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On a Nonlinear Riccati Matrix Eigenproblem

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Abstract

In this paper, a nonlinear Riccati matrix eigenproblem is associated with the algebraic Riccati matrix equation.This idea follows in a similar way as the Lyapunov eigenproblem is associated with the algebraic Lyapunov matrix equation as discussed in [10]. But, the derivation here is more difficult. A computational algorithm is established that computes the positive definite solution R of the new nonlinear EP, and the convergence of the algorithm is analyzed. The solution matrix R is used to determine the optimal feedback matrix of a Riccati control problem. Some computational tests on an initially unstable mechanical system with asymptotically stable closed-loop system matrix are carried out. Further, the results are compared with the case when, instead of the matrix R, the solution matrix P of the algebraic Riccati matrix equation is used. The results indicate that the feedback matrix constructed by means of R is superior to that constructed by means of P. In addition, the solution vector of the optimal control problem in the weighted norm $\|\cdot\|_R$ shows vibration suppression, a feature that is not obtained by using the norm $\|\cdot\|_P$.

Keywords: nonlinear Riccati matrix eigenproblem; algebraic Riccati matrix equation; Riccati control problem; closed-loop system; dynamical system; vibration elimination in weighted norm

1 Introduction

The solution of a control problem

$$\begin{aligned} &\dot{x}(t) = Ax(t) + Bu(t), x(t_0) = x_0 \\ &y(t) = Cx(t) \end{aligned}$$

is well known under the constraint that the integral

$$\int_{t_0}^{t_1} [y^T(t)Q_y y(t) + u^T(t)Q_u u(t)]dt$$

with given positive definite matrices Q_y and Q_u attains its minimum for $t_1 \to \infty$. This constraint leads to the nonlinear algebraic Riccati matrix equation

$$-P(BQ_u^{-1}B^T)P + A^TP + PA = -C^TQ_yC$$

whereby the feedback gain matrix $F = Q_u^{-1}B^TP$ is obtained leading to the optimal steady-state control law $u(t) = -Fx(t)$ (see [13], [8], and [15]).

The aim of the present paper is to associate with the above Riccati equation a nonlinear eigenproblem whose solution matrix R is employed in the feedback gain matrix F instead of the solution matrix P of the algebraic Riccati equation.

We address some questions of special significance that arise in connection with the nonlinear Riccati eigenproblem. The most important questions are that of the convergence proof of the proposed iterative solution method for the new eigenproblem and that of the existence proof of a solution. A further question is on how to mathematically justify the feedback law in the considered Riccati control problem since we have not the integral optimality criterion available like in the case of the algebraic Riccati matrix equation. To all these questions, answers will be given.

Numerical tests on an initially unstable mechanical system with asymptotically stable closed-loop system matrix are carried out showing the superiority of the new strategy over the previous one.

The paper is structured as follows. In Section 2, optimal control based on the algebraic Riccati equation is revisited. First, the known unconstrained integral criterion is recapitulated. Then, as a new result, the optimal feedback law is obtained by an equivalent constrained derivative criterion. The advantage of this derivative criterion is that it can be carried over to optimal control based on the Riccati matrix eigenproblem. Then, in Section 3, the new nonlinear eigenproblem is derived corresponding to the algebraic Riccati matrix equation. In Section 4, optimal control based on the nonlinear matrix Riccati eigenproblem is studied. Section 5 describes an algorithm for the solution of the nonlinear Riccati matrix eigenproblem. In Section 6, the convergence of the algorithm and the existence of the solution of the Riccati eigenproblem are investigated. Section 7 presents an application to a dynamical system with vibration behavior. Finally, in Section 8, conclusions are drawn followed by the references.

2 Optimal control based on the algebraic Riccati matrix equation

In this section, we show that the known unconstrained integral criterion to obtain the optimal feedback law can be replaced by a constrained derivative criterion leading to the same result.

2.1 Unconstrained integral optimality criterion

We consider a dynamical system described by the equations

$$\dot{x}(t) = Ax(t) + Bu(t),\ x(t_0) = x_0 \tag{1}$$

$$y(t) = Cx(t), \tag{2}$$

(see [13, (3-213)]) where $A \in I\!R^{n\times n}$ is called the *system matrix*, $B \in I\!R^{n\times m}$ the *input matrix*, $C \in I\!R^{p\times n}$ the *output matrix*, $u(t) \in I\!R^m$ the *input vector* or *control vector*, $x(t) \in I\!R^n$ the *state vector*, and $y(t) \in I\!R^p$ the *output vector* or *controlled vector*; for the denotations, see [4].

The problem (1), (2) is also called a *control system.*

In addition to (1), (2), we consider the criterion (see [13, (3-214)])

$$\begin{aligned} & \textstyle\int_{t_0}^{t_1} [y^T(t)Q_y y(t) + u^T(t)Q_u u(t)]dt \\ = & \textstyle\int_{t_0}^{t_1} [(Q_y y(t), y(t)) + (Q_u u(t), u(t))]dt \end{aligned} \tag{3}$$

for $t_1 \to \infty$, where $(\cdot,\cdot)$ denotes the Euclidian inner product and where $Q_y \in I\!R^{p\times p}$ and $Q_u \in I\!R^{m\times m}$ are positive definite.

One has the following result (see [1], [5], [8] and [9], [13], [14], and [15]).

Theorem 1:

Consider the time-invariant control problem for the system (1), (2). Further, let the pair $[A,B]$ be stabilizable *and the pair $[A,C]$ be* observable.

Then, the control $u^*(\cdot)$ that minimizes the quadratic cost functional

$$J(x_0; u(\cdot)) = \int_{t_0}^{\infty} [(Q_y y(t), y(t)) + (Q_u u(t), u(t))]dt \tag{4}$$

is given by the linear feedback law

$$u^*(t) = -Q_u^{-1} B^T P\, x(t) = -Fx(t) \tag{5}$$

with

$$F = Q_u^{-1} B^T P \tag{6}$$

where P is the unique positive definite solution of the algebraic Riccati matrix equation

$$-P(BQ_u^{-1}B^T)P + A^TP + PA = -C^TQ_yC. \tag{7}$$

In addition, P has the property

$$J(x_0; u^*(\cdot)) = \min_{u(\cdot)} J(x_0; u(\cdot)) = x_0^T P\, x_0 = (P\, x_0, x_0). \tag{8}$$

◇

Remark:

It should be noted that the result holds true for observility replaced by detectability except that P can then only be shown to be positive semidefinite. As we will use P to define an energy norm later on, we need here the positive definiteness and hence the observability assumption. ◇

Remark: The algebraic Riccati matrix equation is nonlinear and inhomogeneous; it can be *generalized to complex matrices* by

$$-P(BQ_u^{-1}B^*)P + A^*P + PA = -C^*Q_yC. \tag{9}$$

Using the *substitutions*

$$\left.\begin{array}{l} BQ_u^{-1}B^* \to G \\ \\ C^*Q_yC \to Q \end{array}\right\} \tag{10}$$

leads to the *generic form of the algebraic Riccati matrix equation*

$$-PGP + A^*P + PA = -Q. \tag{11}$$

We mention that, for $G = 0$, one obtains the *Lyapunov matrix equation*

$$A^*P + PA = -Q,$$

which is a linear inhomogeneous matrix equation. ◇

Closed-loop form of the control problem

One has, with $u(t) = u^*(t)$,

$$u(t) = -Fx(t) = -Q_u^{-1}B^TPx(t)$$

leading in the more general case of complex matrices to

$$Bu(t) = -BQ_u^{-1}B^*Px(t) = -GPx(t). \tag{12}$$

Inserting this into (1) implies

$$\dot{x}(t) = (A - GP)x(t),\ x(t_0) = x_0, \tag{13}$$

where P is the solution of (7) resp. (11). This is the closed-loop form of the control problem.

2.2 Constrained derivative optimality criterion

In this section, we shall first show why the unconstrained integral criterion is not applicable in the case of the Riccati matrix eigenproblem. By investigating the integrand for the associated functional, we are led to consider an optimality criterion for a certain derivative.

Now, we take a closer look to the integrand of the criterion (4) with the optimal control vector (5). We have

$$\begin{aligned} & y^T Q_y\, y(t) + u^T(t)\, Q_u\, u(t) \\ = \;& x^T(t)\, C^T Q_y C\, x(t) + x^T(t)\, (-Q_u^{-1} B^T P)^T Q_u (-Q_u^{-1} B^T P)\, x(t) \\ = \;& x^T(t)\, [\, Q + PGP\,]\, x(t)\,. \end{aligned} \tag{14}$$

Turning to the more general complex case and subtracting PGP on both sides of (11), leads to

$$\begin{aligned} & (Q_y y(t), y(t)) + (Q_u u(t), u(t)) \\ = \;& -([\,-Q - PGP\,]\, x(t), x(t)) \\ = \;& -([\,-2\, PGP + A^* P + P\, A]\, x(t), x(t)) \\ = \;& -([\,(A - G\, P)^*\, P + P\,(A - G\, P)]\, x(t), x(t)) \\ = \;& -\frac{d}{dt}\,(P\, x(t), x(t)) =: -\frac{d}{dt}\, \|x(t)\|_P^2, \end{aligned}$$

that is,

$$(Q_y y(t), y(t)) + (Q_u u(t), u(t)) = -\frac{d}{dt}\, \|x(t)\|_P^2. \tag{15}$$

Since in the case of the Riccati matrix eigenproblem we do not have the matrix Q, we do not have the quantity $y^T\, Q_y\, y(t)$ and thus cannot minimize the quantity $J(x_0; u(\cdot))$. But, relation (15) leads to the idea of minimizing the quantity

$$\frac{d}{dt}\, \|x(t)\|_P^2 \tag{16}$$

instead of the quantity

$$\int_{t_0}^{\infty} [(Q_y y(t), y(t)) + (Q_u u(t), u(t))] dt$$

with the optimal value

$$J(x_0; u^*(\cdot)) = \int_{t_0}^{\infty} [-\frac{d}{d\tau}\, \|x(\tau)\|_P^2] d\tau = \|x_0\|_P^2 = (P\, x_0, x_0)$$

if the closed-loop system is asymptotically stable.

