Multiplicity of Solutions of Quasilinear Subelliptic Equations on Heisenberg Group

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Abstract

In this paper, a class of quasilinear elliptic equations on the Heisenberg Group is concerned. Under some suitable assumptions, by virtue of the nonsmooth critical point theory, the existence of infinitely many weak solutions of the problems is obtained.

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1 Introduction

Let $\mathbb{H}^N$ be the space $\mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}$ equipped with the following group operation: $\eta \circ \eta' = (x, y, t) \circ (x', y', t') = (x + x', y + y', t + t' + 2(x' \cdot y - x \cdot y'))$, where \( \cdot \) denotes the usual inner-product in $\mathbb{R}^N$. This operation endows $\mathbb{H}^N$ with the structure of a Lie group. The vector fields $X_1, \cdots, X_N, Y_1, \cdots, Y_N, T$

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Given by $X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}$, $Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}$, $T = \frac{\partial}{\partial t}$ form a basis for the tangent space at $\eta = (x, y, t)$.

**Definition 1** The Heisenberg Laplacian is defined by $\Delta_H = \sum_{j=1}^{N} (X_j^2 + Y_j^2)$ and let $\nabla_H u$ denote the $2N$-vector $(X_1 u, \cdots, X_N u, Y_1 u, \cdots, Y_N u)$. If $\tilde{F} = (F_1, \cdots, F_N, G_1, \cdots, G_N)$, we set $\text{div}_H \tilde{F} = X_1 F_1 + \cdots + X_N F_N + Y_1 G_1 + \cdots + Y_N G_N$.

In this paper, we study the existence and multiplicity of solutions for the problem

$$-\text{div}_H(A(\eta, u)\nabla_H u) + \frac{1}{2} A_s(\eta, u)|\nabla_H u|^2 + (b(\eta) - \lambda)u = f(\eta, u), \quad \eta \in \mathbb{H}^N,$$

(1)

where $N > 2$, $\lambda \in \mathbb{R}$ and $b(\cdot)$ denotes some continuous function with $b(\eta) \geq 0$ for all $\eta \in \mathbb{H}^N$ and $\lim_{|\eta|_{\mathbb{H}^N} \to \infty} b(\eta) = +\infty$. We set

$$E = \{u \in L^2(\mathbb{H}^N) | \int_{\mathbb{H}^N} |b(\eta)u|^2 + |\nabla_H u|^2 < \infty\}.$$

For the sake of simplicity, in the following, we use $\int$ as $\int_{\mathbb{H}^N}$.

In fact, the condition in $\mathbb{R}^N$ had been studied by Sami Aouaoui [1], Addolorata Salvatore [8], and so on. In this paper we study the problem on the Heisenberg group $\mathbb{H}^N$.

We use the variation methods to solve the problem (1). Finding weak solutions of (1) in $E$ is equivalent to finding critical points of the functional $I : E \to \mathbb{R}$

$$I(u) = \frac{1}{2} \int A(\eta, u)|\nabla_H u|^2 + \frac{1}{2} \int (b(\eta) - \lambda)u^2 - \int F(\eta, u),$$

(2)

where $F(\eta, \xi) = \int_0^\xi f(\eta, t)dt$. The first difficulty in this problem is that the functional is continuous but not differentiable in whole space $E$. Nevertheless, the derivatives of $I$ exist along directions of $E \cap L^\infty(\mathbb{H}^N)$.

**Remark 1** $E \leftrightarrow L^p(\mathbb{H}^N), 2 \leq p < 2^*, \text{ where } 2^* = \frac{2Q}{Q - 2}, Q = 2N + 2$.

Now, we introduce our eigenvalue problem. Letting $\mathcal{L} u = -\Delta_H u + b(\eta)u$, we consider the following eigenvalue problem: $\mathcal{L} u = \lambda u$. By virtue of the spectral theory for compact operators, we get a sequence of eigenvalues $0 < \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots$, such that $\lambda_n \to +\infty$ as $n \to \infty$. The first eigenvalue $\lambda_1$ has the variational characterization $\lambda_1 = \inf \{\|u\|^2 ; u \in E, \|u\|_{L^2(\mathbb{H}^N)} = 1\}$.

