called universe. For $x \in U$, the fuzzy value $\mu_A(x)$ determines the degree to which $x$ belongs to the fuzzy set $A$. Since that time, the theory of fuzzy set has evolved significantly in many directions and is finding applications in a wide variety of fields. In 1971, Rosenfeld [5] introduced the concept of fuzzy subgroups. A lot of basic properties in group theory are found to be carried over to fuzzy groups. In 1981, Wu [7] studied the fuzzy normal subgroups. Recently fuzzy subgroups have been studied for example by Anthony and Sherwood in [1, 2] and by Sivaramakrishna Das in [6].

In 2002 Ramot et al. [4] introduced the concept of the complex fuzzy sets, where the real-valued membership function $\mu_A$ with the range of $[0,1]$ is replaced by a complex-valued function of the form $r_A(x)e^{i\omega_A(x)}$, where $i = \sqrt{-1},$
\( r_A(x) \), and \( \omega_A(x) \) are both real-valued. Ramot et al.\cite{3}, also introduced different fuzzy complex operations and relations, like union, intersection and complement.

In this paper we defined the complex fuzzy subgroups and introduced some new concepts like \( \pi \)-fuzzy sets, \( \pi \)-fuzzy subgroups, homogeneous complex fuzzy sets. Then we investigated some of characteristics of complex fuzzy subgroups. Finally we defined the image and inverse image of complex fuzzy subgroups under group homomorphism and we studied their elementary properties.

2 Preliminaries

**Definition 2.1.**\cite{5} Let \( G \) be a group and \( A \) be a fuzzy set. Then \( A \) is said to be a fuzzy subgroup if the following hold

1. \( \mu(xy) \geq \min\{\mu(x), \mu(y)\} \) for all \( x, y \) in \( G \).
2. \( \mu(x^{-1}) \geq \mu(x) \) for all \( x \) in \( G \).

**Proposition 2.2.**\cite{5} Let \( A \) be a fuzzy subgroup. Then \( \mu(x^{-1}) = \mu(x) \).

**Definition 2.3.**\cite{4} A complex fuzzy set, defined on a universe of discourse \( U \), is characterized by a membership function \( \mu_A(x) \) that assigns any element a complex-valued grade of membership in \( A \). By definition, the values \( \mu_A(x) \) may receive all lie within the unit circle in the complex plane, and are thus of the form \( r_A(x)e^{i\omega_A(x)} \), where \( i = \sqrt{-1} \), \( r_A(x) \) and \( \omega_A(x) \) are both real-valued, and \( r_A(x) \in [0, 1], \omega_A(x) \in [0, 2\pi] \). The complex fuzzy set may be represented as the set of ordered pairs

\[
A = \{(x, \mu_A(x)) : x \in U \}.
\]

**Definition 2.4.**\cite{3} Let \( A = \{(x, \mu_A(x)) : x \in U \} \), and \( B = \{(x, \mu_B(x)) : x \in U \} \) be two complex fuzzy sets of the same universe \( U \) with membership functions \( \mu_A(x) = r_A(x)e^{i\omega_A(x)} \) and \( \mu_B(x) = r_B(x)e^{i\omega_B(x)} \) respectively. Then

1. \( \mu_{A \cap B}(x) = r_{A \cap B}(x)e^{i\omega_{A \cap B}(x)} = \min\{r_A(x), r_B(x)\} e^{i\min\{\omega_A(x), \omega_B(x)\}} \).
2. \( \mu_{A^c}(x) = (1-r_A(x))e^{i(2\pi-\omega_A(x))} \), where \( A^c \) denotes the complement of \( A \).

**Definition 2.5.**\cite{7} A fuzzy subgroup \( A = \{(x, \mu(x)) : x \in G \} \) is said to be a normal fuzzy subgroup of \( G \) if and only if \( \mu_A(xy) = \mu_A(yx) \).
# 3 Complex Fuzzy Subgroups

**Definition 3.1.** Let $A = \{(x, \mu_A(x)) : x \in U\}$ be a fuzzy set. Then the set $A_\pi = \{(x, \gamma_{A_\pi}(x)) : \gamma_{A_\pi}(x) = 2\pi \mu_A(x), x \in U\}$ is said to be a $\pi$–fuzzy set.