The question is as to whether the minimization of (16) will lead to the same optimal feedback law (5), (6) as for the criterion (4).

In the sequel, we will show that this indeed the case.

So, let (1), (2) be given and $P \in \mathbb{C}^{n\times n}$ be any positive definite matrix. We consider the functional

$$j(u) := \frac{d}{dt}\,\|x(t)\|_P^2$$

that depends implicitly on u. One obtains

$$\begin{aligned} j(u) &= \frac{d}{dt}\,\|x(t)\|_P^2 = \frac{d}{dt}\,(P\,x(t), x(t)) = (P\,\dot{x}, x) + (P\,x, \dot{x}) \\ &= (P\,[A\,x + B\,u], x) + (P\,x, A\,x + B\,u) \\ &= (P\,A\,x, x) + (P\,B\,u, x) + (P\,x, A\,x) + (P\,x, B\,u) \\ &= ([A^*\,P + P\,A]\,x, x) + (u, B^*\,P\,x) + (B^*\,P\,x, u) \\ &= ([A^*\,P + P\,A]\,x, x) + (u, Q_u^{-1}\,B^*\,P\,x)_{Q_u} + (Q_u^{-1}\,B^*\,P\,x, u)_{Q_u}\,. \end{aligned}$$

Now, we minimize $j(u)$ subject to $u \in \mathbb{C}^n$ with $\|u\|_{Q_u} \le \|Q_u^{-1}\,B^*\,P\,x\|_{Q_u}$.

The minimum is attained for $u^* = -Q_u^{-1}\,B^*\,P\,x$, so that relation (12) follows. Thus, we see that the constrained derivative minimization criterion leads to the same result as the unconstrained integral minimization criterion. Further, we obtain

$$\begin{aligned} &\min_{\substack{u\in\mathbb{C}^n \\ \|u\|_{Q_u} \le \|Q_u^{-1}\,B^*\,P\,x\|_{Q_u}}} j(u) = j(u^*) \\ &= ([A^*\,P + P\,A]\,x, x) - 2\,(Q_u^{-1}\,B^*\,P\,x, Q_u^{-1}\,B^*\,P\,x)_{Q_u} \\ &= ([A^*\,P + P\,A]\,x, x) - 2\,(P\,G\,P\,x, x) \\ &= ([(A - G\,P)^*\,P + P\,(A - G\,P))]\,x, x)\,. \end{aligned} \tag{17}$$

3 Derivation of the nonlinear Riccati matrix eigenproblem

In this section, we describe the transition from the algebraic Riccati matrix equation with positive definite solution matrix P to a corresponding nonlinear Riccati matrix eigenproblem with nonlinearity in the eigenmatrix and positive definite solution matrix R. This transition is not self-evident and is done in several steps exhibiting the idea behind each step. The procedure is similar to that employed in [10] for the linear case, that is, when $G = 0$. The first two steps are of preliminary kind paving the way for the third step.

Step 1:

This step is similar to the transition from $A^*P + PA = -S$ to $A^*P + PA = \rho R$ in [10] in that we denote the solution of (11) by R instead of P and by making *substitution 1*

$$-Q \to \rho R. \tag{18}$$

This leads to the possible nonlinear Riccati matrix eigenproblem

$$-RGR + A^*R + RA = \rho R. \tag{19}$$

This is a nonlinear eigenproblem with nonlinearity in the eigenmatrix R. This eigenproblem is strongly different from the nonlinear EP $(\lambda^2 M + \lambda B + K)w = 0$ with nonlinearity in the eigenvalue.

Remark: In the special case $G = 0$, we get back the eigenproblem $A^*R + RA = \rho R$ studied in [10]. $\diamond$

We suppose for the moment that the EP (19) has a solution pair (R, ρ), where R is positive definite and $\rho < 0$.

As in [10], we investigate the solution $x(t)$ in the weighted norm $\|\cdot\|_R$ generated by the weighted scalar product $(\cdot,\cdot)_R$.

According to (13) with P replaced by R and since $G^* = G$,

$$\begin{aligned}
\tfrac{d}{dt}\|x(t)\|_R^2 &= \tfrac{d}{dt}(R\,x(t), x(t)) = (R\,\dot{x}(t), x(t)) + (R\,x(t), \dot{x}(t)) \\
&= (R\,[A - G\,R]\,x(t), x(t)) + (R\,x(t), [A - G\,R]\,x(t)) \\
&= ([-2\,RGR + A^*R + RA]\,x(t), x(t)) \\
&= ([-RGR + A^*R + RA]\,x(t), x(t)) - (RGR\,x(t), x(t)) \\
&= (\rho Rx(t), x(t)) - (RGR\,x(t), x(t)) \\
&= \rho\|x(t)\|_R^2 - (RGR\,x(t), x(t)),
\end{aligned}$$

that is,

$$\frac{d}{dt}\|x(t)\|_R^2 = \rho\|x(t)\|_R^2 - (RGR\,x(t), x(t)) \tag{20}$$

leading to

$$\frac{d}{dt}\|x(t)\|_R^2 \le \rho\|x(t)\|_R^2 \tag{21}$$

instead of the desired

$$\frac{d}{dt}\|x(t)\|_R^2 = \rho\|x(t)\|_R^2. \tag{22}$$

Step 2:

In order to obtain (22) instead of (20), we have to replace the substitution (18) by *Substitution 2*

$$-Q \to \rho R + RGR. \tag{23}$$

Then instead of (19), we get

$$-RGR + A^*R + RA = \rho R + RGR \tag{24}$$

or

$$(A - G\,R)^*R + R(A - G\,R) = \rho R. \tag{25}$$

If the nonlinear EP (19) has a solution pair (R, ρ) with positive definite R and negative ρ, then evidently relation (22) holds.

Remark: Problems (19) or (25) are eigenproblems on their own. But, we do not investigate their solutions. Instead, we modify (25) in a way resembling that in [10] and study the solution of the emerging EP.

Step 3:

In [10], in the first step, the EP $A^*R + RA = \rho R$ was considered. Then, in a second step, this EP was replaced by the EPs $A^*R_i + R_iA = \rho_i R_i$, $i = 1, \ldots, n$, where, in the case of diagonalizable matrices A, the positive semi-definite eigenmatrices R_i were given by $R_i = u_i^* u_i$ with u_i^* being the (right column) eigenvectors of the EP $A^*u_i^* = \lambda(A^*)u_i^* = \overline{\lambda_i(A)}u_i^*$ and u_i being the (left row) eigenvectors of the EP $u_iA = \lambda_i(A)u_i$. The matrix $R := \sum_{i=1}^n R_i$ is positive definite.

Now, we want to carry over this method to the EP (25).

If we replace in (22) the matrix R by R_i, we obtain

$$\frac{d}{dt}\|x(t)\|_{R_i}^2 = \rho_i \|x(t)\|_{R_i}^2,\ i = 1, 2, \ldots, n \tag{26}$$

valid for $A^*R_i + R_iA = \rho_i R_i$. The corresponding identities for $\dot{x}(t) = (A - GR)x(t)$ according to (13), with R instead of P in the feedback matrix, read

$$\begin{aligned}
\tfrac{d}{dt}\|x(t)\|_{R_i}^2 &= \tfrac{d}{dt}(R_i\,x(t), x(t)) = (R_i\,\dot{x}(t), x(t)) + (R_i\,x(t), \dot{x}(t)) \\
&= (R_i\,(A - G\,R)\,x(t), x(t)) + (R_i\,x(t), (A - G\,R)\,x(t)) \qquad (27) \\
&= (((A - G\,R)^*R_i + R_i(A - G\,R))\,x(t), x(t)).
\end{aligned}$$

Thus, if we want to obtain (26), we are led to the equations

$$(A - G\,R)^*R_i + R_i(A - G\,R) = \rho_i R_i,\ i = 1, 2, \ldots, n \tag{28}$$

$$R := \sum_{k=1}^n R_k. \tag{29}$$

Relations (28), (29) define a system of coupled nonlinear Riccati matrix eigenproblems.

Then with (28), relation (27) turns into (26) and further,

$$\|x(t)\|_R^2 = \sum_{i=1}^{n} \|x(t)\|_{R_i}^2 \tag{30}$$

(as in [10] for $A^* R_i + R_i A = \rho_i R_i$) because of (29).

If we replace P in (12), (13) by R in (28), (29), then

$$Bu(t) = -GRx(t), \tag{31}$$

and we obtain the *closed-loop system*

$$\dot{x}(t) = (A - GR)x(t),\ x(t_0) = x_0. \tag{32}$$

◇

4 Optimal control based on the Riccati matrix eigenproblem

In this section, both the integral criterion and the derivative criterion are carried over from the case when the optimal control is based on the algebraic Riccati matrix equation to the case when it is based on the Riccati matrix eigenproblem.

4.1 Feedback law by analogy

In Section 3, we have replaced the algebraic Riccati matrix equation (11), i.e.,

$$-P\,G\,P + A^*\,P + P\,A = -Q$$

or

$$-2\,P\,G\,P + A^*\,P + P\,A = -(Q + P\,G\,P)$$

resp.

$$(A - G\,P)^*\,P + P\,(A - G\,P) = -(Q + P\,G\,P) \tag{33}$$

by the nonlinear Riccati matrix eigenproblem (28), (29). The feedback law (12) was then taken by analogy as

$$B\,u(t) = -G\,R\,x(t)\,, \tag{34}$$

that is, by replacing P in (12) by R. However, since the feedback law for R was obtained only by analogy, it is not clear in what sense it is optimal.

