Some New Generalized Double Sequence Spaces

Defined by a Double Orlicz Function

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Abstract

In this paper we introduce some new generalized double sequence spaces by using a double Orlicz function. We also examine some properties of these double sequence spaces.

Keywords: Double sequence, double Orlicz function, non-regular matrix

1. Introduction

The sequence spaces are generalized in many directions by different mathematicians. The single sequence spaces studied by Esi [5], Esi and Et [9], Savaş [2], and many others, also the double sequence spaces studied by Esi [10], Savaş and Patterson [3], Tripathy [7] and many others for more details.

The Orlicz function has been founded by Prof. Wlayshaw Roman Orlicz from Poland and carried his name, so he was constructed the Orlicz space [6]. Throughout the paper \((x, y) = (x_{k,l}, y_{k,l})\) for all \(k, l \in \mathbb{N}\) a double infinite array of elements \((x_{k,l}, y_{k,l})\), when \(x = (x_{k,l}), y = (y_{k,l})\).
2. Basic definitions and symbols

Let us define the \(\mathbb{N}\)-function \(M(x,y)\) in the term of a double sequence spaces

**Definition 2.1.** A double Orlicz function is a function

\[
M: [0,\infty) \times [0,\infty) \to [0,\infty) \times [0,\infty)
\]

where

\[
M(x,y) = \left(M_1(x), M_2(y)\right), \quad M_1: [0,\infty) \to [0,\infty) \quad \text{and} \quad M_2: [0,\infty) \to [0,\infty),
\]

such that \(M_1, M_2\) are Orlicz functions which is continuous, non-decreasing, even, convex and satisfies the following conditions

1) \(M_1(0) = 0, M_2(0) = 0 \Rightarrow M(0,0) = (M_1(0), M_2(0)) = (0,0),\)
2) \(M_1(x) > 0, M_2(y) > 0 \Rightarrow M(x,y) = (M_1(x), M_2(y)) > (0,0)\) for \(x > 0, y > 0\), we mean by \(M(x,y) > (0,0)\), that \(M_1(x) > 0, M_2(y) > 0\)
3) \(M_1(x) \to \infty, M_2(y) \to \infty\) as \(x, y \to \infty\), then \(M(x,y) = (M_1(x), M_2(y)) \to (\infty, \infty)\) as \((x,y) \to (\infty, \infty)\), we mean by \(M(x,y) \to (\infty, \infty)\), that \(M_1(x) \to \infty, M_2(y) \to \infty\).

Lindenstrauss and Tzafriri [8] used the idea of Orlicz function to construct the Orlicz sequence space, for that idea we will construct a double sequence space as follows:

\[
\ell^2_M = \left\{(x,y) \in w^2: \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \sup \left\{M_1 \left(\frac{|x_{k,l}|}{\rho}\right), M_2 \left(\frac{|y_{k,l}|}{\rho}\right)\right\} < \infty, \right. \right.
\]

for some \(\rho > 0\)

The space \(\ell^2_M\) with the norm

\[
\| (x,y) \|_M = \inf \left\{\rho > 0: \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \sup \left\{M_1 \left(\frac{|x_{k,l}|}{\rho}\right), M_2 \left(\frac{|y_{k,l}|}{\rho}\right)\right\} \leq 1, \right. \right.
\]

becomes a Banach space which is called a double Orlicz of double sequence space where \(w^2\) is a family of all \(\mathbb{R}^2\) or \(\mathbb{C}^2\) double sequences (i.e, \(x_{k,l}\) and \(y_{k,l}\) are complex or real double sequences.

According the definition of Pringsheim [11] we get the following:

**Definition 2.2.** A double sequences \(x = (x_{k,l}), y = (y_{k,l})\) have limits \(\ell_1, \ell_2\) in Pringsheim’s sense denoted by \((P - \lim x = \ell_1, P - \lim y = \ell_2)\), so, \((x,y) = (x_{k,l}, y_{k,l})\) has limit \((\ell_1, \ell_2)\) denoted by \(P - \lim (x,y) = (\ell_1, \ell_2)\) provided that given \(\epsilon > 0\), there exists \(N \in \mathbb{N}\) such that \(|(x_{k,l} - \ell_1, y_{k,l} - \ell_2)| < \epsilon\) whenever \(k,l > N\). We shall describe such an \(x,y\) more briefly as “\(P\)-convergent”.
**Definition 2.3.** Let \( x = (x_{k,l}) \), \( y = (y_{k,l}) \) be a double sequences. We say that \((x,y) = (x_{k,l},y_{k,l})\) be a bounded, if there exists a positive number \( C \) such that 
\[
|(x_{k,l},y_{k,l})| < C \quad \text{for all } k, l.
\]

**Definition.**[14] The four dimensional matrix \( A = (a_{n,m,k,l}) \) is said to be RH-

-regular if it maps every bounded \( P \)-convergent sequences into a \( P \)-convergent sequence with the same \( P \)-limit.