**Definition 3.2.** Let $G$ be a group and $A_\pi$ be a $\pi$–fuzzy set. Then $A_\pi$ is said to be a $\pi$–fuzzy subgroup if the following hold:

1. $\gamma(xy) \geq \min\{\gamma(x), \gamma(y)\}$ for all $x, y$ in $G$.
2. $\gamma(x^{-1}) \geq \gamma(x)$ for all $x$ in $G$.

**Proposition 3.3.** A $\pi$–fuzzy set $A_\pi$ is a $\pi$–fuzzy subgroup if and only if $A$ is a fuzzy subgroup.

**Proof.** Clear.

**Definition 3.4.** Let $A = \{(x, \mu_A(x)) : x \in G\}$ and $B = \{(x, \mu_B(x)) : x \in G\}$ be complex fuzzy subsets of $G$, with membership functions $\mu_A(x) = r_A(x)e^{i\omega_A(x)}$ and $\mu_B(x) = r_B(x)e^{i\omega_B(x)}$, respectively. Then

1. A complex fuzzy subsets $A$ is said to be a homogeneous complex fuzzy set if for all $x, y \in G$

   $$r_A(x) \leq r_A(y) \text{ if and only if } \omega_A(x) \leq \omega_A(y).$$

2. A complex fuzzy subsets $A$ is said to be homogeneous with $B$, if for all $x, y \in G$

   $$r_A(x) \leq r_B(y) \text{ if and only if } \omega_A(x) \leq \omega_B(y) \quad \forall x, y \in G.$$

Throughout this paper every complex fuzzy set is a homogeneous complex fuzzy set.

**Proposition 3.5.** A complex fuzzy set $A$ is a homogeneous complex fuzzy set if and only if $A^c$ is a homogeneous complex fuzzy set.

**Proof.** Clear.

**Definition 3.6.** Let $G$ be a group and $A = \{(x, \mu_A(x)) : x \in G\}$ be a homogeneous complex fuzzy set. Then $A$ is said to be a complex fuzzy subgroup of $G$ if the following hold:

1. $\mu_A(xy) \geq \min\{\mu_A(x), \mu_A(y)\}$ for all $x, y$ in $G$.
2. $\mu_A(x^{-1}) \geq \mu_A(x)$ for all $x$ in $G$. 
**Theorem 3.7.** Let $G$ be a group and $A = \{(x, \mu_A(x)) : x \in G\}$ be a homogeneous complex fuzzy set with membership function $\mu_A(x) = r_A(x)e^{i\omega_A(x)}$. Then $A$ is a complex fuzzy subgroup of $G$ if and only if:

1. The set $\overline{A} = \{(x, r_A(x)) : x \in G, r_A(x) \in [0, 1]\}$ is a fuzzy subgroup.
2. The set $\underline{A} = \{(x, \omega_A(x)) : x \in G, \omega_A(x) \in [0, 2\pi]\}$ is a $\pi$–fuzzy subgroup.

**Proof.** Let $A$ be a complex fuzzy subgroup and $x, y \in G$. Then we have

\[
 r_A(xy)e^{i\omega_A(xy)} = \mu_A(xy) \\
 \geq \min \{\mu_A(x), \mu_A(y)\} \\
 = \min \{r_A(x)e^{i\omega_A(x)}, r_A(y)e^{i\omega_A(y)}\} \\
 = \min \{r_A(x), r_A(y)\} e^{i\min\{\omega_A(x), \omega_A(y)\}}
\]

(since $A$ is homogeneous).

So $r_A(xy) \geq \min \{r_A(x), r_A(y)\}$ and $\omega_A(xy) \geq \min \{\omega_A(x), \omega_A(y)\}$. On the other hand

\[
 r_A(x^{-1})e^{i\omega_A(x^{-1})} = \mu_A(x^{-1}) \\
 \geq \mu_A(x) \\
 = r_A(x)e^{i\omega_A(x)}
\]

which implies

$r_A(x^{-1}) \geq r_A(x)$ and $\omega_A(x^{-1}) \geq \omega_A(x)$

So $\overline{A}$ is a fuzzy subgroup and $A$ is a $\pi$–fuzzy subgroup.