4.2 Feedback law by minimization criterion

The situation is different for the derivative minimization criterion. Replacing P in (16) by R and considering now

$$j(u) := \frac{d}{dt}\,\|x(t)\|_R$$

with R from (28), (29), we obtain

$$\min_{\substack{u\in\mathbb{C}^n \\ \|u\|_{Q_u}\le\|Q_u^{-1}B^*Rx\|_{Q_u}}} j(u) = j(u^*)$$

for

$$u^* = -Q_u^{-1}\,B^*\,R\,x$$

in the same way as in Section 2.2 leading to the optimal feedback law (34).

So, here, we not only have a formal analogy of the cases P and R, but also a mathematical justification for the feedback law (34) in the case R.

5 Algorithm for the solution of the nonlinear Riccati matrix eigenproblem

In this section, we present an algorithm for the solution of the nonlinear Riccati matrix eigenproblem. The idea behind this algorithm is that the nonlinear EP can be formally cast into the form of the Lyapunov EP from [10] when the closed-loop system matrix is used and when our knowledge from [10] is employed how to solve the Lyapunov EP. We apply an iterative method.

5.1 Initial guesses $\rho_i^{(0)}$, $R_i^{(0)}$, $R^{(0)}$ for ρ_i, R_i, R in problem (28), (29)

We have in mind the case where the real parts of the eigenvalues $\lambda_i(A)$, $i = 1,\ldots,n$ are all positive.

To determine $\rho_i^{(0)}$, $R_i^{(0)}$, $R^{(0)}$, we choose the substitution

$$G \to \gamma R^{-1}$$

with an appropriate $\gamma > 0$. Then, with the identity matrix I, (28), (29) are replaced by

$$(A-\gamma I)^* R_i^{(0)} + R_i^{(0)}(A-\gamma I) = \rho_i^{(0)} R_i^{(0)},\ i = 1,\ldots,n \qquad (35)$$

$$R^{(0)} = \sum_{k=1}^{n} R_k^{(0)}. \qquad (36)$$

Similarly as in [10],

$$\rho_i^{(0)} = 2\,Re\,\lambda_i(A-\gamma I) = 2\,Re\,\lambda_i(A) - 2\gamma. \tag{37}$$

If we choose γ such that

$$Re\,\lambda_i(A) < \gamma,\ i = 1,\ldots,n, \tag{38}$$

then

$$\rho_i^{(0)} < 0,\ \ i = 1,\ldots,n. \tag{39}$$

Further, from [10], we have that

$$R_i^{(0)} = u_i^{*(0)} u_i^{(0)}, \tag{40}$$

where

$$(A-\gamma I)^*\, u_i^{*(0)} = \lambda_i((A-\gamma I)^*)\, u_i^{*(0)},\ \ i = 1,\ldots,n \tag{41}$$

and $R^{(0)} = \sum_{k=1}^n R_k^{(0)}$ is positive definite if A is diagonalizable.

An alternative to (32) is given by

$$(A - G\,I)^* R_i^{(0)} + R_i^{(0)}(A - G\,I) = \rho_i^{(0)} R_i^{(0)},\ \ i = 1,\ldots,n \tag{42}$$

Both (35) and (42) will lead to the same results in Section 8.6.

5.2 Determination of the further iterates $\rho_{\mathbf{i}}^{(\mathbf{j+1})}$, $\mathbf{R}_{\mathbf{i}}^{(\mathbf{j+1})}$, $\mathbf{R}^{(\mathbf{j+1})}$ from $\rho_{\mathbf{i}}^{(\mathbf{j})}$, $\mathbf{R}_{\mathbf{i}}^{(\mathbf{j})}$, $\mathbf{R}^{(\mathbf{j})}$, $\mathbf{j} = \mathbf{0}, \mathbf{1}, \mathbf{2}, \ldots$

This is done under the assumption that all matrices $A - GR^{(j)}$ are diagonalizable.

The iterates $\rho_i^{(j+1)}$, $R_i^{(j+1)}$, $R^{(j+1)}$ are computed from

$$[A - GR^{(j)}]^* R_i^{(j+1)} + R_i^{(j+1)}[A - GR^{(j)}] = \rho_i^{(j+1)} R_i^{(j+1)},\ \ i = 1,\ldots,n \tag{43}$$

or

$$-\,[R^{(j)} G R_i^{(j+1)} + R_i^{(j+1)} G R^{(j)}] + A^* R_i^{(j+1)} + R_i^{(j+1)} A = \rho_i^{(j+1)} R_i^{(j+1)}, \tag{44}$$

$i = 1,\ldots,n$ with

$$R^{(j+1)} = \sum_{k=1}^{n} R_k^{(j+1)} \tag{45}$$

by the method described in [10] for $A^* R_i + R_i A = \rho_i R_i,\ i = 1,\ldots,n$ with $R = \sum_{k=1}^n R_k$, when replacing A by $A - GR^{(j)}$, R_i by $R_i^{(j+1)}$, and ρ_i by $\rho_i^{(j+1)}$ in (43) for $j = 0, 1, 2, \ldots$.

6 Convergence of algorithm and existence of solution for Riccati matrix eigenproblem

In this section, we obtain a convergence result for a subsequence $I\!N' \subset I\!N$. This also leads to the existence proof for the solution to the nonlinear Riccati eigenproblem.

Consider the problem (28), (29), where $A \in I\!R^{n\times n}$ and $G \in I\!R^{p\times p}$ is positive semi-definite. We seek positive semi-definite solution matrices $R_i \in \mathbb{C}^{n\times n},\ i = 1,\ldots,n$ such that $R = \sum_{i=1}^{n} R_i$ is positive definite.

First, we prove

Lemma 2: (Boundedness of some sequences)

Let $A \in I\!R^{n\times n}$, *let* $G \in I\!R^{p\times p}$ *be positive semi-definite, and let* $A^{(j)} := A - G\,R^{(j)},\ j = 0,1,2,\ldots$ *be diagonalizable. Further, let* $u_i{}^*(A^{(j)*})$ *be the normed (right column) eigenvectors of* $A^{(j)*}$ *and* $\lambda_i(A^{(j)*})$ *be the associated eigenvalues as well as* $u_i(A^{(j)})$ *be the normed (left row) eigenvectors of* $A^{(j)}$ *and* $\lambda_i(A^{(j)})$ *be the pertinent eigenvalues, that is,*

$$A^{(j)*}\,u_i{}^*(A^{(j)*}) = \lambda_i(A^{(j)*})\,u_i{}^*(A^{(j)*}), \tag{46}$$

and

$$u_i(A^{(j)})\,A^{(j)} = \lambda_i(A^{(j)})\,u_i(A^{(j)}), \tag{47}$$

$i = 1,\ldots,n,\ j = 0,1,2,\ldots.$ *Moreover, let* $\|\cdot\|$ *be any submultiplicative matrix norm.*

Then, there exists a constant $c_R > 0$ *such that*

$$\left.\begin{array}{rcl} |\lambda_i(A^{(j)*})| & \leq & c_R,\ i = 1,\ldots,n \\ |\lambda_i(A^{(j)})| & \leq & c_R,\ i = 1,\ldots,n \\ |\rho_i^{(j+1}| & \leq & c_R,\ i = 1,\ldots,n \\ \|R_i^{(j+1)}\| & \leq & c_R,\ i = 1,\ldots,n \\ \|R^{(j)}\| & \leq & c_R, \end{array}\right\} \tag{48}$$

$j = 0,1,2,\ldots.$

Proof: One has

$$\begin{array}{rcl} |\lambda_i(A^{(j)*})| = |\overline{\lambda_i(A^{(j)})}| = |\lambda_i(A^{(j)})| & \leq & \|A^{(j)}\| = \|A - G\,R^{(j)}\| \\ & \leq & (\|A\| + \|G\|\,\sum_{i=1}^{n}\|R_i^{(j)}\|)\,. \end{array} \tag{49}$$

Now, from [10, Theorem 5],

$$\begin{aligned} R_i^{(j+1)} &= {u_i}^*(A^{(j)*})\, u_i(A^{(j)}) \\ &= \begin{bmatrix} {u_{i,1}}^*(A^{(j)*}) \\ {u_{i,2}}^*(A^{(j)*}) \\ \vdots \\ {u_{i,n}}^*(A^{(j)*}) \end{bmatrix} \Big[u_{i,1}(A^{(j)}),\ u_{i,2}(A^{(j)}),\ \ldots,\ u_{i,n}(A^{(j)})\Big] \\ &= \Big({u_{i,k}}^*(A^{(j)*})\, u_{i,s}(A^{(j)})\Big)_{k,s=1,\ldots,n}\,. \end{aligned} \tag{50}$$

This entails

$$\begin{aligned} |R_i^{(j+1)}|_\infty &:= \max_{k,s=1,\ldots,n} |(R_i^{(j+1)})_{ks}| = \max_{k,s=1,\ldots,n} |{u_{i,k}}^*(A^{(j)*})|\, |u_{i,s}(A^{(j)})| \\ &\le \|{u_i}^*(A^{(j)*})\|_2\, \|u_i(A^{(j)})\|_2 = 1\,, \end{aligned} \tag{51}$$

$j = 0, 1, 2, \ldots$. Due to the equivalence of norms in finite-dimensional spaces, there exists a positive constant $\tilde{c}_R$ such that

$$\|R^{(j+1)}\| \le \sum_{i=1}^{m} \|R_i^{(j+1)}\| \le \tilde{c}_R,\ j = 0, 1, 2, \ldots\ . \tag{52}$$