Lemma [12,13], the four dimensional matrix \( A \) is RH-

-regular if and only if

\( RH_1: P - \lim_{n,m} a_{n,m,k,l} = 0 \) for each \( k \) and \( l \);

\( RH_2: P - \lim_{n,m} \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} a_{n,m,k,l} = 1; \)

\( RH_3: P - \lim_{n,m} \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} |a_{n,m,k,l}| = 0, \) for each \( l \);

\( RH_4: P - \lim_{n,m} \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} |a_{n,m,k,l}| = 0, \) for each \( k \);

\( RH_5: \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} |a_{n,m,k,l}| \) is \( P \)-convergent;

\( RH_6: \) There exists finite positive integers \( E \) and \( F \) such that \( \sum_{k,l>E} a_{n,m,k,l} < E \).

Let \( M(x,y) = (M_1(x), M_2(y)) \) be a double Orlicz function, and

\((x,y) = (x_{k,l},y_{k,l}),\) be a double infinite array of elements \((x_{k,l},y_{k,l}),\) where \( x = (x_{k,l}), y = (y_{k,l}), \) so let \( p = (p_{k,l}) \) be a factorable double sequence of positive real numbers and \( A = (a_{n,m,k,l}) \) be a nonnegative RH-

-regular summability matrix method. We now define the following double sequence spaces:

\[
W^2(A,M,p) = \left\{ (x,y) \in w^2: P - \lim_{n,m} \sum_k \sum_l a_{n,m,k,l} \sup \left\{ \left( M_1 \left( \frac{|x_{k,l}|}{\rho} \right) \right)^{p_{k,l}}, \left( M_2 \left( \frac{|y_{k,l}|}{\rho} \right) \right)^{p_{k,l}} \right\} = 0, \right\}
\]

for some \( \rho, \ell_1, \ell_2 > 0 \)

this means \( W^2(A,M,p) = (2W(A,M_1,p), 2W(A,M_2,p)) \).

\[
W^2_0(A,M,p) = \left\{ (x,y) \in w^2: P - \lim_{n,m} \sum_k \sum_l a_{n,m,k,l} \sup \left\{ \left( M_1 \left( \frac{|x_{k,l}|}{\rho} \right) \right)^{p_{k,l}}, \left( M_2 \left( \frac{|y_{k,l}|}{\rho} \right) \right)^{p_{k,l}} \right\} = 0, \right\}
\]

for some \( \rho > 0 \)

this means \( W^2_0(A,M,p) = (2W_0(A,M_1,p), 2W_0(A,M_2,p)) \).
\[ W_\infty^2(A, M, p) = \]
\[ \left\{ (x, y) \in w^2 : \sup_{n,m} \sum_k \sum_i a_{n,m,k,i} \sup \left\{ M_1 \left( \frac{|x_{k,i}|}{\rho} \right)^{p_{k,i}}, M_2 \left( \frac{|y_{k,i}|}{\rho} \right)^{p_{k,i}} \right\} < \infty \right\} \]

for some \( \rho > 0 \)

this means \( W_\infty^2(A, M, p) = (2W_\infty(A, M_1, p), 2W_\infty(A, M_2, p)) \).

When \( A = (C, 1,1) \), we have the following double sequence spaces

1) \( W^2(M, p) = \)
\[ \left\{ (x, y) \in w^2 : P - \lim_{n,m} \frac{1}{nm} \sum_{k=1}^n \sum_{l=1}^m \sup \left\{ M_1 \left( \frac{|x_{k,l} - \ell_1|}{\rho} \right)^{p_{k,l}}, M_2 \left( \frac{|y_{k,l} - \ell_2|}{\rho} \right)^{p_{k,l}} \right\} = 0 \right\} , \]

for some \( \rho > 0 \) and \( \ell_1, \ell_2 > 0 \)

this means \( W^2(M, p) = (2W(M_1, p), 2W(M_2, p)) \).

2) \( W_0^2(M, p) = \)
\[ \left\{ (x, y) \in w^2 : P - \lim_{n,m} \frac{1}{nm} \sum_{k=1}^n \sum_{l=1}^m \sup \left\{ M_1 \left( \frac{|x_{k,l} - \ell_1|}{\rho} \right)^{p_{k,l}}, M_2 \left( \frac{|y_{k,l} - \ell_2|}{\rho} \right)^{p_{k,l}} \right\} = 0 \right\} , \]

for some \( \rho > 0 \)

this means \( W_0^2(M, p) = (2W_0(M_1, p), 2W_0(M_2, p)) \) and

3) \( W_\infty^2(M, p) = \)
\[ \left\{ (x, y) \in w^2 : \sup_{n,m} \frac{1}{nm} \sum_{k=1}^n \sum_{l=1}^m \sup \left\{ M_1 \left( \frac{|x_{k,l}|}{\rho} \right)^{p_{k,l}}, M_2 \left( \frac{|y_{k,l}|}{\rho} \right)^{p_{k,l}} \right\} < \infty \right\} , \]

for some \( \rho > 0 \)

this means \( W_\infty^2(M, p) = (2W_\infty(M_1, p), 2W_\infty(M_2, p)) \).