**Definition 2** A critical point $u$ of the functional $I$ is defined as a function $u \in E$ such that $\langle I'(u), h \rangle = 0, \forall h \in E \cap L^\infty(\mathbb{H}^N)$. 

To state and prove the main results in this paper, we make the following assumptions

(A − 1) For each \( s \in \mathbb{R} \), \( A(\eta, s) \) is measurable with respect to \( \eta \); for a.e. \( x \in \mathbb{H}^N \), \( A(\eta, s) \) is a function of class \( C^1 \) with respect to \( s \).

(A − 2) There exist \( 0 < \alpha < \beta < +\infty \) such that \( \alpha \leq A(\eta, s) \leq \beta \), \( |A_s(\eta, s)| \leq \beta \), a.e. \( \eta \in \mathbb{H}^N \) and \( \forall s \in \mathbb{R} \).

(A − 3) There exist \( \theta > 2, 1 < \gamma < \frac{\theta}{2} \) and \( \alpha_1 > 0 \) such that \( 0 \leq \frac{2}{\gamma} A_s(\eta, s) s \leq \left( \frac{\theta}{2} - \gamma \right) A(\eta, s) - \alpha_1 \), a.e. \( \eta \in \mathbb{H}^N \) and \( \forall s \in \mathbb{R} \).

(A − 4) \( A(\eta, -s) = A(\eta, s) \), a.e. \( \eta \in \mathbb{H}^N \) and \( \forall s \in \mathbb{R} \).

(f − 1) Let \( \theta \) be as in (A − 3). We assume that \( f(\cdot, \cdot) : \mathbb{H}^N \times \mathbb{R} \to \mathbb{R} \) is a Carathéodory function such that \( f(\eta, 0) = 0 \), a.e. \( \eta \in \mathbb{H}^N \) and \( 0 < \theta F(\eta, s) \leq f(\eta, s)s \), a.e. \( \eta \in \mathbb{H}^N \) and \( \forall s \neq 0 \) in \( \mathbb{R} \).

(f − 2) There exists \( 1 \leq p < \frac{Q + 2}{Q - 2} \) such that \( \exists c_0 > 0 \), \( |f(\eta, s)| \leq c_0 |s|^p \), a.e. \( \eta \in \mathbb{H}^N \) and \( \forall s \in \mathbb{R} \).

(f − 3) \( f(\eta, -s) = -f(\eta, s) \), a.e. \( \eta \in \mathbb{H}^N \) and \( \forall s \in \mathbb{R} \).

**Remark 2** Under assumptions (A − 1) − (A − 4), (f − 1) − (f − 3), we have for \( I \) the following assertions: \( I : E \to \mathbb{R} \) is continuous, and for every \( u \in E \) and \( h \in E \cap L^\infty(\mathbb{H}^N) \), one holds
\[
\langle I'(u), h \rangle = \int A(\eta, u) \nabla u \nabla h + \frac{1}{2} \int A_s(\eta, u) |\nabla h|^2 h
+ \int (b(\eta) - \lambda) uh - \int f(\eta, u) h. \tag{3}
\]
Moreover, for \( h \in E \cap L^\infty(\mathbb{H}^N) \), the function \( u \to \langle I'(u), h \rangle \) is continuous.

Next we can state some nonsmooth critical theory (see [3−5]).

**Definition 3** Let \( f : X \to \mathbb{R} \) be a continuous function and let \( u \in X \). We denote by \( |df|(u) \) the supremum of the \( \sigma' \) in \( [0, +\infty) \) such that there exist \( \delta > 0 \) and a continuous map \( H : B(u, \delta) \times [0, \delta] \to X \) such that for all \( (v, t) \in B(u, \delta) \times [0, \delta] \),
\[ d(H(v, t), v) \leq t \text{ and } f(H(v, t)) \leq f(v) - \sigma t. \]
The extended real number \( |df|(u) \) is called the weak slope of \( f \) at \( u \).