From (49) and (52),

$$|\lambda_i(A^{(j)*})| = |\overline{\lambda_i(A^{(j)})}| = |\lambda_i(A^{(j)})| \le \|A\| + \|G\|\, \tilde{c}_R,\ j = 0, 1, 2, \ldots\ . \tag{53}$$

Let

$$c_R := \max\{\tilde{c}_R,\ 2\,(\|A\| + \|G\|\,\tilde{c}_R),\ \|R^{(0)}\|\}\,. \tag{54}$$

Then, on the whole, (48) follows. ◇

Further, we have

Theorem 3: (Existence of convergent subsequence; existence of solution)
Let $A \in I\!R^{n\times n}$*, let* $G \in I\!R^{p\times p}$ *be positive semi-definite. Let* $A^{(j)} = A - G\,R^{(j)},\ j = 0, 1, 2, \ldots$ *be diagonalizable.*

Then, there exists a subsequence $I\!N' \subset I\!N$ *as well as elements* ${u_i}^* := {u_i}^*((A - G\,R)^*),\ u_i := u_i\,(A - G\,R),\ \lambda_i((A - G\,R)^*) = \overline{\lambda_i(A - G\,R)},\ \lambda_i((A -$

$G\,R)$, ρ_i, R_i, and S_i, $i=1,\dots,n$ such that, as ($j \longrightarrow \infty$),

$$\left.\begin{array}{lll} {u_i}^*(A^{(j)*}) & \longrightarrow & {u_i}^*\ (j\in I\!N'),\ i=1,\dots,n \\ u_i(A^{(j)}) & \longrightarrow & u_i\ (j\in I\!N'),\ i=1,\dots,n \\ \lambda_i(A^{(j)*}) & \longrightarrow & \lambda_i((A-G\,R)^*),\ i=1,\dots,n \\ \lambda_i(A^{(j)}) & \longrightarrow & \lambda_i((A-G\,R),\ i=1,\dots,n \\ \rho_i^{(j+1} & \longrightarrow & \rho_i\ (j\in I\!N'),\ i=1,\dots,n \\ R_i^{(j)} & \longrightarrow & R_i\ (j\in I\!N'),\ i=1,\dots,n \\ R_i^{(j+1)} & \longrightarrow & S_i\ (j\in I\!N'),\ i=1,\dots,n \end{array}\right\} \tag{55}$$

where the matrices R_i and S_i are positive semi-definite.

As a consequence,

$$(A-G\,R)^*\cdot {u_i}^* = \overline{\lambda_i(A-G\,R)}\,{u_i}^* \tag{56}$$

and

$$u_i\cdot(A-G\,R) = \lambda_i(A-G\,R)\,u_i, \tag{57}$$

so that $\lambda_i((A-G\,R)^*) = \overline{\lambda_i(A-G\,R)}$ are the eigenvalues of $(A-G\,R)^*$ as well as ${u_i}^* := {u_i}^*((A-G\,R)^*)$ associated normed (right column) eigenvectors and $\lambda_i((A-G\,R)$ the eigenvalues of $A-G\,R$ as well as $u_i := u_i\,(A-G\,R)$ normed (left row) eigenvectors of $A-G\,R$ implying $\rho_i = 2\,Re\,\lambda_i((A-G\,R)$ for $i=1,\dots,n$. Further, also

$$R^{(j)} = \sum_{i=1}^n R_i^{(j)} \longrightarrow R := \sum_{i=1}^n R_i \qquad (j\in I\!N')$$

and

$$R^{(j+1)} = \sum_{i=1}^n R_i^{(j+1)} \longrightarrow S := \sum_{i=1}^n S_i \qquad (j\in I\!N')$$

so that

$$(A-G\,R)^*S_i + S_i(A-G\,R) = \rho_i S_i, \quad i=1,2,\dots,n. \tag{58}$$

The matrix S is positive definite, and the matrix $A-G\,R$ is diagonalizable. If

$$S_i = R_i,\ i=1,\dots,n, \tag{59}$$

then the eigenproblem (28), (29) has a positive definite solution.

Proof: Relations (55) follow from Lemma 2; and relations (56), (57) from the first four relations of (55). Then, also (58) follows. Further, from [10, Theorem 5], $S_i = u_i{}^* u_i$. Since $u_i{}^*$ and u_i are eigenvectors for $i = 1, \ldots, n$, they are linearly independent. Thus, $S = \sum_{i=1}^{n} S_i$ is positive definite according to [10, Theorem 6]. The matrix $A - G\,R$ is diagonalizable since the eigenvectors $u_i,\ i = 1, \ldots, n$ are linearly independent. $\diamond$

Remark: Even if it should turn out that $S_i \neq R_i$ for some $i \in \{1, \ldots, m\}$, this poses no problem. The reason is as follows: we have

$$\frac{d}{dt}\|x(t)\|_{S_i}^2 = \frac{d}{dt}(S_i\,x, x) = \|x\|_{S_i}^2$$

so that then

$$\|x(t)\|_{S_i}^2 = \|x_0\|_{S_i}^2\, e^{\rho_i\,(t-t_0)}$$

and thus

$$\|x(t)\|_S^2 = \sum_{i=1}^{m} \|x(t)\|_{S_i}^2 = \sum_{i=1}^{m} \|x_0\|_{S_i}^2\, e^{\rho_i\,(t-t_0)}$$

so that then the weighted norm $\|\cdot\|_S$ takes the role of the weighted norm $\|\cdot\|_R$. $\diamond$

7 Applications to a vibration problem

In Subsection 7.1, we begin with a one-DOF model (one-degree-of-freedom model) taken from Hagedorn [7] with Coulomb-like friction force (leading to self excitation) with additional applied external force. In Subsection 7.2, this model is extended to a three-DOF model and in Subsection 7.3 to an ℓ-DOF model. Then, in Subsection 7.4, the state-space description $\dot{x}_d = A\,x_d + g(t), x_d(0) = x_{d,0}$ for the mechanical ℓ-DOF model is established and the associated external excitation vector $g(t)$ is assumed to be of the form $g(t) = Bu(t)$, where the input matrix B is chosen in such a way that the resulting IVP pertains to a physical model of half matrix dimension and of second order like the ℓ-DOF model we departed from. In Subsection 7.5, data are specified. In Subsection 7.6, computational results are presented where first the vibration behavior of the free system $\dot{x}_d = A\,x_d, x_d(0) = x_{d,0}$ is studied, followed by the system with the closed-loop system matrix $A - G\,P$ and then that with the closed-loop system matrix $A - G\,R$. In all cases, $y = \|x_d(t)\|_2$ is plotted and, in the case of closed-loop matrix $A - G\,P$, also $y = \|x_d(t)\|_P$ as well as, in the case of the closed-loop matrix $A - G\,R$, also $y = \|x_d(t)\|_R$. Further, $y = \|x_d(t)\|_R$ is plotted in the case of the closed-loop system matrix $A - G\,P$ where R is computed from the associated Lyapunov eigenproblem.

We describe all steps of the development towards the final model in Sections 7.1 - 7.4 in order to make the paper easily understandable for a large readership.

Similarly, Sections 7.5 - 7.6 contain numerous computational results in order that the readers may check their own calculations.

Finally, we say something about the used computer equipment.

The following hardware was available: an Intel Core2 Duo Processor at 3166 GHz, a 500 GB mass storage facility, and two 2048 MB high-speed memories. As software package for the computations, we used MATLAB, Version 7.11.

7.1 One-DOF model with self excitation

We begin with a one-DOF model from Hagedorn [7, pp.128-132] with Coulomb-like friction force (leading to self excitation) with additional applied external force. The pertinent model is given in *Fig.1*.

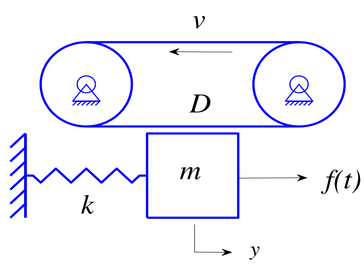

Fig.1: One-DOF model with Coulomb-like friction force D generated by conveyor moving with velocity v, according to [7, Fig.3.3]

We mention that we write D instead of R in [7] since the symbol R is used later for a positive definite matrix. In Fig.1, m is a mass, k a stiffness constant, D a friction force function, v the velocity of the conveyor, $f(t)$ an applied external force and y the displacement of the dynamical system.

The free-body diagram pertinent to Fig.1 delivers the equation of motion

$$m\,\ddot{y} + D + k\,y = f(t) \tag{60}$$

with

$$D = D(v_{rel}) = D(\dot{y} - v).$$

As in [7], equation (60) will be linearized. The linearization formula for a general function f reads

$$f(x_0 + h) \approx f(x_0) + \frac{df}{dx}_{|x=x_0} \cdot h.$$

When this formula is applied to $D(\dot{y} - v)$, we obtain

$$D = D(v_{rel}) = D(\dot{y} - v) = D(\underbrace{-v}_{x_0} + \underbrace{\dot{y}}_{h}) = D(-v) + \dot{y}\,\frac{dD}{dv_{rel}}_{|v_{rel}=-v}$$

or

$$D = D(\dot{y} - v) = D(-v) - d\,\dot{y}$$

with

$$-d = \frac{dD}{dv_{rel}}_{|_{v_{rel}=-v}} < 0$$

i.e., $d > 0$; see [7, p.131, Fig.3.7 and p.132, first line]). This leads to the linearized differential equation

$$m\,\ddot{y} + D(-v) - d\,\dot{y} + k\,y = f(t)$$

or

$$m\,\ddot{y} - d\,\dot{y} + k\,(y - y_s) = f(t)$$

with

$$y_s = -\frac{D(-v)}{k}.$$

Let the shifted displacement y_d be defined by

$$y_d := y - y_s.$$

Then,

$$\dot{y}_d = \dot{y},$$

and

$$\ddot{y}_d = \ddot{y}.$$

This implies

$$m\,\ddot{y}_d + b\,\dot{y}_d + k\,y_d = f(t) \tag{61}$$

with

$$b = -d < 0.$$

The linearized model (61) will be symbolically described by *Fig.2*.