The double sequence spaces \( W^2(A, M, p), W_0^2(A, M, p) \) and \( W_\infty^2(A, M, p) \) are generalization of the double sequence spaces \( W^2(M, p), W_0^2(M, p) \) and \( W_\infty^2(M, p) \) respectively.

When \( M(x) = x, M(y) = y \) and \( M(x, y) = (x, y) \), we have the following sequence spaces:

\[ W^2(A, p) = \]
\[ \left\{ (x, y) \in w^2 : P - \lim_{nm} \sum_k \sum_i a_{n,m,k,i} \sup \left\{ |x_{k,i} - \ell_1|^{p_{k,i}}, |y_{k,i} - \ell_2|^{p_{k,i}} \right\} = 0 \right\} , \]

for some \( \rho, \ell_1, \ell_2 > 0 \)
this means $W^2(A, p) = \left(2W(A, p), 2W(A, p)\right)$.

\[ W_0^2(A, p) = \left\{ (x, y) \in w^2 : P - \lim_{n,m} \sum_k \sum_l a_{n,m,k,l} \sup \left\{ |x_{k,l}|^{p_{k,l}}, |y_{k,l}|^{p_{k,l}} \right\} = 0 \right\} \]

for some $\rho > 0$

this means $W_0^2(A, p) = \left(2W_0(A, p), 2W_0(A, p)\right)$.

\[ W_\omega^2(A, p) = \left\{ (x, y) \in w^2 : \sup_{n,m} \sum_k \sum_l a_{n,m,k,l} \sup \left\{ |x_{k,l}|^{p_{k,l}}, |y_{k,l}|^{p_{k,l}} \right\} < \infty \right\} \]

for some $\rho > 0$

this means $W_\omega^2(A, p) = \left(2W_\omega(A, p), 2W_\omega(A, p)\right)$.

When $M(x) = x, M(y) = y, M(x, y) = (x, y)$ and $A = (C, 1,1)$, we obtain the sequence spaces $(C, 1,1, p)^2, (C, 1,1, p)_0^2$ and $(C, 1,1, p)_\omega^2$,

this means $(C, 1,1, p)^2 = ((C, 1,1, p), (C, 1,1, p))$

$(C, 1,1, p)_0^2 = ((C, 1,1, p)_0, (C, 1,1, p)_0)$ and

$(C, 1,1, p)_\omega^2 = ((C, 1,1, p)_\omega, (C, 1,1, p)_\omega)$.

According the definition of a double Orlicz function, we get the following:

A double Orlicz function $M$ satisfies the $\Delta_2$-condition for all values of $u_1, u_2$, if there exists a constant $K > 0$, such that $M_1(2u_1) \leq KM_1(u_1), u_1 \geq 0$,

$M_2(2u_2) \leq KM_2(u_2), u_2 \geq 0$, and consequently $M(2u_1, 2u_2) = (M_1(2u_1), M_2(2u_2)) \leq K(\min(u_1, u_2), u_1, u_2 \geq 0, then$ $M(2u_1, 2u_2) \leq KM(u_1, u_2), u_1, u_2 \geq 0$.

We know that when $M$ be an Orlicz function the sequence spaces

$W(M, p), W_0(M, p)$ and $W_\omega(M, p)$ were defined by Parashar and Choudhary[4] and generalized by Esi[1] defined as below:

\[ W(A, M, p) = \left\{ x \in w : \lim_m \sum_k a_{m,k} \left( M \left( \frac{|x_k - L|}{r} \right) \right)^{p_k} = 0, for some r > 0 \right\} \]

\[ W_0(A, M, p) = \left\{ x \in w : \lim_m \sum_k a_{m,k} \left( M \left( \frac{|x_k|}{r} \right) \right)^{p_k} = 0, for some r > 0 \right\} \]

\[ W_\omega(A, M, p) = \left\{ x \in w : \sup_m \sum_k a_{m,k} \left( M \left( \frac{|x_k|}{r} \right) \right)^{p_k} < \infty, for some r > 0 \right\} \]

where $A = (a_{m,k})$ is nonnegative regular matrix and $p = (p_k)$ is any sequence of positive real numbers (see Esi and Et[9], p.970).
Note. 2W(A, M₁, p), 2W₀(A, M₁, p) and 2W∞(A, M₁, p) represent the generalized double sequence spaces defined by Orlicz function M₁.

3. Main results

Theorem 3.1. Let p = (pᵢ,j) be bounded. The classes of sequences W²(A, M, p), W₀²(A, M, p) and W∞²(A, M, p) are linear spaces over the set of complex numbers ℂ².