**Remark 3** For every \( u \in E \) yields:
\[ |dI|(u) \geq \sup\{ \langle I'(u), h \rangle; h \in E \cap L^\infty(\mathbb{H}^N), \| h \| \leq 1 \}, \tag{4}\]
where \( |dI|(u) \) denotes the weak slope of \( J \) at \( h \).

**Definition 4** Let \( f : X \to \mathbb{R} \) be a continuous function and let \( c \in \mathbb{R} \). We say that \( f \) satisfies \((PS)_c\), i.e. the Palais-Smale condition at level \( c \), if every sequence \( \{u_n\} \) in \( X \) with \( |df|(u_n) \to 0 \) and \( f(u_n) \to c \) admits a strongly convergent subsequence.
The main result of our paper is given by the following theorem:

**Theorem 1** Assume \((A - 1) - (A - 4), (f - 1)\) and \((f - 3)\). Then there exists a sequence \(\{u_n\} \subset E \cap L^\infty(\mathbb{H}^N)\) of weak solutions of problem (1) with \(J(u_n) \to +\infty\).

## 2 Preliminaries and Fundamental Lemmas

To reach our result of multiplicity of solutions for the problem (1), we need some lemmas. First we introduce the following fundamental theorem [3, Theorem 1.4] which is an extension of a well-known result for \(C^1\) functionals (see [6, Theorem 9.12]).

**Lemma 1** Let \(X\) be an infinite-dimensional Banach space and let \(f : X \to \mathbb{R}\) be continuous, even and satisfy \((PS)_c\) for every \(c \in \mathbb{R}\). Assume, also, that:

(a) For every finite-dimensional subspace \(W \subset E\), there exists \(R > 0\) such that:
\[
\forall u \in W : \|u\| = R \Rightarrow f(u) \leq f(0).
\]

(b) There exist \(\rho > 0, \alpha > f(0)\) and a subspace \(V \subset X\) of finite codimension such that:
\[
\forall u \in V : \|u\| = \rho \Rightarrow f(u) \geq \alpha.
\]

Then there exists a sequence \(\{c_n\}\) of critical values of \(f\) with \(c_n \to +\infty\).

Now, in order to prove that the functional \(I\) satisfies the Palais-Smale condition, let us introduce an auxiliary notion.

**Definition 5** Let \(c\) be a real number. We say that \(I\) satisfies the concrete Palais-Smale condition at level \(c\) (denoted by \((C - P - S)_c\)) if from every sequence \(\{u_n\} \subset E\) satisfying:
\[
\lim_{n \to \infty} I(u_n) = c \quad \text{and} \quad |\langle I'(u_n), h \rangle| \leq \varepsilon_n \|h\|,
\]  

\(0\) for all \(h \in E \cap L^\infty(\mathbb{H}^N)\),

where \(\{\varepsilon_n\}\) is a real sequence converging to zero, it is possible to extract a subsequence strongly convergent in \(E\).

**Lemma 2** Let \(c\) be a real number. If \(I\) satisfies \((C - P - S)_c\), then \(I\) satisfies \((PS)_c\).

**Lemma 3** Let \(u \in E\) be a critical point of \(I\), then \(u \in L^\infty(\mathbb{H}^N)\).

The proofs of Lemma 2 and Lemma 3 are standard, so we omit here.

**Lemma 4** Let \(\{u_n\} \subset E\) be a bounded sequence in \(E\) satisfying
\[
|\langle I'(u_n), h \rangle| \leq \varepsilon_n \|h\| \quad \forall h \in E \cap L^\infty(\mathbb{H}^N),
\]  

\(0\) for all \(h \in E \cap L^\infty(\mathbb{H}^N)\),

with \(\{\varepsilon_n\}\) being a real number sequence converging to zero. Then there exists \(u \in E\) such that \(\nabla_{\mathbb{H}} u_n \to \nabla_{\mathbb{H}} u\) a.e. in \(\mathbb{H}^N\) and, up to a subsequence, \(\{u_n\}\) is weakly convergent to \(u\) in \(E\). Moreover, we have
\[
\langle I'(u_n), h \rangle = 0 \quad \forall h \in E \cap L^\infty(\mathbb{H}^N),
\]
i.e. $u$ is a critical point of $I$.