We mention that the coefficient $b = -d$ is described by the usual symbol for a viscous damper with the additional sign $(-)$ indicating that the damping coefficient is negative.

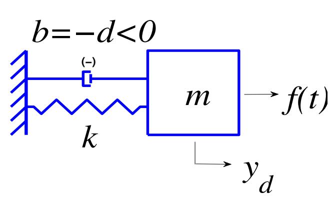

Fig.2: Model equivalent to that in Fig.1 after linearization, leading to shifted displacement and self excitation due to negative viscous damping

7.2 Three-DOF model with self excitation

Next, we generalize the one-DOF model in Fig.1 to the three-DOF model in *Fig.3*.

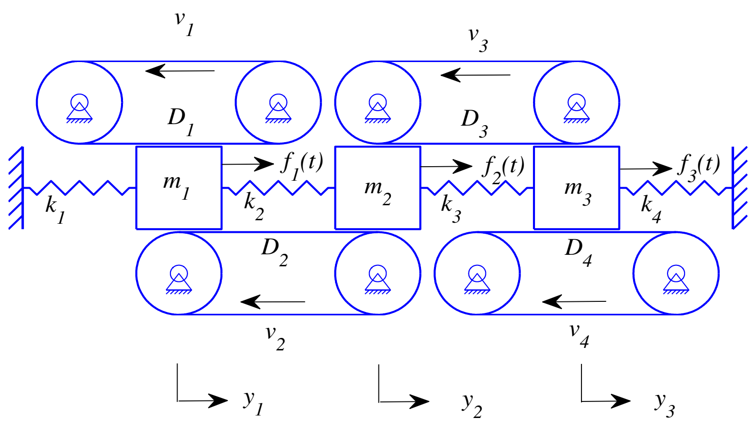

Fig.3: Three-DOF model with Coulomb-like friction force functions D_1, D_2, D_3, D_4 generated by conveyors moving with velocities v_1, v_2, v_3, v_4, respectively

One has

$$\begin{aligned} D_1 &= D_1(\dot{y}_1 - v_1), \\ D_2 &= D_2(\dot{y}_2 - \dot{y}_1 - v_2), \\ D_3 &= D_3(\dot{y}_3 - \dot{y}_2 - v_3), \\ D_4 &= D_4(-\dot{y}_3 - v_4). \end{aligned} \tag{62}$$

The free-body diagram for the system in Fig.3 leads to the equations of motion

$$\begin{aligned} m\,\ddot{y}_1 + D_1 + k_1\,y_1 - D_2 - k_2\,(y_2 - y_1) &= f_1(t), \\ m\,\ddot{y}_2 + D_2 + k_2\,(y_2 - y_1) - D_3 - k_3\,(y_3 - y_2) &= f_2(t), \\ m\,\ddot{y}_3 + D_3 + k_3\,(y_3 - y_2) - D_4 - k_4\,(-y_3) &= f_3(t). \end{aligned} \tag{63}$$

Linearization of the functions (62) leads to

$$\begin{aligned}
D_1 = D_1(\dot{y}_1 - v_1) \quad &\approx \quad D_1(-v_1) - d_1\,\dot{y}_1\\
D_2 = D_2(\dot{y}_2 - \dot{y}_1 - v_2) \quad &\approx \quad D_2(-v_2) - d_2\,(\dot{y}_2 - \dot{y}_1),\\
D_3 = D_3(\dot{y}_3 - \dot{y}_2 - v_3) \quad &\approx \quad D_3(-v_3) - d_3\,(\dot{y}_3 - \dot{y}_2),\\
D_4 = D_4(-\dot{y}_3 - v_4) \quad &\approx \quad D_1(-v_4) - d_4\,(-\dot{y}_3)
\end{aligned}$$

with

$$-d_i = \frac{dD_i}{dv_{i,rel}}_{|v_{i,rel}=-v_i} < 0,\ i = 1,2,3,4.$$

Thus, the linearized equations of (63) take the matrix form

$$\boldsymbol{M}\,\ddot{\boldsymbol{y}} - \boldsymbol{D}\,\dot{\boldsymbol{y}} + \boldsymbol{K}\,\boldsymbol{y} + \boldsymbol{f}_s = \boldsymbol{f}(t) \tag{64}$$

with

$$\boldsymbol{M} = \begin{bmatrix} m_1 & & \\ & m_2 & \\ & & m_3 \end{bmatrix},$$

$$\boldsymbol{D} = \begin{bmatrix} d_1 + d_2 & -d_2 & \\ -d_2 & d_2 + d_3 & -d_3 \\ & -d_3 & d_3 + d_4 \end{bmatrix},$$

$$\boldsymbol{K} = \begin{bmatrix} k_1 + k_2 & -k_2 & \\ -k_2 & k_2 + k_3 & -k_3 \\ & -k_3 & k_3 + k_4 \end{bmatrix},$$

$$\boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix},$$

$$\boldsymbol{f}_s = \begin{bmatrix} D_1(-v_1) - D_2(-v_2) \\ D_2(-v_2) - D_3(-v_3) \\ D_3(-v_3) - D_4(-v_4) \end{bmatrix}.$$

Let $\boldsymbol{y}_s$ be the solution of the matrix equation

$$\boldsymbol{K}\,\boldsymbol{y}_s = \boldsymbol{f}_s.$$

Here, $\boldsymbol{y}_s$ is interpreted as displacement due to the static force $\boldsymbol{f}_s$, leading to a shift of the equilibrium point. Inserting this into (64) implies

$$\boldsymbol{M}\,\ddot{\boldsymbol{y}} - \boldsymbol{D}\,\dot{\boldsymbol{y}} + \boldsymbol{K}\,(\boldsymbol{y} - \boldsymbol{y}_s) = \boldsymbol{f}(t). \tag{65}$$

Let

$$\boldsymbol{y}_d := \boldsymbol{y} - \boldsymbol{y}_s \tag{66}$$

be the shifted displacement. Then,

$$\dot{\boldsymbol{y}}_d = \dot{\boldsymbol{y}},$$

$$\ddot{\boldsymbol{y}}_d = \ddot{\boldsymbol{y}}.$$

Setting

$$\boldsymbol{B} := -\boldsymbol{D} \tag{67}$$

leads to

$$\boldsymbol{M}\,\ddot{\boldsymbol{y}}_d + \boldsymbol{B}\,\dot{\boldsymbol{y}}_d + \boldsymbol{K}\,\boldsymbol{y}_d = \boldsymbol{f}(t). \tag{68}$$

We describe this differential equation symbolically by *Fig.4*.

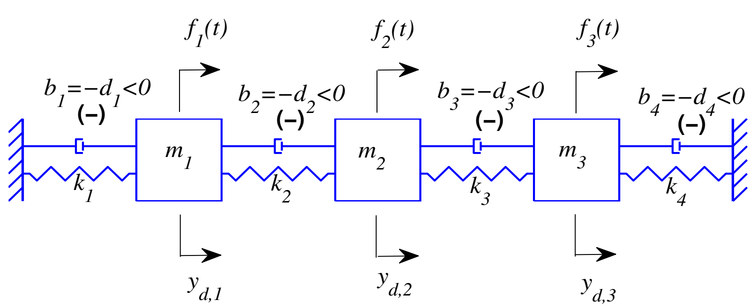

Fig.4: Model equivalent to that in Fig.3 after linearization, leading to shifted displacements and self excitation due to negative viscous damping coefficients

Remark: Equation (68) has the same form as

$$\boldsymbol{M}\,\ddot{\boldsymbol{y}} + \boldsymbol{B}\,\dot{\boldsymbol{y}} + \boldsymbol{K}\,\boldsymbol{y} = \boldsymbol{f}(t) \tag{69}$$

for the model in *Fig.5*.

