A New Approach to Homotopy Perturbation Method for Solving Emden–Fowler Equations

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Abstract

In this article, we presented a new approach deduced from He’s homotopy perturbation method (HPM) to obtain exact and/or approximate analytical solutions of Emden–Fowler type equations. The results reveal that the proposed algorithm is very effective and in most cases yield exact solutions in the first approximation. Some examples are given to illustrate the new approach.

Keywords: Emden–Fowler equations, homotopy perturbation method, Theory of stellar structure

1. Introduction

Singular initial value problems in the second order ordinary differential equations, linear and nonlinear, homogeneous and inhomogeneous, have attracted the attention of many mathematicians and physicists. One of the equation describing this type in mathematical physics and astrophysics can be modeled by the so-called initial value problems (IVPs) of Emden–Fowler type [1-3]

\[ y'' + \alpha x y' + f(x) g(y) = h(x), \quad 0 < x \leq 1, \]  

subject to the conditions
\[ y(0) = A \quad \text{and} \quad y'(0) = 0 \quad (1.1b) \]

where \( \alpha \) and \( A \) are constants, and \( f(x) \), \( g(x) \) are real-valued continuous functions and \( h(x) \in C[0,1] \). A substantial amount of work has been done on this type of equation for various structures of \( f(x), g(y) \) and \( h(x) \). When \( f(x) = 1 \) and \( h(x) = 0 \) Eq. (1.1) reduces to the Lane–Emden equation which, with specified \( g(y) \), was used to model several phenomena in mathematical physics and astrophysics such as the theory of stellar structure, the thermal behavior of a spherical cloud of gas, isothermal gas sphere and theory of thermionic currents [1-3]. Bender et al. [4] handled the solution of Lane–Emden equation as well as those of a variety of nonlinear problems in quantum mechanics and astrophysics by means of perturbation methods based on the existence of a small parameter. The solution to the Emden–Fowler problem were presented by Wazwaz [5-6] using the Adomian decomposition method (ADM) [7]. Also in [8] Wazwaz applied ADM to solve time-dependent Emden–Fowler type equations. Sometimes it is a very intricate problem to calculate the so-called Adomian polynomials involved in ADM. Bataineh et al. [9] solved Emden–Fowler type equations by a powerful and more convenient analytical technique named homotopy analysis method (HAM) was first proposed by Laio [10]. Also Singh et al. [11] solved this type of equations by a modified homotopy analysis method. Another efficient and powerful analytical technique, called the homotopy perturbation method, was first proposed by He [12] and further developed and improved by himself [13–17]. Chowdhury and Hashim [18-19] gave the solutions of class of singular second-order initial value problem by using He’s homotopy perturbation method. Ahmet and Turgut [20] presented a new scheme, deduced from He’s homotopy perturbation method to solve this type of equations. Later in 2009 they [21] solved this type of equations by variational iteration method.

Homotopy perturbation method has been used to solve many singular initial value and boundary value problems with diverse variations. In this work, we present a new approach of He’s homotopy perturbation method (HPM) to obtain exact and/or approximate solutions of the Emden–Fowler type equations of the form (1.1).

2. Basic ideas of He’s HPM

Now, for convenience, consider the following general nonlinear differential equation

\[ A(u) - f(r) = 0, \quad r \in \Omega \quad (2.1) \]
A new approach to homotopy perturbation method

with boundary conditions;

\[ B \left( u, \frac{\partial u}{\partial n} \right) = 0, \quad r \in \Gamma \]  \hspace{1cm} (2.2)

where \( A \) is a general differential operator, \( B \) is a boundary operator, \( f (r) \) is a known analytic function, \( \Gamma \) is the boundary of the domain \( \Omega \).