**Proof** We argue as in [3, Lemma 2.3] or [7, Lemma 3].

We state, in the next lemma a vectorial version of Brezis-Browder’s theorem [8]. The proof of this result can be found in [9] under easy adapting to our

**Lemma 5** Denoting by $E^*$ the dual of the space $E$, let $v \in E$ and $T \in E^* \cap L^1_{\text{loc}}(\mathbb{H}^N)$ be such that $T(\eta)v(\eta) \geq h(\eta)$, a.e. $\eta \in \mathbb{H}^N$ for some $\eta \in L^1(\mathbb{H}^N)$. Then $Tv \in L^1(\mathbb{H}^N)$ and $(T,v)_{E^*,E} = \int Tv$.

**Lemma 6** Let $c \in \mathbb{R}$ and $\{u_n\}$ be a sequence satisfying (6) and

$$\lim_{n \to \infty} I(u_n) = c. \quad (8)$$

Then $\{u_n\}$ is bounded in $E$.

**Proof** By (6), we have

$$\left| \int \left( \frac{1}{2} A_s(\eta,u_n)|\nabla_{\mathbb{H}} u_n|^2 h - \lambda u_n h - f(\eta,u_n) h \right) \right| \leq \varepsilon_n \|h\| + (1 + \beta) \|u_n\| \|h\|.$$ 

Hence $W_n = \frac{1}{2} A_s(\eta,u_n)|\nabla_{\mathbb{H}} u_n|^2 - \lambda u_n - f(\eta,u_n) \in E^*$. Now, by (A - 3)

$$W_n(\eta)u_n(\eta) \geq -\lambda u_n^2(\eta) - f(\eta,u_n(\eta))u_n(\eta) \quad \text{a.e. } \eta \in \mathbb{H}^N.$$

By $(-\lambda u_n^2 - f(\eta,u_n)u_n) \in L^1(\mathbb{H}^N)$, we deduce from Lemma 5 that $W_n u_n \in L^1(\mathbb{H}^N)$ and

$$\langle W_n,u_n \rangle_{E^*,E} = \frac{1}{2} \int A_s(\eta,u_n)|\nabla_{\mathbb{H}} u_n|^2 u_n - \lambda \int u_n^2 - \int f(\eta,u_n) u_n.$$

Hence, the following inequality holds true

$$\left| \int (A(\eta,u_n)|\nabla_{\mathbb{H}} u_n|^2 + \frac{1}{2} A_s(\eta,u_n)|\nabla_{\mathbb{H}} u_n|^2 u_n + (b(\eta) - \lambda)u_n^2 - f(\eta,u_n) u_n) \right| \leq \varepsilon_n \|u_n\|. \quad (9)$$

By (8) and simple calculating, we have $\theta I(u_n) - \gamma \cdot (9) \leq c_6(1 + \|u_n\|)$. From (A - 3) and (f - 1), it follows

$$\alpha_1 \int |\nabla_{\mathbb{H}} u_n|^2 + (\theta \frac{2}{2} - \gamma) \int (b(\eta) - \lambda)u_n^2 + \theta(\gamma - 1) \int F(\eta, u_n) \leq c_6(1 + \|u_n\|). \quad (10)$$

There exist $M > 0$ and $c_7(M,\lambda) > 0$ such that

$$\left( \frac{\theta}{2} - \gamma \right) \int (b(\eta) - \lambda)u_n^2 \geq \left( \frac{\theta}{2} - \gamma \right) \int \frac{b(\eta)}{2} u_n^2 - c_7 \int_{\{\eta \in \mathbb{H}^N < M\}} u_n^2.$$
We obtain by (10)
\[
\alpha_1 \int |\nabla H u_n|^2 + \left(\frac{\theta}{4} - \frac{\gamma}{2}\right) \int b(\eta) u_n^2 + \theta(\gamma - 1) \int F(\eta, u_n) \leq c_6 (1 + \|u_n\|) + c_7 \|u_n\|_{L^2(\{|\eta|_{\mathbb{H}^N} < M\})}^2.
\] (11)