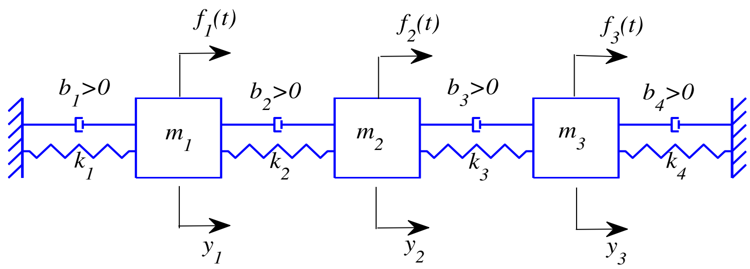

Fig.5: Three-DOF model with (positive) viscous damping coefficients corresponding to that of Fig.4

7.3 ℓ-DOF model with self excitation

The three-DOF model in Fig.3 can be generalized to an ℓ-DOF model that, after linearization, can be described by

$$\boldsymbol{M}\,\ddot{\boldsymbol{y}} - \boldsymbol{D}\,\dot{\boldsymbol{y}} + \boldsymbol{K}\,\boldsymbol{y} + \boldsymbol{f}_s = \boldsymbol{f}(t) \tag{70}$$

with

$$\boldsymbol{M} = \begin{bmatrix} m_1 & & & & \\ & m_2 & & & \\ & & m_3 & & \\ & & & \ddots & \\ & & & & m_\ell \end{bmatrix}, \tag{71}$$

$$\boldsymbol{D} = \begin{bmatrix} d_1+d_2 & -d_2 & & & \\ -d_2 & d_2+d_3 & -d_3 & & \\ & -d_3 & d_3+d_4 & -d_4 & \\ & \ddots & \ddots & \ddots & \\ & & -d_{\ell-1} & d_{\ell-1}+d_\ell & -d_\ell \\ & & & -d_\ell & d_\ell+d_{\ell+1} \end{bmatrix}, \tag{72}$$

$$\boldsymbol{K} = \begin{bmatrix} k_1+k_2 & -k_2 & & & \\ -k_2 & k_2+k_3 & -k_3 & & \\ & -k_3 & k_3+k_4 & -k_4 & \\ & \ddots & \ddots & \ddots & \\ & & -k_{\ell-1} & k_{\ell-1}+k_\ell & -k_\ell \\ & & & -k_\ell & k_\ell+k_{\ell+1} \end{bmatrix}, \tag{73}$$

$$\boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_\ell \end{bmatrix}, \tag{74}$$

$$\boldsymbol{f}_s = \begin{bmatrix} D_1(-v_1) - D_2(-v_2) \\ D_2(-v_2) - D_3(-v_3) \\ \vdots \\ D_n(-v_\ell) - D_{\ell+1}(-v_{\ell+1)}) \end{bmatrix}. \tag{75}$$

Let $\boldsymbol{y}_s$ be the solution of

$$\boldsymbol{K}\,\boldsymbol{y}_s = \boldsymbol{f}_s. \tag{76}$$

Inserting this into (70) implies

$$\boldsymbol{M}\,\ddot{\boldsymbol{y}} + \boldsymbol{B}\,\dot{\boldsymbol{y}} + \boldsymbol{K}\,(\boldsymbol{y} - \boldsymbol{y}_s) = \boldsymbol{f}(t) \tag{77}$$

with

$$\boldsymbol{y}_d = \boldsymbol{y} - \boldsymbol{y}_s \tag{78}$$

and

$$\boldsymbol{B} := -\boldsymbol{D}, \tag{79}$$

and leads to

$$\boldsymbol{M}\,\ddot{\boldsymbol{y}}_d + \boldsymbol{B}\,\dot{\boldsymbol{y}}_d + \boldsymbol{K}\,\boldsymbol{y}_d = \boldsymbol{f}(t). \tag{80}$$

For the initial conditions, we chose

$$\boldsymbol{y}_d(0) = \boldsymbol{y}_{d,0},\ \dot{\boldsymbol{y}}_d(0) = \dot{\boldsymbol{y}}_{d,0}. \tag{81}$$

The pertinent IVP is given by (80), (81). The pertinent model is described in symbolic form by *Fig.6*.

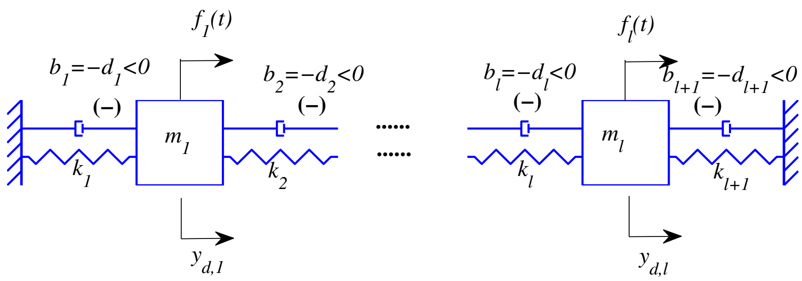

Fig.6: l-DOF model corresponding to that in Fig.3 after linearization, leading to shifted displacements and self excitation due to negative viscous damping coefficients

7.4 State-space description and feedback

In state-space description, relations (80), (81) can be written as

$$\dot{x}_d = A\,x_d + g(t),\, t \geq 0,\; x_d(0) = x_{d,0}, \tag{82}$$

where

$$x_d \;=\; \begin{bmatrix} \boldsymbol{y}_d \\ \boldsymbol{z}_d \end{bmatrix} = \begin{bmatrix} \boldsymbol{y}_d \\ \dot{\boldsymbol{y}}_d \end{bmatrix} \tag{83}$$

is the *state vector*, where the *system matrix* A is given by

$$A \;=\; \left[\begin{array}{c|c} \boldsymbol{O} & \boldsymbol{I} \\ \hline -\boldsymbol{M}^{-1}\boldsymbol{K} & -\boldsymbol{M}^{-1}\boldsymbol{B} \end{array}\right], \tag{84}$$

and where the *excitation vector* has the from

$$g(t) \;=\; \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{M}^{-1}\boldsymbol{f}(t) \end{bmatrix}. \tag{85}$$

This vector can be cast into the form

$$g(t) = B\,u(t) \tag{86}$$

with

$$B \;=\; \left[\begin{array}{c|c} \boldsymbol{O} & \boldsymbol{O} \\ \hline \boldsymbol{O} & \boldsymbol{M}^{-1} \end{array}\right] \tag{87}$$

and

$$u(t) \;=\; \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{f}(t) \end{bmatrix}. \tag{88}$$

In the case of the optimal Riccati controller using matrix R in the nonlinear EP (28), (29), from (12), the expression $B\,u(t)$ is given by the feedback

$$B\,u(t) \;=-G\,R\,x_d(t) \tag{89}$$

with

$$G \;=\; B\,Q_u^{-1}\,B^T \;=\; B\,Q_u^{-1}\,B^*$$

where R is the positive definite solution matrix of the nonlinear Riccati matrix eigenproblem (28), (29).

In order to get a simple expression for G, we choose

$$Q_u^{-1} \;=\; \gamma \left[\begin{array}{c|c} \boldsymbol{M} & \boldsymbol{O} \\ \hline \boldsymbol{O} & \boldsymbol{M} \end{array}\right] \tag{90}$$

or

$$Q_u \;=\; \gamma^{-1} \left[\begin{array}{c|c} \boldsymbol{M}^{-1} & \boldsymbol{O} \\ \hline \boldsymbol{O} & \boldsymbol{M}^{-1} \end{array}\right] \tag{91}$$

leading to

$$G \;=\; B\,Q_u^{-1}\,B^T \;=\; \gamma\,B. \tag{92}$$

Combining (86) and (89), we get

$$g(t) \;=\; B\,u(t) \;=-\,G\,R\,x_d(t) \tag{93}$$

so that then

$$\dot{x}_d = (\,A - G\,R\,)\,x_d,\; t \ge 0, \tag{94}$$

with

$$x_d(0) = x_{d,0}. \tag{95}$$

In this case,

$$\begin{aligned} g(t) \;=\; B\,u(t) \;&=\; -G\,R\,x_d(t) \;=\; -\gamma\,B\,R\,x_d(t) \\ &=\; -\gamma \left[\begin{array}{c|c} \boldsymbol{O} & \boldsymbol{O} \\ \hline \boldsymbol{O} & \boldsymbol{M}^{-1} \end{array}\right] \left[\begin{array}{c|c} \boldsymbol{R}_{11} & \boldsymbol{R}_{12} \\ \hline \boldsymbol{R}_{21} & \boldsymbol{R}_{22} \end{array}\right] \left[\begin{array}{c} \boldsymbol{y}_d(t) \\ \dot{\boldsymbol{y}}_d(t) \end{array}\right] \\ &=\; -\gamma \left[\begin{array}{c|c} \boldsymbol{O} & \boldsymbol{O} \\ \hline \boldsymbol{M}^{-1}\,\boldsymbol{R}_{21} & \boldsymbol{M}^{-1}\,\boldsymbol{R}_{22} \end{array}\right] \left[\begin{array}{c} \boldsymbol{y}_d(t) \\ \dot{\boldsymbol{y}}_d(t) \end{array}\right] \\ &=\; \left[\begin{array}{c} \mathbf{0} \\ \boldsymbol{M}^{-1}[\,-\gamma\,(\,\boldsymbol{R}_{21}\,\boldsymbol{y}_d(t) + \boldsymbol{R}_{22}\,\dot{\boldsymbol{y}}_d(t)\,)\,] \end{array}\right] \end{aligned} \tag{96}$$

so that

$$g(t) = g_c(t) = \left[\begin{array}{c} \mathbf{0} \\ \boldsymbol{M}^{-1}\boldsymbol{f}_c(t) \end{array}\right] \tag{97}$$

with the control force

$$\boldsymbol{f}_c(t) = -\gamma\,(\,\boldsymbol{R}_{21}\,\boldsymbol{y}_d(t) + \boldsymbol{R}_{22}\,\dot{\boldsymbol{y}}_d(t)\,). \tag{98}$$

This feedback vector can usually be realized by actuators.

Remark: In the case when the optimal controller is defined by the use of matrix P satisfying the algebraic Riccati matrix equation (9), then in the preceding derivations, one has to replace $R = \left[\begin{array}{c|c} \boldsymbol{R}_{11} & \boldsymbol{R}_{12} \\ \hline \boldsymbol{R}_{21} & \boldsymbol{R}_{22} \end{array}\right]$ by $P = \left[\begin{array}{c|c} \boldsymbol{P}_{11} & \boldsymbol{P}_{12} \\ \hline \boldsymbol{P}_{21} & \boldsymbol{P}_{22} \end{array}\right]$. ◇

7.5 Data

We first remark that since we are not interested in the shift vector y_s defined by (76), neither the velocities v_i, $i = 1, \ldots, \ell + 1$ of the conveyors nor the components of the vector f_s in (75) will be specified. This means that we only study the shifted displacement vector $\boldsymbol{y}_d$ in (78) satisfying (80), (81).