Conversely, let $\overline{A}$ be a fuzzy subgroup and $\underline{A}$ be a $\pi$–fuzzy subgroup. So we have

\[
 r_A(xy) \geq \min \{r_A(x), r_A(y)\} \\
 \omega_A(xy) \geq \min \{\omega_A(x), \omega_A(y)\} \\
 r_A(x^{-1}) \geq r_A(x) \\
 \omega_A(x^{-1}) \geq \omega_A(x).
\]

Now

\[
 \mu_A(xy) = r_A(xy)e^{i\omega_A(xy)} \geq \min \{r_A(x), r_A(y)\} e^{i\min\{\omega_A(x), \omega_A(y)\}} \\
 = \min \{r_A(x)e^{i\omega_A(x)}, r_A(y)e^{i\omega_A(y)}\} (\text{homogeneity}) \\
 = \min \{\mu_A(x), \mu_A(y)\}.
\]
On other hand
\[
\mu_A(x^{-1}) = r_A(x^{-1})e^{i\omega_A(x^{-1})} \\
\geq r_A(x)e^{i\omega_A(x)} \\
= \mu_A(x).
\]

So \( A \) is a complex fuzzy subgroup.

**Theorem 3.8.** Let \( \{A_i\} \) be a collection of complex fuzzy subgroups of a group \( G \) such that \( A_j \) is homogeneous with \( A_k \) for all \( j, k \in I \). Then \( \bigcap_{i \in I} A_i \) is a complex fuzzy subgroup.

**Proof.** For all \( i \in I \) we have \( r_{A_i}(x) \) is a fuzzy subgroup and \( \omega_{A_i}(x) \) is a \( \pi \)-fuzzy subgroup (Theorem 3.7). Now, let \( x, y \in G \). Then
\[
\mu_{\bigcap_{i \in I} A_i}(xy) = r_{\bigcap_{i \in I} A_i}(xy)e^{i\omega_{\bigcap_{i \in I} A_i}(xy)} \\
= \min_{i \in I} \left\{ r_{A_i}(xy) \right\} e^{i \min_{i \in I} \left\{ \omega_{A_i}(xy) \right\}} \\
\geq \min_{i \in I} \left\{ \min \left\{ r_{A_i}(x), r_{A_i}(y) \right\} \right\} e^{i \min_{i \in I} \left\{ \min \left\{ \omega_{A_i}(x), \omega_{A_i}(y) \right\} \right\}} \\
= \min \left\{ \min_{i \in I} \left\{ r_{A_i}(x) \right\}, \min_{i \in I} \left\{ r_{A_i}(y) \right\} \right\} \\
= \min \left\{ \min_{i \in I} \left\{ r_{A_i}(x) \right\} e^{i \min_{i \in I} \left\{ \omega_{A_i}(x) \right\}} \right\} \\
= \min \left\{ \mu_{\bigcap_{i \in I} A_i}(x), \mu_{\bigcap_{i \in I} A_i}(y) \right\}.
\]

On other hand
\[
\mu_{\bigcap_{i \in I} A_i}(x^{-1}) = r_{\bigcap_{i \in I} A_i}(x^{-1})e^{i\omega_{\bigcap_{i \in I} A_i}(x^{-1})} \\
= \min_{i \in I} \left\{ r_{A_i}(x^{-1}) \right\} e^{i \min_{i \in I} \left\{ \omega_{A_i}(x^{-1}) \right\}} \\
= \min_{i \in I} \left\{ r_{A_i}(x) \right\} e^{i \min_{i \in I} \left\{ \omega_{A_i}(x) \right\}} \\
= \mu_{\bigcap_{i \in I} A_i}(x).
\]

**Theorem 3.9.** A complex fuzzy set \( A \) is a complex fuzzy subgroup of a group \( G \) if and only if \( A^c \) is a complex fuzzy subgroup of a group \( G \).