By \((f - 1)\), there exist \(a_0 > 0\) and \(b_0 > 0\) such that
\[
F(\eta, s) \geq a_0 |s|^\theta - b_0, \quad \text{a.e. } \eta \in \{|\eta|_{\mathbb{H}^N} < M\}, \forall s \in \mathbb{R}.
\] (12)

From (11) and (12), it follows
\[
\min(\alpha_1, \frac{\theta}{4} - \frac{\gamma}{2}) \|u_n\|^2 + \theta(\gamma - 1) \|u_n\|_{L^\theta(\{|\eta|_{\mathbb{H}^N} < M\})}^\theta \leq c_6 (1 + \|u_n\|) + c_7 \|u_n\|_{L^2(\{|\eta|_{\mathbb{H}^N} < M\})}^2 + b_0 \theta(\gamma - 1)m(\{|\eta|_{\mathbb{H}^N} < M\}).
\] (13)

It is easy to see that \(\|u_n\|_{L^\theta(\{|\eta|_{\mathbb{H}^N} < M\})}^\theta \leq c(\varepsilon) + \varepsilon \|u_n\|_{L^\theta(\{|\eta|_{\mathbb{H}^N} < M\})}^\theta\). By (13), choosing \(\varepsilon\) enough small, we have
\[
\min(\alpha_1, \frac{\theta}{4} - \frac{\gamma}{2}) \|u_n\|^2 \leq c_6 (1 + \|u_n\|) + c_7 \|u_n\|_{L^\theta(\{|\eta|_{\mathbb{H}^N} < M\})}^\theta + b_0 \theta(\gamma - 1)m(\{|\eta|_{\mathbb{H}^N} < M\}).
\]

It implies that \(\{u_n\}\) is bounded in \(E\). \(\square\)

**Lemma 7** Let \(\{u_n\}\) be a subsequence as in Lemma 4. Then \(\{u_n\}\), possessing a subsequence, converges strongly to \(u\) in \(E\).

**Proof** By Lemma 4 we know that \(u\) is a critical point of the functional \(I\). Then, using Lemma 3, we get \(u \in L^\infty(\mathbb{H}^N)\) and we can take \(u\) as test function in (7)
\[
\int (A(\eta, u)|\nabla \mathbb{H} u|^2 + \frac{1}{2} A_s(\eta, u)|\nabla \mathbb{H} u|^2 u^2 + (b(\eta) - \lambda)u^2 - f(\eta, u)u) = 0.
\] (14)

By Lemma 4, \(\nabla \mathbb{H} u_n \rightarrow \nabla \mathbb{H} u\) a.e. in \(\mathbb{H}^N\), then by virtue of Fatou’s lemma, we have
\[
\int A_s(\eta, u)|\nabla \mathbb{H} u|^2 u \leq \liminf_{n \rightarrow \infty} \int A_s(\eta, u_n)|\nabla \mathbb{H} u_n|^2 u_n.
\] (15)

Moreover, by the compactness of the embedding of \(E\) into \(L^2(\mathbb{H}^N)\) and \(L^{p+1}(\mathbb{H}^N)\),
\[
\lim_{n \rightarrow \infty} \int f(\eta, u_n) u_n = \lim_{n \rightarrow \infty} \int f(\eta, u) u
\] (16)

and
\[
\lim_{n \rightarrow \infty} \int u_n^2 = \int u^2.
\] (17)
Using (14)-(17), and by (9), we have
\[
\limsup_{n \to \infty} \left( \int A(\eta, u_n) |\nabla_{H^\infty} u_n|^2 + \int b(\eta) u_n^2 \right) \leq \int A(\eta, u) |\nabla_{H^\infty} u|^2 + \int b(\eta) u^2.
\] (18)

By Lebesgue's dominated convergence theorem and the weak convergence of \(\{u_n\}\) to \(u\) in \(E\), we get
\[
\lim_{n \to \infty} \int A(\eta, u_n) \nabla_{H^\infty} u_n \nabla u = \int A(\eta, u_n) |\nabla_{H^\infty} u|^2, \tag{19}
\]
\[
\lim_{n \to \infty} \int b(\eta) u_n u = \int b(\eta) u^2, \tag{20}
\]
\[
\lim_{n \to \infty} \int A(\eta, u_n) |\nabla_{H^\infty} u|^2 = \int A(\eta, u) |\nabla_{H^\infty} u|^2. \tag{21}
\]