The values m_j, $j = 1, \ldots, \ell$ and k_j, $j = 1, \ldots, \ell + 1$ are specified as in [10], whereas the values of b_j, $j = 1, \ldots, \ell + 1$ are negative. They read

$$\begin{aligned} m_j &= 1, \; j = 1, \ldots, \ell \\ k_j &= 1, \; j = 1, \ldots, \ell + 1 \end{aligned}$$

and

$$b_j = \begin{cases} -0.005, & j \; even, \\ -0.0025, & j \; odd. \end{cases}$$

We set $d_j = -b_j$, $j = 1, \ldots, \ell + 1$. Further, we choose $\ell = 5$ in this paper so that the state-space vector has dimension $n = 2\ell = 10$. For the initial time, we specify

$$t_0 = 0.$$

Moreover, the initial conditions for $\boldsymbol{y}_d(t)$ and $\dot{\boldsymbol{y}}_d(t)$ are chosen as $\boldsymbol{y}_d(0) = \boldsymbol{y}_{d,0}$ and $\dot{\boldsymbol{y}}_d(0) = \dot{\boldsymbol{y}}_{d,0}$ with

$$\boldsymbol{y}_d(0) = \boldsymbol{y}_{d,0} = [-1, 1, -1, 1, -1]^T$$

and

$$\dot{\boldsymbol{y}}_d(0) = \dot{y}_{d,0} = [-1, -1, -1, -1, -1]^T.$$

For γ, we choose always values from $\gamma = 0.3$.

7.6 Computational results

We first consider the problem pertinent to Fig.6. when there is no excitation, that is, when $\boldsymbol{f}(t) = 0$ or $g(t) = 0$ resp. $\dot{x}_d = A\,x_d, x_d(0) = x_{d,0}$. Next, the case when $g(t) = -G\,P\,x_d(t)$ is studied where P is the positive definite solution matrix of the algebraic Riccati matrix equation $-P\,G\,P + A^*P + P\,A = -Q$, that is, when $\dot{x}_d = (A - G\,P)\,x_d,\ x_d(0) = x_{d,0}$. Further, the case when $g(t) = -G\,R\,x_d(t)$ is studied where R is the positive definite solution matrix of the nonlinear Riccati matrix eigenproblem $(A-G\,R)^*R_i + R_i(A-G\,R) = \rho_i R_i, i = 1, 2, \ldots, n,\ R = \sum_{i=1}^n R_i$, that is, when $\dot{x}_d = (A - G\,R)\,x_d,\ x_d(0) = x_{d,0}$.

(i) The free dynamical system matrix $\dot{x}_d = A\,x_d,\ x_d(0) = x_{d,0}$

The eigenvalues of the system matrix A are given by

$$
\begin{aligned}
\lambda_1(A) &= 0.006997595264192 + 1.931838543426845i,\\
\lambda_2(A) &= 0.005625000000015 + 1.732039643921802i,\\
\lambda_3(A) &= 0.003750000000000 + 1.414208590519800i,\\
\lambda_4(A) &= 0.001874999999985 + 0.999999414066447i,\\
\lambda_5(A) &= 0.000502404735808 + 0.517637963136842i,
\end{aligned}
$$

$$\lambda_{j+5}(A) = \overline{\lambda_j(A)},\ j = 1, \ldots, 5.$$

Since $Re\,\lambda_i(A) > 0,\ i = 1, \ldots, n = 2\ell = 10$, the dynamical system is unstable.

In *Fig.7* for the time range $[0; 100]$. As can be seen, $\|x_d(t)\|_2$ tends to infinity for increasing t.

In *Fig.8* and *Fig.9*, $y = \|x_d(t)\|$ is shown for the norms $\|\cdot\| = \|\cdot\|_2$ and $\|\cdot\| = \|\cdot\|_R$ as well as for the value $\gamma = 0.3$. More precisely, in Fig.8, $\gamma = 0.3$, $\|\cdot\| = \|\cdot\|_2$, and in Fig.9, $\gamma = 0.3$, $\|\cdot\| = \|\cdot\|_R$. Now, we comment on the figures.

In Fig.8, there is overshooting of $y = \|x_d(t)\|_2$ in the initial time interval; for increasing γ, the overshooting decreases and the decay in the terminal interval increases.

In Fig.9, the vibration behavior is eliminated, and there is no overshooting of $y = \|x_d(t)\|_R$; if γ increases, the decay in the terminal interval increases.

An advantage of R over P can also be seen if we compare $y = x_d(t)$ in the norms $\|\cdot\|_R$ and $\|\cdot\|_P$.

In Fig.9, we have plotted $y = x_d(t)$ in the norm $\|\cdot\|_R$ where R is the positive definite matrix computed from the nonlinear Riccati matrix eigenproblem (28), (29). The course of the associated curve is qualitatively like that of the function $y = e^{-t}$.

Now, we want to see what happens when R is replaced by the positive definite matrix P computed from the algebraic Riccati matrix equation (11). For the given data, we plot $y = \|x_d(t)\|_P$; the corresponding graph can be seen in *Fig.10*.

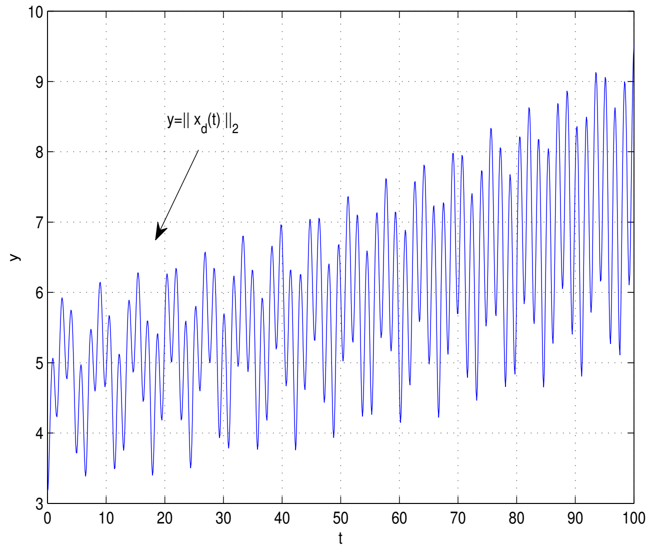

Fig.7: Graph of $y = \|x_d(t)\|_2$ for unstable system $\dot{x}_d = A\,x_d,\ x_d(0) = x_{d,0}$ with $t \in [0; 100]$

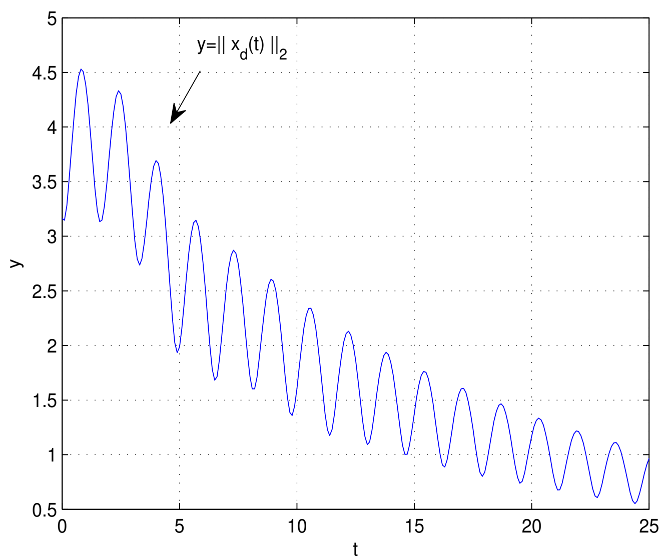

Fig.8: Graph of $y = \|x_d(t)\|$ for stabilized system $\dot{x}_d = (A - G\,R)\,x_d,\ x_d(0) = x_{d,0}$ for $\gamma = 0.3,\ \|\cdot\| = \|\cdot\|_2$

As is obvious, $y = \|x_d(t)\|_P$ decreases monotonically, but vibration is not suppressed, as opposed to the case of the norm plotted in Fig.12. So, in this respect, feedback constructed by means of R is superior to that constructed by P.

We mention that the vibration-suppression property of the norm $\|\cdot\|_R$ has recently been discovered for the simpler problem of free dynamical systems $\dot{x} = Ax,\ x(t_0) = x_0$ in [12].

(ii) Numerical details

First, we give some additional numerical details for $\gamma = 0.3$ with $tol = 0.5\,e - 12$ on the non-monotonicity of the sequences $(\rho_i^{(j)})_{j\in I\!N}$ for $i = 1\ldots,\ell$ with $\ell = 5$. The results indicate that the whole sequence converges. The quantities $\rho_i^{(0)},\ i = 1,\ldots,5$ are determined from (35). The numbering of the next approximations is such that $|\rho_i^{(j)}| \le |\rho_k^{(j)}|,\ i \le k,\ i,k = 1,\ldots,5,\ j \ge 1$.