**Proof.** Let \( A \) be a complex fuzzy group. Then:
\[ \mu_{A^c}(xy) = (1 - r_A(xy)) e^{i(2\pi - \omega_A(xy))} \]
\[ \geq (1 - \min \{r_A(x), r_A(y)\}) e^{i(2\pi - \min\{\omega_A(x), \omega_A(y)\})} \]
\[ = \max \{ (1 - r_A(x)), (1 - r_A(y)) \} e^{i\max\{\omega_A(x), \omega_A(y)\}}, \omega_A(y)\} \]
\[ \geq \min \{ (1 - r_A(x)), (1 - r_A(y)) \} e^{i\min\{\omega_A(x), \omega_A(y)\}} \]
\[ = \min \{ (1 - r_A(x)) e^{i(2\pi - \omega_A(x))}, (1 - r_A(y)) e^{i(2\pi - \omega_A(y))} \} \]
\[ = \min \{ \mu_{A^c}(x), \mu_{A^c}(y) \}. \]

On the other hand
\[ \mu_{A^c}(x^{-1}) = (1 - r_A(x^{-1})) e^{i(2\pi - \omega_A(x^{-1}))} \]
\[ = (1 - r_A(x)) e^{i(2\pi - \omega_A(x))} \quad \text{(Proposition 2.2)} \]
\[ = \mu_{A^c}(x) \]

Conversely, let \( A^c \) be a complex fuzzy subgroup. Then
\[ \mu_A(xy) = r_A(xy) e^{i\omega_A(xy)} \]
\[ = 1 - (1 - r_A(xy)) e^{i(2\pi - (2\pi - \omega_A(xy)))} \]
\[ \geq (1 - \min \{ (1 - r_A(x)), (1 - r_A(y)) \} e^{i(2\pi - \min\{\omega_A(x), \omega_A(y)\})} \]
\[ = \max \{ r_A(x), r_A(y) \} e^{i\max\{\omega_A(x), \omega_A(y)\}} \]
\[ \geq \min \{ r_A(x), r_A(y) \} e^{i\min\{\omega_A(x), \omega_A(y)\}} \]
\[ = \min \{ r_A(x) e^{i\omega_A(x)}, r_A(y) e^{i\omega_A(y)} \} \]
\[ = \min \{ \mu_A(x), \mu_A(y) \}. \]

On the other hand
\[ \mu_A(x^{-1}) = r_A(x^{-1}) e^{i\omega_A(x^{-1})} \]
\[ = 1 - (1 - r_A(x^{-1})) e^{i(2\pi - (2\pi - \omega_A(x^{-1})))} \]
\[ = 1 - (1 - r_A(x)) e^{i(2\pi - (2\pi - \omega_A(x)))} \quad \text{(Proposition 2.2)} \]
\[ = r_A(x) e^{i\omega_A(x)} \]
\[ = \mu_A(x). \]

**Definition 3.10.** A complex fuzzy subgroup \( A = \{(x, \mu(x)) : x \in G\} \) is said to be a normal complex fuzzy subgroup of \( G \) if and only if \( \mu_A(xy) = \mu_A(yx) \).

**Theorem 3.11.** Let \( G \) be a group and \( A = \{(x, \mu_A(x)) : x \in G\} \) be a homogeneous complex fuzzy set with membership function \( \mu_A(x) = r_A(x) e^{i\omega_A(x)} \). Then \( A \) is a normal complex fuzzy subgroup of \( G \) if and only if:
1. The set $\overline{A} = \{(x, r_A(x)) : x \in G, r_A(x) \in [0, 1]\}$ is a normal fuzzy subgroup.

2. The set $\underline{A} = \{(x, \omega_A(x)) : x \in G, \omega_A(x) \in [0, 2\pi]\}$ is a normal $\pi$–fuzzy subgroup.

**Proof.** In (Theorem 3.7) we proved that $A$ is a complex fuzzy subgroup if and only if $\overline{A}$ is a fuzzy subgroup and $\underline{A}$ is a $\pi$–fuzzy subgroup. Now we need to prove the normality.