By (18)-(21), it follows
\[
\limsup_{n \to \infty} \left( \int A(\eta, u_n) |\nabla_{H^\infty} u_n - \nabla_{H^\infty} u|^2 + \int b(\eta) |u_n - u|^2 \right) \leq 0.
\]

Then, \(u_n\) converges strongly to \(u\) in \(E\). \(\square\)

**Remark 4** By Lemma 6 and Lemma 7, we have for every \(c\), the functional \(I\) satisfies the \((C - P - S)_c\) condition.

### 3 Proof of Theorem 1

**Proof of Theorem 1** The functional \(I\) is continuous and even. Moreover, by Lemma 2 we know \(I\) satisfies \((PS)_c\) for every \(c \in \mathbb{R}\).

Firstly, we consider the condition (a) in Lemma 1. Let \(W\) be a finite-dimensional subspace of \(E\), and \(u \in W\) such that \(I(u) \geq 0\). Then from (A − 2) we have
\[
\max(1, \beta) \|u\|^2 - \lambda \|u\|_{L^2(H^\infty)}^2 - \int F(\eta, u) \geq 0. \tag{22}
\]

By (f − 1) and (f − 2), there exist \(m(\eta) \in L^\infty(H^\infty)\) with \(m(\eta) > 0\) a.e. \(\eta \in H^\infty\) and a positive constant \(c\) such that \(F(\eta, s) \geq m(\eta)|s|^\theta - cs^2\) a.e. \(\eta \in H^\infty, \forall s \in \mathbb{R}\). Using (22), this implies that
\[
\max(1, \beta) \|u\|^2 \geq \int m(\eta)|s|^\theta - c \int u^2. \tag{23}
\]

Since \(W\) is finite-dimensional subspace of \(E\), \((\int m(\eta)|s|^\theta)^{\frac{1}{\theta}}\) is a norm on \(W\), then, by (23), there exists \(c > 0\) such that \(\|u\|^\theta \leq c \|u\|^2\). Taking into account
that $\theta > 2$, we deduce that the set $\{u \in W, I(u) \geq 0\}$ is bounded in $E$ and the condition (a) in Lemma 1 holds.

Next, we consider the condition (b) in Lemma 1. By $(A - 2)$ and $(f - 2)$, we have for $u \in E$,

$$I(u) \geq \min\{1, \alpha\} \frac{1}{2} \|u\|^2 - \frac{\lambda}{2} \int u^2 - c_9 \|u\|^{p+1}. \quad (24)$$

**Case A:** If $\lambda \min\{1, \alpha\} < \lambda_1$, then by (24), we have

$$I(u) \geq \min\{1, \alpha\} \frac{1}{2} \|u\|^2 - c_9 \|u\|^{p+1}, \text{ if } \lambda \leq 0,$$

or

$$I(u) \geq \frac{1}{2} \min\{1, \alpha\} - \frac{\lambda}{\lambda_1} \|u\|^2 - c_9 \|u\|^{p+1}, \text{ if } \lambda > 0.$$  

Hence, there exist $\rho > 0$ small enough and $\delta > 0$ such that $I(u) \geq \delta > I(0)$, for $\|u\| = \rho$.

**Case B:** If $\lambda \min\{1, \alpha\} \in [\lambda_k, \lambda_{k+1})$, for some $k \geq 1$, we denote by $v_j$ an orthonormal basis of eigenvectors of the operator $L(u) = -\Delta u + b(\cdot) u$. Taking $V_k = (\text{span}\{v_1, \cdots, v_k\})^\perp$, by (24), we have

$$I(u) \geq \frac{1}{2} \min\{1, \alpha\} \frac{\lambda}{\lambda_{k+1}} \|u\|^2 - c \|u\|^{p+1}.$$  

This yields that there exist $\rho > 0$ and $\delta > 0$ such that $I(u) \geq \delta > I(0), \text{ } u \in V_k, \|u\| = \rho$. Hence condition (b) of Lemma 1 holds with $V = V_k$. Which achieves the proof. \Box

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