The operator \( A \) can, generally speaking, be divided into two parts \( L \) and \( N \), where \( L \) is linear and \( N \) is nonlinear, therefore Eq. (2.1) can be written as,

\[ L(u) + N(u) - f(r) = 0 \]  \hspace{1cm} (2.3)

By using homotopy technique, one can construct a homotopy \( v(r, p): \Omega \times [0, 1] \rightarrow \mathbb{R} \) which satisfies

\[ H(v, p) = (1 - p)[L(v) - L(u_0)] + p[A(u) - f(r)] = 0 \]  \hspace{1cm} (2.4)

or

\[ H(v, p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(r)] = 0 \]  \hspace{1cm} (2.5)

where \( p \in [0, 1] \) is an embedding parameter, and \( u_0 \) is the initial approximation of Eq. (2.1) which satisfies the boundary conditions. Clearly, we have

\[ H(v, 0) = L(v) - L(u_0) = 0, \]  \hspace{1cm} (2.6)

\[ H(v, 1) = A(v) - f(r) = 0 \]  \hspace{1cm} (2.7)

the changing process of \( p \) from zero to unity is just that of \( v(r, p) \) changing from \( u_0(r) \) to \( u(r) \). This is called deformation, and also \( L(v) - L(u_0) \) and \( A(u) - f(r) \) are called homotopic in topology. Here the embedding parameter is introduced much more naturally, unaffected by artificial factors; further it can be considered as a small parameter for \( 0 \leq p \leq 1 \). So it is very natural to assume that the solution of (2.6) and (2.7) can be expressed as

\[ v = v_0 + pv_1 + p^2v_2 + \cdots \]  \hspace{1cm} (2.8)

and setting \( p = 1 \) results in the approximate solution of Eq. (2.1) as;

\[ u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \cdots \]  \hspace{1cm} (2.9)

The convergence of series (2.9) has been proved by He in his paper [13].
3. Basic idea behind the new approach

To show the basic idea, let us consider the Emden–Fowler equation of type (1.1). In the new approach, we split \( f(x) \) and \( h(x) \) into infinite sums as follows
\[
f(x) = \sum_{i=0}^{\infty} k_i(x) ; \quad h(x) = \sum_{i=0}^{\infty} l_i(x),
\]
where \( p \in [0,1] \) is an embedding parameter. At \( p = 0 \), \( \varphi(x; 0) = k_0 \) and \( \sigma(x; 0) = l_0 \) where as \( \varphi(x; 1) = f(x) \) and \( \sigma(x; 1) = h(x) \) when \( p = 1 \). Now, we construct a homotopy \( Y(x; p): \mathbb{R} \times [0,1] \to \mathbb{R} \) which satisfies the following equation
\[
Y'' + \frac{\alpha}{x} Y' + p \varphi(x; p) g(Y) = \sigma(x; p)
\]
where \( Y(x; p) = Y_0 + pY_1 + p^2Y_2 + \cdots \) (3.4)

Thus as \( p \) changes from 0 to 1, \( Y(x; p) \) deforms continuously from \( Y_0(x) \) to the solution \( y(x) = \lim_{p \to 1} Y(x; p) \). Substituting Eq. (3.4) into Eq. (3.3) and equating the terms with identical powers of \( p \), we can obtain a series of linear equations. Solving these linear equations, we can get the approximate solutions.

4. Applications of the new approach

In this section, we apply an alternative approach of He’s HPM to obtain exact and/or approximate solutions of the Emden–Fowler equations of type (1.1). Some examples are given to show the potential of this modification.