We obtain *Table 1:*

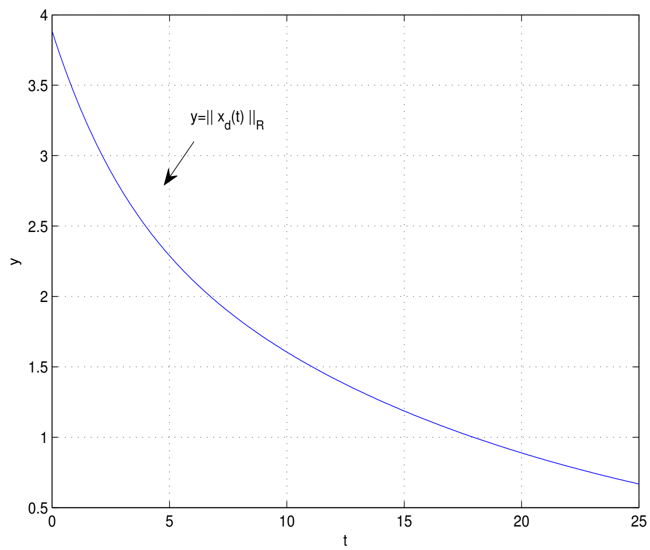

Fig.9: Graph of $y = \|x_d(t)\|_R$ for stabilized system
$\dot{x}_d = (A - G\,R)\,x_d,\ x_d(0) = x_{d,0}$ for $\gamma = 0.3$

j	$\rho_1^{(j)}$	$\rho_2^{(j)}$	$\rho_3^{(j)}$	$\rho_4^{(j)}$	$\rho_5^{(j)}$
0	−0.586004809471617	−0.598995190528384	−0.592500000000000	−0.596250000000030	−0.588749999999969
1	−0.112799737886242	−0.138750065207853	−0.192499692897521	−0.296249285578237	−0.472199850160670
2	−0.112823544820325	−0.138782009893091	−0.192549834267286	−0.296333355538621	−0.472288570213166
3	−0.112608394339580	−0.138360950912393	−0.191224576189990	−0.289726993242286	−0.433874028209329
4	−0.112608660307832	−0.138361776732197	−0.191232483482684	−0.289866000983903	−0.436745582439665
5	−0.112609028008919	−0.138362985372483	−0.191241068884471	−0.290002309831195	−0.439228293651434
6	−0.112609033218213	−0.138363023453315	−0.191241017710577	−0.289996457824372	−0.438850465973104
7	−0.112609031084859	−0.138363011509708	−0.191240976899617	−0.289993764621754	−0.438701061745265
8	−0.112609027675355	−0.138362998378734	−0.191240949259319	−0.289993960641283	−0.438738428290658
			⋮		
149	−0.112609029256882	−0.138363002953763	−0.191240954804824	−0.289993990007971	−0.438743057463690
150	−0.112609029256881	−0.138363002953763	−0.191240954804825	−0.289993990007972	−0.438743057463691
151	−0.112609029256882	−0.138363002953762	−0.191240954804824	−0.289993990007973	−0.438743057463692
152	−0.112609029256882	−0.138363002953763	−0.191240954804824	−0.289993990007972	−0.438743057463692
153	−0.112609029256881	−0.138363002953761	−0.191240954804824	−0.289993990007972	−0.438743057463691
154	−0.112609029256882	−0.138363002953762	−0.191240954804824	−0.289993990007972	−0.438743057463690
155	−0.112609029256881	−0.138363002953761	−0.191240954804824	−0.289993990007972	−0.438743057463690

This is in good accordance with

$$\begin{array}{rcl} \rho_1 = 2\,Re\,\lambda_1(A - G\,R) & = & -0.112609029256881, \\ \rho_2 = 2\,Re\,\lambda_2(A - G\,R) & = & -0.138363002953763, \\ \rho_3 = 2\,Re\,\lambda_3(A - G\,R) & = & -0.191240954804823, \\ \rho_4 = 2\,Re\,\lambda_4(A - G\,R) & = & -0.289993990007975, \\ \rho_5 = 2\,Re\,\lambda_5(A - G\,R) & = & -0.438743057463689. \end{array}$$

The remaining numbering is such that

$$\rho_{i+5} = \rho_i,\ i = 1, \ldots, 5.$$

It took $it = 156$ iterations with (35) and $it = 158$ iterations with (42). We have checked the result and observed that it satisfies the system (28), (29) with a precision of at least 12 decimal places. One has also that $R^* = R^T = R$.

Since the eigenvalues are given by

$$\begin{array}{rcl|rcl} \lambda_1(R) & = & 0.4215, & \lambda_6(R) & = & 1.1435, \\ \lambda_2(R) & = & 0.4340, & \lambda_7(R) & = & 1.3434, \\ \lambda_3(R) & = & 0.4975, & \lambda_8(R) & = & 1.5025, \\ \lambda_4(R) & = & 0.6566, & \lambda_9(R) & = & 1.5661, \\ \lambda_5(R) & = & 0.8565, & \lambda_{10}(R) & = & 1.5785, \end{array}$$

matrix R is positive definite.

The matrix $A - G\,R$ is asymptotically stable. This is numerically confirmed since $||x(t = 500)||_R = 1.61005e - 012$.

(iii) <u>The closed-loop system $\dot{x}_d = (A - G\,P)\,x_d,\ x_d(0) = x_{d,0}$</u>

We made a calculation for $Q = \left[\begin{array}{c|c} \boldsymbol{O} & \boldsymbol{O} \\ \hline \boldsymbol{O} & I \end{array}\right] =: Q_0 \in I\!R^{10\times 10}$ and $\gamma = 0.3$; the solution matrix P of (11) is positive definite with

$$\begin{array}{rcl|rcl} \lambda_1(P) & = & 12.3357, & \lambda_6(P) & = & 3.2781, \\ \lambda_2(P) & = & 9.8305, & \lambda_7(P) & = & 3.2384, \\ \lambda_3(P) & = & 6.4764, & \lambda_8(P) & = & 3.2001, \\ \lambda_4(P) & = & 0.8500, & \lambda_9(P) & = & 3.1722, \\ \lambda_5(P) & = & 3.3055, & \lambda_{10}(P) & = & 3.1983. \end{array}$$

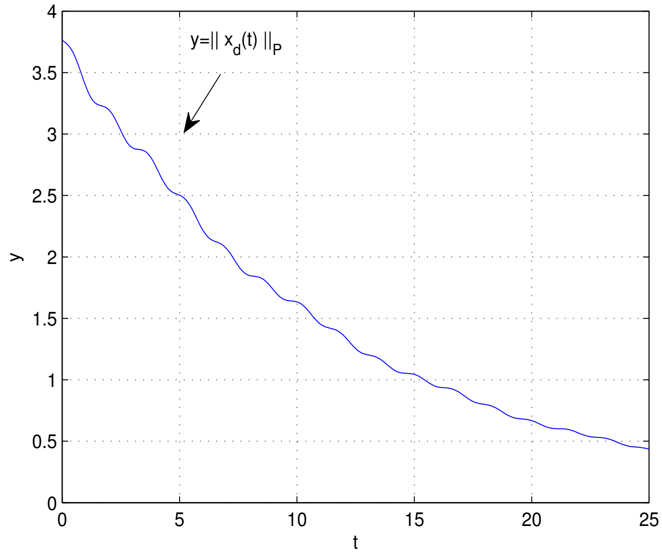

Fig.10: Graph of $y = \|x_d(t)\|_P$ for stabilized system
$\dot{x}_d = (A - G\,P)\,x_d,\ x_d(0) = x_{d,0}$ for $\gamma = 0.3$, $Q = 0.1Q_0$

8 Conclusions

In this paper, we first revisit the optimal control based on the algebraic Riccati matrix equation. The pertinent solution matrix P is positive definite. As a new result, the optimal feedback law is obtained by a constrained derivative criterion that is equivalent to the known unconstrained integral criterion. The advantage of the derivative criterion is that it can be carried over to the optimal control based on the Riccati matrix eigenproblem. Then, the new nonlinear eigenproblem is derived corresponding to the algebraic Riccati matrix equation. This is followed by optimal control based on the nonlinear Riccati eigenproblem. An algorithm for the positive definite solution of this new nonlinear eigenproblem is proposed based on the idea to cast it into the form of a Lyapunov eigenproblem, the solution method is known for. We could show the convergence of a subsequence and prove the existence of a solution. Applications are made to a mechanical vibration problem that − after linearization − is unstable, but where the closed-loop system matrices $A - G\,P$ resp. $A - G\,R$ are asymptotically stable. Numerical tests with specified data indicate that the feedback matrix $G\,R$ based on matrix R is superior to the feedback matrix $G\,P$ based on P. Especially, in the weighted norm $\|\cdot\| = \|\cdot\|_R$, the solution $x_d(t)$ of the IVP $\dot{x}_d(t) = (A - G\,R)\,x_d(t),\ t \geq 0,\ x_d(0) = x_{d,0}$, behaves essentially like $y = e^{-t}$, whereas the solution of $\dot{x}_d(t) = (A - G\,P)\,x_d(t),\ t \geq 0,\ x_d(0) = x_{d,0}$, in the norm $\|\cdot\|_P$ is only monotonically decreasing, but does not eliminate vibration. We mention that a weighted norm is also used in Section 5.2.3 of [16] in connection with the stability investigation of time-invariant dynamical systems based on the Lyapunov matrix equation, but no vibration-suppression property is studied there like in [12].

The numerical example works with a system matrix $A \in I\!R^{10\times 10}$, that is, of relatively small size. If A is very large, then condensation methods can be used to reduce the dimension of the involved matrices, see, e.g., [3] and [6]. Further, we mention that, in our case, system matrix A and some other matrices are diagonalizable. However, the method can be extended to non-diagonalizable matrices like in the solution of the Lyapunov eigenproblem, described in [10], [11] since the solution alorithm is essentially based on the idea to rewrite the nonlinear Riccati eigenproblem into a Lyapunov eigenproblem. The example given above is chosen in order to test the theory. In order to make the paper easily understandable also for a large readership, the application section contains a detailed and comprehensive example.

Acknowledgements.
Starting point for this paper was a question posed by Prof. Peter Benner, namely whether it was possible to avoid overshooting in a common control problem. As has been seen above, overshooting can at least be avoided in a special weighted norm which is of value on its own and has potential applications. Apart from initializing the project, Prof. Benner made some useful comments. For all this, the author would like to thank Prof. Benner.

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