Let $A$ be a normal complex fuzzy subgroup and $x, y \in G$, so $\mu(xy) = \mu(yx)$. Then $r_A(xy)e^{i\omega_A(xy)} = r_A(yx)e^{i\omega_A(yx)}$, which implies that $r_A(xy) = r_A(yx)$ and $\omega_A(xy) = \omega_A(yx)$, thus $\overline{A}$ is fuzzy subgroup and $\underline{A}$ is $\pi$–fuzzy subgroup. Conversely, let $x, y \in G$, $\overline{A}$ be a fuzzy subgroup and $\underline{A}$ be a $\pi$–fuzzy subgroup. Then $r_A(xy) = r_A(yx)$ and $\omega_A(xy) = \omega_A(yx)$, so $r_A(xy)e^{i\omega_A(xy)} = r_A(yx)e^{i\omega_A(yx)}$. This implies that $\mu_A(xy) = \mu_A(yx)$.

**Theorem 3.12.** A complex fuzzy subgroup $A$ is a normal complex fuzzy subgroup if and only if $A^c$ is a normal complex fuzzy subgroup.

**Proof.** Clear.

**Theorem 3.13.** Let $A$ and $B$ are two normal complex fuzzy subgroups. Then $A \cap B$ is a normal complex fuzzy subgroup.

**Proof.** Clear.

**Definition 3.14.** Let $A = \{ (x, \mu_A(x)) : \mu_A(x) = r_A(x)e^{i\omega_A(x)}, x \in X \}$ be a complex fuzzy subset of $X$. For $\alpha \in [0, 1]$ and $\beta \in [0, 2\pi]$, the set $A_{(\alpha, \beta)} = \{ x \in X : r_A(x) \geq \alpha, \omega_A(x) \geq \beta \}$ is called a level subset of the complex fuzzy subset $A$. In particular if $\beta = 0$. Then we get the level subset $A_\alpha = \{ x \in X : r_A(x) \geq \alpha \}$ and if $\alpha = 0$. Then we get the level subset $A_\beta = \{ x \in X : \omega_A(x) \geq \beta \}$.

**Theorem 3.15.** Let $A = \{ (x, \mu_A(x)) : \mu_A(x) = r_A(x)e^{i\omega_A(x)}, x \in G \}$ be a complex fuzzy subgroup of a group $G$, if $r_A(e) \geq \alpha$ and $\omega_A(e) \geq \beta$. Then the level subset $A_{(\alpha, \beta)}$ is a subgroup of $G$.

**Proof.** $e \in A_{(\alpha, \beta)}$, so $A_{(\alpha, \beta)} \neq \phi$. Let $x, y \in A_{(\alpha, \beta)}$. Then we have $r_A(x) \geq \alpha$ and $\omega_A(x) \geq \beta$, also $r_A(y) \geq \alpha$ and $\omega_A(y) \geq \beta$.

Now,

$$
\mu_A(xy) = r_A(xy)e^{i\omega_A(xy)} \geq \min \{r_A(x)e^{i\omega_A(x)}, r_A(y)e^{i\omega_A(y)}\} = \min \{r_A(x), r_A(y)\} e^{i\min\{\omega_A(x), \omega_A(y)\}}
$$
This implies
\[ r_A(xy) \geq \min \{ r_A(x), r_A(y) \} \geq \min \{ \alpha, \alpha \} = \alpha. \]

And
\[ \omega_A(xy) \geq \min \{ \omega_A(x), \omega_A(y) \} \geq \min \{ \beta, \beta \} = \beta. \]

So \( xy \in A(\alpha, \beta). \)

On the other hand we have
\[ \mu_A(x^{-1}) = r_A(x^{-1})e^{i\omega_A(x^{-1})} \geq r_A(x)e^{i\omega_A(x)}. \]
This implies \( r_A(x^{-1}) \geq r_A(x) \geq \alpha \) and \( \omega_A(x^{-1}) \geq \omega_A(x) \geq \beta. \) So \( x^{-1} \in A(\alpha, \beta). \)

**Corollary 3.16.** Let \( A = \{ (x, \mu_A(x)) : \mu_A(x) = r_A(x)e^{i\omega_A(x)}, x \in G \} \) be a complex fuzzy subgroup of a group \( G, \) if \( r_A(e) \geq \alpha \) and \( \omega_A(e) \geq \beta. \) Then the level subsets \( A_\alpha = \{ x \in G : r_A(x) \geq \alpha \} \) and \( A_\beta = \{ x \in G : \omega_A(x) \geq \beta \} \) are two subgroups of \( G. \)
**Proof.** Clear.