4.1. Example

Now consider the linear, inhomogeneous Emden–Fowler type equation
\[
y'' + \frac{8}{x} y' + xy = x^5 - x^4 + 44x^2 - 30, \quad 0 < x \leq 1,
\]
subject to the initial conditions
\[
y(0) = 0 \text{ and } y'(0) = 0
\] (4.1.2)

having \( y(x) = x^4 - x^3 \) as the exact solution. At first, we split \( f(x) = x \) as
\[
f(x) = \sum_{i=0}^{\infty} k_i(x) \text{ with } k_0 = x, \quad k_i = 0, \quad i > 0
\]
and \( h(x) = x^5 - x^4 + 44x^2 - \cdots \)

30x as \( h(x) = \sum_{i=0}^{\infty} l_i(x) \) with \( l_0 = 44x^2 - 30x \), \( l_1 = x^5 - x^4 \), and \( l_i = 0 \), \( i > 1 \). Now, we construct a homotopy \( Y(x; p): \mathbb{R} \times [0, 1] \rightarrow \mathbb{R} \) which satisfies the following equation

\[
Y'' + \frac{8}{x} Y' + p \varphi(x; p) Y = \sigma(x; p) \quad (4.1.3)
\]

where \( \varphi(x; p) \), \( \sigma(x; p) \) and \( Y(x; p) \) as in Eq. (3.2) and Eq. (3.4) respectively. Substituting Eq. (3.4) into Eq. (4.1.3) and then collecting terms of same power of \( p \), we get

\[
p^0: \begin{cases}
Y_0'' + \frac{8}{x} Y_0' - l_0 = 0 \\
Y_0(0) = 0, \ Y_0'(0) = 0
\end{cases} \quad (4.1.4)
\]

\[
p^1: \begin{cases}
Y_1'' + \frac{8}{x} Y_1' + k_0 Y_0 - l_1 = 0 \\
Y_1(0) = 0, \ Y_1'(0) = 0
\end{cases} \quad (4.1.5)
\]

Solving the above systems, we obtain

\[
Y_0 = x^4 - x^3; \quad (4.1.6)
\]

\[
Y_1 = Y_2 = \cdots = 0; \quad (4.1.7)
\]

and \( y(x) = x^4 - x^3 \). \quad (4.1.8)

### 4.2. Example

Now consider the nonlinear, inhomogeneous Emden–Fowler type equation

\[
y'' + \frac{2}{x} y' + y^3 = 6 + x^6, \quad 0 < x \leq 1, \quad (4.2.1)
\]

subject to the initial conditions

\[
y(0) = 0 \text{ and } y'(0) = 0 \quad (4.2.2)
\]

having \( y(x) = x^2 \) as the exact solution. At first, we split \( f(x) = 1 \) as \( f(x) = \sum_{i=0}^{\infty} k_i(x) \) with \( k_0 = 1 \), \( k_i = 0 \), \( i > 0 \) and \( h(x) = 6 + x^6 \) as \( h(x) = \sum_{i=0}^{\infty} l_i(x) \) with \( l_0 = 6 \), \( l_1 = x^6 \) and \( l_i = 0 \), \( i > 1 \). Now, we construct a homotopy \( Y(x; p): \mathbb{R} \times [0, 1] \rightarrow \mathbb{R} \) which satisfies the following equation
\[ Y'' + \frac{2}{x}Y' + p\varphi(x; p)Y^3 = \sigma(x; p) \quad (4.2.3) \]

where \( \varphi(x; p) \), \( \sigma(x; p) \) and \( Y(x; p) \) as in Eq. (3.2) and Eq. (3.4) respectively. Substituting Eq. (3.4) into Eq. (4.2.3) and then collecting terms of same power of \( p \), we get

\[ p^0: \begin{cases} Y_0'' + \frac{2}{x}Y_0' - l_0 = 0 \\ Y_0(0) = 0, \ Y_0'(0) = 0 \end{cases} \quad (4.2.4) \]

\[ p^1: \begin{cases} Y_1'' + \frac{2}{x}Y_1' + k_0Y_0^3 - l_1 = 0 \\ Y_1(0) = 0, \ Y_1'(0) = 0 \end{cases} \quad (4.2.5) \]

Solving the above systems, we obtain

\[ Y_0 = x^2; \quad (4.2.6) \]

\[ Y_1 = Y_2 = \cdots = 0; \quad (4.2.7) \]

and \( y(x) = x^2 \). \( (4.2.8) \)