Proof. Let x = (xᵢ,j) and a = (aᵢ,j) ∈ 2W₀(A, M₁, p), y = (yᵢ,j) and b = (bᵢ,j) ∈ 2W₀(A, M₂, p), and consequently (x, y) = (xᵢ,j, yᵢ,j), (a, b) = (aᵢ,j, bᵢ,j) ∈ W₀²(A, M, p) and let α, β ∈ ℂ. Then there exists some positive ρ₁, ρ₂ such that

\[ P - \lim_{n,m} \sum_{k,l} \sup \left\{ M_1 \left( \frac{|x_{i,j}|}{\rho_1} \right)^{p_{i,j}}, M_2 \left( \frac{|y_{i,j}|}{\rho_1} \right)^{p_{i,j}} \right\} = 0, \]

and

\[ P - \lim_{n,m} \sum_{k,l} \sup \left\{ M_1 \left( \frac{|a_{i,j}|}{\rho_2} \right)^{p_{i,j}}, M_2 \left( \frac{|b_{i,j}|}{\rho_2} \right)^{p_{i,j}} \right\} = 0. \]

Define ρ₃ = max(2|α|ρ₁, 2|β|ρ₂). Since M₁, M₂ are non-decreasing and convex so is M, where M = (M₁, M₂), therefore

\[ P - \lim_{n,m} \sum_{k,l} \sup \left\{ M_1 \left( \frac{|αx_{i,j} + βa_{i,j}|}{\rho_3} \right)^{p_{i,j}}, M_2 \left( \frac{|αy_{i,j} + βb_{i,j}|}{\rho_3} \right)^{p_{i,j}} \right\} \leq P - \]

\[ \lim_{n,m} \sum_{k,l} \sup \left\{ M_1 \left( \frac{|αx_{i,j} + βa_{i,j}|}{\rho_3} \right)^{p_{i,j}}, M_2 \left( \frac{|αy_{i,j} + βb_{i,j}|}{\rho_3} \right)^{p_{i,j}} \right\} \leq \]

\[ P - \lim_{n,m} \sum_{k,l} \sup \left\{ M_1 \left( \frac{|x_{i,j}|}{\rho_1} + M_1 \left( \frac{|a_{i,j}|}{\rho_2} \right)^{p_{i,j}}, M_2 \left( \frac{|y_{i,j}|}{\rho_1} \right)^{p_{i,j}} \right\} \leq \]

\[ M_2 \left( \frac{|b_{i,j}|}{\rho_2} \right)^{p_{i,j}} \right\} \leq \]

\[ CP - \lim_{n,m} \sum_{k,l} \sup \left\{ M_1 \left( \frac{|x_{i,j}|}{\rho_1} \right)^{p_{i,j}}, M_2 \left( \frac{|y_{i,j}|}{\rho_1} \right)^{p_{i,j}} \right\} \]

\[ \rightarrow 0 \text{ as } n,m \rightarrow \infty, \text{ where } C = \max (1, 2^{h-1}), H = \sup p_{i,j}. \]
Which implies that \((\alpha x + \beta a, \alpha y + \beta b) \in W^2_\delta(A, M, p)\).
Similarly we can prove that \(W^2(A, M, p)\) and \(W^\infty_\delta(A, M, p)\) are linear spaces.

**Theorem 3.2.** Let \(A\) be a nonnegative RH-regular summability matrix method and \(M\) be a double Orlicz function which satisfies \(\Delta_2\)-condition. Then
1) \(W^2_0(A, p) \subset W^2_\delta(A, M, p)\), where \(W^2_0(A, p) = (2W_0(A, p), 2W_0(A, p))\),
2) \(W^2_\delta(A, p) \subset W^2(A, M, p)\), where \(W^2(A, p) = (2W(A, p), 2W(A, p))\),
3) \(W^\infty_\delta(A, p) \subset W^\infty(A, M, p)\), where \(W^\infty(A, p) = (2W_\infty(A, p), 2W_\infty(A, p))\),
where \(0 < c = \inf p_{k,l} \leq \text{supp}_{k,l} = h < \infty\).