## 4 Homomorphism

Here we define the image and inverse image of complex fuzzy subgroups under group homomorphism and study their elementary properties.

**Definition 4.1.**[5] Let \( f : G \rightarrow H \) be a group homomorphism.

Let \( A = \{ (x, \mu_A(x)) : x \in G \} \) and \( B = \{ (x, \mu_B(x)) : x \in H \} \) be fuzzy subgroups.

Then \( C = \{ (y, f(\mu_A)(y)) : y \in H \} \) is called image of \( A, \) where
\[
f(\mu_A)(y) = \begin{cases} \vee \{ \mu_A(x) | x \in G, f(x) = y \} & \text{if } f^{-1}(y) \neq \phi \\ 0, & \text{otherwise} \end{cases}
\]

for all \( y \in H. \)

The set \( D = \{ (x, f^{-1}(\mu_B)(x)) : x \in G \} \) is called inverse image of \( B, \) where
\[ f^{-1}(\mu_B)(x) = \mu_B(f(x)) \]
for all \( x \in G. \)

**Theorem 4.2.**[5] Let \( f : G \rightarrow H \) be a group homomorphism.
Let $A$ be a fuzzy subgroup of $G$ and $B$ be a fuzzy subgroup of $H$, with membership functions $\mu_A(x)$ and $\mu_B(x)$ respectively. Then the image of $A$ is a fuzzy subgroup of $H$, and the inverse image of $B$ is a fuzzy subgroup of $G$.

**Definition 4.3.** Let $f : G \to H$ be a group homomorphism. Let $A = \{(x, \mu_A(x)) : x \in G\}$ and $B = \{(x, \mu_B(x)) : x \in H\}$ be complex fuzzy subgroups and $\mu_A(x) = r_A(x)e^{i\omega_A(x)}$ and $\mu_B(x) = r_B(x)e^{i\omega_B(x)}$ are their membership functions respectively. Then $C = \{(y, f(\mu_A)(y)) : y \in H\}$ is called image of $A$, where

$$f(\mu_A)(y) = \begin{cases} \max \{\mu_A(x) | x \in G, f(x) = y\} & \text{if } f^{-1}(y) \neq \phi \\ 0, & \text{otherwise}\end{cases}$$

for all $y \in H$. The set $D = \{(x, f^{-1}(\mu_B)(x)) : x \in G\}$ is called inverse image of $B$, where $f^{-1}(\mu_B)(x) = \mu_B(f(x))$ for all $x \in G$.

**Lemma 4.4.** Let $f : G \to H$ be a group homomorphism. Let $A$ be a complex fuzzy subgroup of $G$ and $B$ be a complex fuzzy subgroup of $H$, with membership functions $\mu_A(x) = r_A(x)e^{i\omega_A(x)}$ and $\mu_B(x) = r_B(x)e^{i\omega_B(x)}$ respectively. Then

1. $f(\mu_A)(y) = f(r_A)(y)e^{if(\omega_A)(y)}$.
2. $f^{-1}(\mu_B)(x) = f^{-1}(r_B)(x)e^{if^{-1}(\omega_B)(x)}$.

**Proof.**

1. \[f(\mu_A)(y) = \max_{f(x) = y} \mu_A(x) = \max_{f(x) = y} r_A(x)e^{i\omega_A(x)} = \max_{f(x) = y} r_A(x)e^{\max_{f(x) = y} i\omega_A(x)} = f(r_A)(y)e^{if(\omega_A)(y)}.