### 4.3. Example

Now consider another linear, inhomogeneous Emden–Fowler type equation

\[ y'' + \frac{2}{x}y' + y = 6 + 12x + x^2 + x^3, \quad 0 < x \leq 1, \quad (4.3.1) \]

subject to the initial conditions

\[ y(0) = 0 \text{ and } y'(0) = 0 \quad (4.3.2) \]

having \( y(x) = x^2 + x^3 \) as the exact solution. At first, we split \( f(x) = 1 \) as \( f(x) = \sum_{i=0}^{\infty} k_i(x) \) with \( k_0 = 1, \ k_i = 0, \ i > 0 \) and \( h(x) = 6 + 12x + x^2 + x^3 \) as \( h(x) = \sum_{i=0}^{\infty} l_i(x) \) with \( l_0 = 6 + 12x, \ l_1 = x^2 + x^3, \) and \( l_i = 0, \ i > 1 \). Now, we construct a homotopy \( Y(x; p): \mathbb{R} \times [0, 1] \rightarrow \mathbb{R} \) which satisfies the following equation

\[ Y'' + \frac{2}{x}Y' + p\varphi(x; p)Y = \sigma(x; p) \quad (4.3.3) \]
where \( \varphi(x; p) \), \( \sigma(x; p) \) and \( Y(x; p) \) as in Eq. (3.2) and Eq. (3.4) respectively. Substituting Eq. (3.4) into Eq. (4.3.3) and then collecting terms of same power of \( p \), we get the systems of the form like Eq. (4.2.4)-(4.2.5) and then solving them; we obtain

\[
Y_0 = x^2 + x^3; \tag{4.3.4}
\]
\[
Y_1 = Y_2 = \cdots = 0; \tag{4.3.5}
\]

and \( y(x) = x^2 + x^3 \).  \( \tag{4.3.6} \)

4.4. Example

Finally, we consider a linear, homogeneous Emden–Fowler type equation

\[
y'' + \frac{2}{x}y' - (6 + 4x^2)y = 0, \quad 0 < x \leq 1, \tag{4.4.1}
\]

subject to the initial conditions

\[
y(0) = 1 \text{ and } y'(0) = 0 \tag{4.4.2}
\]

having \( y(x) = e^{x^2} \) as the exact solution. In this case \( f(x) = 6 + 4x^2 \) and \( h(x) = 0 \). Now, we split \( f(x) = 6 + 4x^2 \) as \( f(x) = \sum_{i=0}^{\infty} k_i(x) \) with \( k_0 = 6, k_1 = 4x^2 \) and \( k_i = 0, i > 1 \). Now, we construct a homotopy \( Y(x; p) : \mathbb{R} \times [0, 1] \to \mathbb{R} \) which satisfies the following equation

\[
Y'' + \frac{2}{x}Y' - p\varphi(x; p)Y = 0 \tag{4.4.3}
\]

where \( \varphi(x; p) \) and \( Y(x; p) \) as in Eq. (3.2) and Eq. (3.4) respectively. Substituting Eq. (3.4) into Eq. (4.4.3) and then collecting terms of same power of \( p \), we get the systems of the form like Eq. (4.2.4)-(4.2.5) and then solving them; we obtain

\[
Y_0 = 1; \tag{4.4.4}
\]
\[
Y_1 = x^2; \tag{4.4.5}
\]
\[
Y_2 = \frac{x^4}{2}; \tag{4.4.6}
\]

\[\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cd-
Our calculations indicate that the above approximations converges to the exact solution $e^{x^2}$.

5. Conclusion

Many researchers have been solved Emden–Fowler type of equations by several methods. In this paper, we have presented a new approach deduced from He’s homotopy perturbation method to solve this type of equation. Assessment with other methods with ours shows that the approach is simpler and gives exact solution in a few iterations only. Finally, we can conclude that this variant approach is a promising tool for both linear and non-linear singular IVPs of Emden–Fowler type equations.

References


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