**Proof.** Let \(x = (x_{k,l}), y = (y_{k,l}) \in W^2(A, p)\) and \((x_{k,l}, y_{k,l}) \in W^2(A, p)\), then
\[ B_{n,m} = \sum_{k=1}^{n} \sum_{l=1}^{m} a_{n,m,k,l} \sup \{|x_{k,l} - \ell_1|^p_{k,l}, |y_{k,l} - \ell_2|^p_{k,l}\} \to 0 \text{ as } n, m \to \infty. \]

Let \(\epsilon > 0\) and choose \(\delta > 0\) such that
\[ M_1(t_1) < \frac{\epsilon \delta}{2} \text{ for } 0 \leq t_1 \leq \delta \text{ and } M_2(t_2) < \frac{\epsilon \delta}{2} \text{ for } 0 \leq t_2 \leq \delta, \text{ moreover} \]
\[ M(t_1, t_2) < \left(\frac{\epsilon \delta}{2}, \frac{\epsilon \delta}{2}\right) \text{ for } (0,0) \leq (t_1, t_2) \leq (\delta, \delta). \]
Write \(u_{k,l} = |x_{k,l} - \ell_1|\) and \(v_{k,l} = |y_{k,l} - \ell_2|\),
\[ b_{k,l} = |(u_{k,l}, v_{k,l})| = |(x_{k,l} - \ell_1, y_{k,l} - \ell_2)|, \text{ and consider} \]
\[ \sum_{k=1}^{n} \sum_{l=1}^{m} a_{n,m,k,l} \sup \{[M_1(u_{k,l})]^{p_{k,l}}, [M_2(v_{k,l})]^{p_{k,l}}\} = \]
\[ \sum_{k=1}^{n} \sum_{l=1}^{m} a_{n,m,k,l} \sup \{[M_1(u_{k,l})]^{p_{k,l}}, [M_2(v_{k,l})]^{p_{k,l}}\} \]
\[ \sum_{k=1}^{n} \sum_{l=1}^{m} a_{n,m,k,l} \sup \{[M_1(u_{k,l})]^{p_{k,l}}, [M_2(v_{k,l})]^{p_{k,l}}\} \]
\[ \sum_{k=1}^{n} \sum_{l=1}^{m} a_{n,m,k,l} \sup \{[M_1(u_{k,l})]^{p_{k,l}}, [M_2(v_{k,l})]^{p_{k,l}}\} \]
\[ \sum_{k=1}^{n} \sum_{l=1}^{m} a_{n,m,k,l} \sup \{[M_1(u_{k,l})]^{p_{k,l}}, [M_2(v_{k,l})]^{p_{k,l}}\} \]
Since \(M_1, M_2\) and \(M\) are continuous, therefore
\[ \sum_{k=1}^{n} \sum_{l=1}^{m} a_{n,m,k,l} \sup \{[M_1(u_{k,l})]^{p_{k,l}}, [M_2(v_{k,l})]^{p_{k,l}}\} < \epsilon \text{ for } k, l \geq \delta \text{ and for } u_{k,l}, v_{k,l} > \delta \text{ we use the fact} \]
\[ u_{k,l} < \frac{u_{k,l}}{\delta} < 1 + \frac{u_{k,l}}{\delta}, \quad v_{k,l} < \frac{v_{k,l}}{\delta} < 1 + \frac{v_{k,l}}{\delta}, \quad \text{and consequently} \]
\[ (u_{k,l}, v_{k,l}) < \left(\frac{u_{k,l}}{\delta}, \frac{v_{k,l}}{\delta}\right) < \left(1 + \frac{u_{k,l}}{\delta}, 1 + \frac{v_{k,l}}{\delta}\right). \]

Now \(u_{k,l} < 1 + \frac{u_{k,l}}{\delta}, v_{k,l} < 1 + \frac{v_{k,l}}{\delta} \text{ moreover } (u_{k,l}, v_{k,l}) < \left(1 + \frac{u_{k,l}}{\delta}, 1 + \frac{v_{k,l}}{\delta}\right), \)
so \(M_1, M_2\) and \(M\) are non-decreasing, it follows that
$M_1(u_{k,l}) < M_1\left(1 + \frac{u_{k,l}}{\delta}\right) = M_1\left(\frac{2}{2} + \frac{2}{2}\right) \cdot \frac{u_{k,l}}{\delta}$.

This implies that $M_1(u_{k,l}) < M_1\left(\frac{2}{2} + \frac{2}{2}\right) \cdot \frac{u_{k,l}}{\delta}$.

Then $M_1(u_{k,l}) < \frac{1}{2} M_1(2) + \frac{1}{2} M_1\left(\frac{2u_{k,l}}{\delta}\right)$. (Since $M_1$ is convex function).

and $M_2(v_{k,l}) < M_2\left(1 + \frac{v_{k,l}}{\delta}\right) = M_2\left(\frac{2}{2} + \frac{2}{2}\right) \cdot \frac{v_{k,l}}{\delta}$.

This implies that $M_2(v_{k,l}) < M_2\left(\frac{2}{2} + \frac{2}{2}\right) \cdot \frac{v_{k,l}}{\delta}$.

Then $M_2(v_{k,l}) < \frac{1}{2} M_2(2) + \frac{1}{2} M_2\left(\frac{2v_{k,l}}{\delta}\right)$. (Since $M_2$ is convex function).