2. \[f^{-1}(\mu_B)(x) = \mu_B(f(x)) = r_B(f(x))e^{i\omega_B(f(x))} = f^{-1}(r_B)(x)e^{if^{-1}(\omega_B)(x)}.

**Theorem 4.5.** Let $f : G \to H$ be a group homomorphism. Let $A$ be a complex fuzzy subgroup of $G$ and $B$ be a complex fuzzy subgroup of
with membership functions $\mu_A(x) = r_A(x)e^{i\omega_A(x)}$ and $\mu_B(x) = r_B(x)e^{i\omega_B(x)}$ respectively. Then the image of $A$ is a complex fuzzy subgroup of $H$.

**Proof.** Since $A$ is a complex fuzzy subgroup, then by (Theorem 3.7) $r_A(x)$ is a fuzzy subgroup and $\omega_A(x)$ is a $\pi$–fuzzy subgroup. Thus by (Theorem 4.2) and (Proposition 3.3) the image of $r_A(x)$ and $\omega_A(x)$ are fuzzy subgroup and $\pi$–fuzzy subgroup respectively, therefore for all $x, y \in H$ we have:

$$f(r_A)(xy) \geq \min \{f(r_A)(x), f(r_A)(y)\}$$

Now by Lemma 4.4

$$f(\mu_A)(xy) = f(r_A)(xy)e^{if(\omega_A)(xy)}$$

$$\geq \min \{f(r_A)(x), f(r_A)(y)\} e^{im\{f(\omega_A)(x), f(\omega_A)(y)\}}$$

$$= \min \{f(r_A)(x)e^{if(\omega_A)(x)}, f(r_A)(y)e^{if(\omega_A)(y)}\}$$

$$= \min \{f(\mu_A)(x), f(\mu_A)(y)\}.$$  

Also,

$$f(\mu_A)(x^{-1}) = f(r_A)(x^{-1})e^{if(\omega_A)(x^{-1})}$$

$$= f(r_A)(x)e^{if(\omega_A)(x)}$$

$$= f(\mu_A)(x).$$

**Theorem 4.6.** Let $f : G \rightarrow H$ be a group homomorphism. Let $A$ be a complex fuzzy subgroup of $G$ and $B$ be a complex fuzzy subgroup of $H$, with membership functions $\mu_A(x) = r_A(x)e^{i\omega_A(x)}$ and $\mu_B(x) = r_B(x)e^{i\omega_B(x)}$ respectively. Then the inverse image of $B$ is a complex fuzzy subgroup of $G$.

**Proof.** Since $B$ is a complex fuzzy subgroup, then by (Theorem 3.7) $r_B(x)$ is a fuzzy subgroup and $\omega_B(x)$ is a $\pi$–fuzzy subgroup. Thus by (Theorem 4.2) and (Proposition 3.3) the inverse image of $r_B(x)$ and $\omega_B(x)$ are fuzzy subgroup and $\pi$–fuzzy subgroup respectively, therefore for all $x, y \in G$ we have:

$$f^{-1}(r_B)(xy) \geq \min \{f^{-1}(r_B)(x), f^{-1}(r_B)(y)\}$$

Now by Lemma 4.4

$$f^{-1}(\mu_B)(xy) = f^{-1}(r_B)(xy)e^{if^{-1}(\omega_B)(xy)}$$

$$\geq \min \{f^{-1}(r_B)(x), f^{-1}(r_B)(y)\} e^{im\{f^{-1}(\omega_B)(x), f^{-1}(\omega_B)(y)\}}$$

$$= \min \{f^{-1}(r_B)(x)e^{if^{-1}(\omega_B)(x)}, f^{-1}(r_B)(y)e^{if^{-1}(\omega_B)(y)}\}$$

$$= \min \{f^{-1}(\mu_B)(x), f^{-1}(\mu_B)(y)\}.$$
Also

\[ f^{-1}(\mu_B)(x^{-1}) = f^{-1}(r_B)(x^{-1})e^{if^{-1}(\omega_B)(x^{-1})} = f^{-1}(r_B)(x)e^{if^{-1}(\omega_B)(x)} = f^{-1}(\mu_B)(x). \]

References


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