Therefore $M(u_{k,l}, v_{k,l}) < M\left(1 + \frac{u_{k,l}}{\delta}, 1 + \frac{v_{k,l}}{\delta}\right) = M\left(\frac{2}{2} + \frac{2}{2}, \frac{2}{2}\right) \cdot \left(\frac{u_{k,l}}{\delta}, \frac{v_{k,l}}{\delta}\right)$.

This means $M(u_{k,l}, v_{k,l}) < M\left(\frac{2}{2} + \frac{2}{2}, \frac{2}{2}\right) \cdot \left(\frac{u_{k,l}}{\delta}, \frac{v_{k,l}}{\delta}\right)$.

Then

$M(u_{k,l}, v_{k,l}) < \frac{1}{2} M(2,2) + \frac{1}{2} M\left(\frac{2u_{k,l}}{\delta}, \frac{2v_{k,l}}{\delta}\right)$. (Since $M$ is convex function).

Since $M_1$, $M_2$ and $M$ satisfies $\Delta_2$-condition, therefore

$M_1(u_{k,l}) < \frac{1}{2} K \frac{u_{k,l}}{\delta} M_1(2) + \frac{1}{2} K \frac{u_{k,l}}{\delta} M_1(2) < K \frac{u_{k,l}}{\delta} M_1(2)$.

and $M_2(v_{k,l}) < \frac{1}{2} K \frac{v_{k,l}}{\delta} M_2(2) + \frac{1}{2} K \frac{v_{k,l}}{\delta} M_2(2) < K \frac{v_{k,l}}{\delta} M_2(2)$,

hence $M(u_{k,l}, v_{k,l}) < \frac{1}{2} K \left(\frac{u_{k,l}}{\delta}, \frac{v_{k,l}}{\delta}\right) M(2,2) + \frac{1}{2} K \left(\frac{u_{k,l}}{\delta}, \frac{v_{k,l}}{\delta}\right) M(2,2) < K \left(\frac{u_{k,l}}{\delta}, \frac{v_{k,l}}{\delta}\right) M(2,2)$.

This implies that

$\Sigma_{k=1}^n \Sigma_{l=1}^m a_{n,m,k,l} \sup \{ [M_1(u_{k,l})]^{p_{k,l}}, [M_2(v_{k,l})]^{p_{k,l}} \} < max(1, (K\delta^{-1} M_1(2) M_2(2))^n) B_{n,m}$,

therefore,

$\Sigma_{k=1}^n \Sigma_{l=1}^m a_{n,m,k,l} \sup \{ [M_1(|x_{k,l} - \ell_1|)]^{p_{k,l}}, [M_2(|y_{k,l} - \ell_1|)]^{p_{k,l}} \} =
\Sigma_{k=1}^n \Sigma_{l=1}^m a_{n,m,k,l} \sup \{ [M_1(|x_{k,l} - \ell_2|)]^{p_{k,l}}, [M_2(|y_{k,l} - \ell_2|)]^{p_{k,l}} \} + \Sigma_{k=1}^n \Sigma_{l=1}^m a_{n,m,k,l} \sup \{ [M_1(|x_{k,l} - \ell_1|)]^{p_{k,l}}, [M_2(|y_{k,l} - \ell_1|)]^{p_{k,l}} \} + \Sigma_{k=1}^n \Sigma_{l=1}^m a_{n,m,k,l} \sup \{ [M_1(|x_{k,l} - \ell_2|)]^{p_{k,l}}, [M_2(|y_{k,l} - \ell_2|)]^{p_{k,l}} \} \to 0$ as $n, m \to \infty$. 
Hence \((x_{k,l}, y_{k,l}) \in W^2(A, M, p)\).
Similarly we can proof that \(2[A, p]_0 \subset 2W_0(A, M, p), [A, p]^2_\infty \subset W^2_\infty(A, M, p)\).

**Theorem 3.3.**
(i) if \(0 < c = \inf p_{k,l} < p_{k,l} \leq 1\), then \(W^2(A, M, p) \subset W^2(A, M)\).

(ii) if \(1 \leq p_{k,l} \leq \sup p_{k,l} < \infty\), then \(W^2(A, M) \subset W^2(A, M, p)\).

Proof. (i) Let \(x = (x_{k,l}) \in 2W(A, M_1, p), y = (y_{k,l}) \in 2W(A, M_2, p)\), therefore 
\((x, y) = (x_{k,l} , y_{k,l}) \in W^2(A, M, p)\), since \(0 < c = \inf p_{k,l} < p_{k,l} \leq 1\), we get 
\[\sum_k \sum_l a_{n,m,k,l} \sup \left\{ \left[ M_1 \left( \frac{|x_{k,l} - x_1|}{\rho} \right) \right], \left[ M_2 \left( \frac{|y_{k,l} - y_2|}{\rho} \right) \right] \right\} \leq \]
\[\sum_k \sum_l a_{n,m,k,l} \sup \left\{ \left[ M_1 \left( \frac{|x_{k,l} - x_1|}{\rho} \right) \right]^{p_{k,l}}, \left[ M_2 \left( \frac{|y_{k,l} - y_2|}{\rho} \right) \right]^{p_{k,l}} \right\}, \]
thus \((x_{k,l}, y_{k,l}) \in W^2(A, M)\).

(ii) Let \(p_{k,l} \geq 1\) for each \(k, l\) and \(\sup p_{k,l} < \infty\) and let \(x \in 2W(A, M_1, p), y \in 2W(A, M_2, p)\), so \((x, y) = (x_{k,l}, y_{k,l}) \in W^2(A, M)\). Then for each \(\varepsilon \) \((0 < \varepsilon < 1)\), there exists a positive integer \(N\) such that 
\[\sum_k \sum_l a_{n,m,k,l} \sup \left\{ \left[ M_1 \left( \frac{|x_{k,l} - x_1|}{\rho} \right) \right], \left[ M_2 \left( \frac{|y_{k,l} - y_2|}{\rho} \right) \right] \right\} \leq \varepsilon < 1, \text{ for all } n, m \leq N. \]
This implies that 
\[\sum_k \sum_l a_{n,m,k,l} \sup \left\{ \left[ M_1 \left( \frac{|x_{k,l} - x_1|}{\rho} \right) \right]^{p_{k,l}}, \left[ M_2 \left( \frac{|y_{k,l} - y_2|}{\rho} \right) \right]^{p_{k,l}} \right\} \leq \]
\[\sum_k \sum_l a_{n,m,k,l} \sup \left\{ \left[ M_1 \left( \frac{|x_{k,l} - x_1|}{\rho} \right) \right], \left[ M_2 \left( \frac{|y_{k,l} - y_2|}{\rho} \right) \right] \right\}. \]
Thus we obtain \((x_{k,l}, y_{k,l}) \in W^2(A, M, p)\). this complete the proof. □

**Corollary 3.1.** Let \(A = (C, 1,1)\) and \(M\) be a double Orlicz function. Then  
\(W^2(A, M, p)\) satisfies \(\Delta_2\)-condition then 
\((C, 1,1)^2_0 \subset W^2_0(M), (C, 1,1)^2_\infty \subset W^2_\infty(M)\).

**Proof.** Consider \((C, 1,1)^0_0 = 2W_0\) and \((C, 1,1)^0_\infty = W^2_0\), 
\((C, 1,1) = 2W\) hence \((C, 1,1)^2 = W^2\), and \((C, 1,1)_\infty = W^2_\infty\), hence \((C, 1,1)^2_\infty = W^2_\infty\).
Let \(x = (x_{k,l}), y = (y_{k,l}) \in 2W, \text{ and } (x, y) = (x_{k,l}, y_{k,l}) \in W^2, \text{ then} \)}
\[ B_{n,m} = \frac{1}{nm} \sum_{k=1}^{n} \sum_{l=1}^{m} \sup \left\{ \left| x_{k,l} - \ell_1 \right|, \left| y_{k,l} - \ell_2 \right| \right\} \rightarrow 0 \text{ as } n, m \rightarrow \infty. \]

Let \( \epsilon > 0 \) and choose \( \delta \) with \( 0 < \delta < 1 \) such that \( M_1(t_1) < \epsilon \) for \( 0 \leq t_1 \leq \delta \), \( M_2(t_2) < \epsilon \) for \( 0 \leq t_2 \leq \delta \), hence \( M(t_1, t_2) < (\epsilon, \epsilon) \) for \( (0, 0) \leq (t_1, t_2) \leq (\delta, \delta) \).

Write \( u_{k,l} = |x_{k,l} - \ell_1| \) and \( v_{k,l} = |y_{k,l} - \ell_2| \).

Let \( b_{k,l} = |(u_{k,l}, v_{k,l})| = |(x_{k,l} - \ell_1, y_{k,l} - \ell_2)| \), and consider

\[
\sum_{k=1}^{n} \sum_{l=1}^{m} \sup \left\{ \left[ M_1(u_{k,l}), [M_2(v_{k,l})] \right] \right\} = \\
\sum_{k=1}^{n} \sum_{l=1 & b_{k,l} \leq \delta} \sup \left\{ \left[ M_1(u_{k,l}), [M_2(v_{k,l})] \right] \right\} + \\
\sum_{k=1}^{n} \sum_{l=1 & b_{k,l} > \delta} \sup \left\{ \left[ M_1(u_{k,l}), [M_2(v_{k,l})] \right] \right\}.
\]

Since \( M_1, M_2 \) and \( M \) are continuous, hence

\[
\sum_{k=1}^{n} \sum_{l=1 & b_{k,l} \leq \delta} \sup \left\{ \left[ M_1(u_{k,l}), [M_2(v_{k,l})] \right] \right\} < (nm)\epsilon \text{ and for } u_{k,l} > \delta, v_{k,l} > \delta, \text{ we use the fact}
\]

\[
u_{k,l} < \frac{u_{k,l}}{\delta} < 1 + \frac{u_{k,l}}{\delta}, \text{ and } v_{k,l} < \frac{v_{k,l}}{\delta} < 1 + \frac{v_{k,l}}{\delta},
\]

moreover \((u_{k,l}, v_{k,l}) < \left( \frac{u_{k,l}}{\delta}, \frac{v_{k,l}}{\delta} \right) < \left( 1 + \frac{u_{k,l}}{\delta}, 1 + \frac{v_{k,l}}{\delta} \right).\)

Since \( M_1, M_2 \) and consequently \( M \) are non-decreasing, it follows that

\[
M_1(u_{k,l}) < 1 + M_1 \left( \frac{u_{k,l}}{\delta} \right) = M_1 \left( \frac{2}{2} + \frac{2}{2} \cdot \frac{u_{k,l}}{\delta} \right),
\]

implies that \( M_1(u_{k,l}) < \frac{1}{2} M_1(2) + \frac{1}{2} M_1 \left( \frac{2}{2} + \frac{2}{2} \cdot \frac{u_{k,l}}{\delta} \right), \) (Since \( M_1 \) is convex function)

and \( M_2(v_{k,l}) < 1 + M_2 \left( \frac{v_{k,l}}{\delta} \right) = M_2 \left( \frac{2}{2} + \frac{2}{2} \cdot \frac{v_{k,l}}{\delta} \right), \)

thus \( M_2(v_{k,l}) < \frac{1}{2} M_2(2) + \frac{1}{2} M_2 \left( \frac{2}{2} + \frac{2}{2} \cdot \frac{v_{k,l}}{\delta} \right), \) (Since \( M_2 \) is convex function),

therefore, \( M(u_{k,l}, v_{k,l}) < 1 + M \left( \frac{u_{k,l}}{\delta}, \frac{v_{k,l}}{\delta} \right) = M \left( \frac{2}{2} + \frac{2}{2} \cdot \frac{u_{k,l}}{\delta}, \frac{2}{2} + \frac{2}{2} \cdot \frac{v_{k,l}}{\delta} \right) \cdot \left( \frac{u_{k,l}}{\delta}, \frac{v_{k,l}}{\delta} \right), \)

this means

\[
M(u_{k,l}, v_{k,l}) < \frac{1}{2} M(2,2) + \frac{1}{2} M \left( 2, \frac{2}{2} + \frac{2}{2} \cdot \frac{u_{k,l}}{\delta} \right). \] (Since \( M \) is convex function)

Since \( M = (M_1, M_2) \) satisfies \( \Delta_2 \)-condition, therefore

\[
M_1(u_{k,l}) < \frac{1}{2} K \frac{u_{k,l}}{\delta} M_1(2) + \frac{1}{2} K \frac{u_{k,l}}{\delta} M_1(2) < K \frac{u_{k,l}}{\delta} M_1(2),
\]
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and \( M_2(v_{k,l}) < \frac{1}{2} K \frac{v_{k,l}}{\delta} M_2(2) + \frac{1}{2} K \frac{v_{k,l}}{\delta} M_2(2) < K \frac{v_{k,l}}{\delta} M_2(2) \),

therefore \( M(u_{k,l}, v_{k,l}) < \frac{1}{2} K \left( \frac{u_{k,l}}{\delta}, \frac{v_{k,l}}{\delta} \right) M(2,2) + \frac{1}{2} K \left( \frac{u_{k,l}}{\delta}, \frac{v_{k,l}}{\delta} \right) M(2,2) < K \left( \frac{u_{k,l}}{\delta}, \frac{v_{k,l}}{\delta} \right) M(2,2) \).

This implies that \( \sum_{k=1}^{n} \sum_{l=1}^{m} \sup \left[ \left( M_1(u_{k,l}), M_2(v_{k,l}) \right) \right] < \max(1, K \delta^{-1} M(2,2)) B_{n,m} \),

this means \( \sum_{k=1}^{n} \sum_{l=1}^{m} \sup \left[ M_1 \left( |x_{k,l} - \ell_1|, |y_{k,l} - \ell_2| \right) \right] = \sum_{k=1}^{n} \sum_{l=1}^{m} \sup \left[ M_1 \left( |x_{k,l} - \ell_1|, |y_{k,l} - \ell_2| \right) \right] + \sum_{k=1}^{n} \sum_{l=1}^{m} \sup \left[ M_2 \left( |x_{k,l} - \ell_1|, |y_{k,l} - \ell_2| \right) \right] \rightarrow 0 \) as \( n, m \rightarrow \infty \).

Thus \( (x_{k,l}, y_{k,l}) \in W^2(M) \).

Similarly we can prove that \( (C, 1, 1)_{0} \subset W^2_0(M), (C, 1, 1)_{\infty} \subset W_\infty^2(M) \).

References


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