Comparing Bonus Malus Premiums of Two Types

Claim which is Assessed Using Bayesian Method

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Abstract

Some automobile insurances have special feature that is bonus malus system. The calculation of the bonus or malus depends on claims history of previous periods. Determining the premium using a bonus malus system requires the number of claims and the claim sizes. It is not enough to determine premium on bonus malus system only based on the number of claims and claim sizes, because it is not fair to give the same punishment to the policyholders having small and large claims. Therefore, there is should be a limit value of the claim size, called critical value. This critical value helps us to determine the punishment for policyholders having claim sizes below this critical value and the ones having claim sizes above it. We propose an alternative method for determining premium payment of bonus malus system using Bayesian method and we calculated the premium payment if the limit value is changed frequently. We assume that the number of claims follows Poisson distribution, while the total number of claims with the claim size larger than limit value follows binomial distribution. We used exponential and beta as prior distribution. The result is the limit value does not significantly affect the amount of the premium.
Keywords: bonus-malus system, Bayesian method, critical value

1 Introduction

Calculation of the premium in bonus malus system depends on the number of claims of policyholders during one period. If the policyholders make claim during this period, then they must pay more premium in the next period. On the other hand, if the policyholders do not make any claim during this period, their premium payment for the next period will be decreased.

The insurance premium on bonus malus system had been studied using several methods, including the Bayesian method. Bayesian method is used when the data for calculating the premium for each policyholder are available, whether based on claims history of previous periods or based on the profile of the driver. Some studies which related to bonus malus are Lemaire [4], Neuhaus [5], Denuit [3], and Déniz [2]. Déniz studied about the bivariate credibility bonus malus premium based on the number of claims and the number of claims with claim size under and above critical value.

In this paper, we proposed new model by modifying the number of claims distribution which is assumed having Poisson ($\theta$) where $\theta$ is a realization of an exponential random variable. The aim of this study is to formulate premium decision based on two categories, they are the claim sizes above or below critical value ($\psi$) which is assessed using Bayesian methods. In addition, we calculated the premium payment if the limit value is changed frequently. We employed 274 and 3755 for the critical value.

2 Results

Bonus malus premium can be calculated regarding its number of claims and claim sizes. We assume that the number of claims ($X$) having Poisson distribution with parameter $\theta > 0$ and $Z_i$ is the claim corresponds to a claim size larger or smaller than critical value ($\psi$), which is assumed having bernoulli distribution. $Z_i = 1$ if the $i$th claim corresponds to a claim size larger then $\psi$ and $Z_i = 0$ if otherwise. Then, $Z = \sum_{i=1}^{x} Z_i$ is the total number of claims with the claim size larger than critical value ($\psi$) which is having binomial distribution ($x, p$). Joint distribution of the number of claims ($X$) and the total number of claims with the claim size larger than critical value $\psi$ ($Z$) has a probability mass function as follows

$$f(x, z; \theta, p) = f_Z(z; x, p)f_X(x; \theta) = \frac{x!}{z!} p^x(1 - p)^{x-z} \frac{\theta^x e^{-\theta}}{x!}.$$ (1)

There are two assumptions in this study. The first assumption is that all policyholders have the same probability of making a claim. Then, $\theta$ which is the parameter of the Poisson distribution becomes a realization of $\Theta$ which assumed an exponential random variable with parameter $\lambda > 0$, as the prior distribution. The second assumption is all policyholders with $x$ claims have the same
probability of making a claim with its size larger than critical value ($\psi$). Then, parameter $p$ which is one of the binomial parameter becomes a realization of $P$ which assumed a beta random variable with parameter $\alpha > 0$ and $\beta > 0$. Joint prior distribution between exponential and beta, with assumption that $\theta$ and $p$ are independent, can be determined as follows

$$\pi(\theta, p) = \pi_1(\theta) \pi_2(p) = \lambda e^{-\lambda \theta} \frac{1}{B(\alpha, \beta)} p^{\alpha-1} (1 - p)^{\beta-1}. \quad (2)$$

Given sample of size $t$: $(\tilde{x}, \tilde{z}) = (\tilde{x}_1, \tilde{z}_1), (\tilde{x}_2, \tilde{z}_2), \ldots, (\tilde{x}_t, \tilde{z}_t)$, the likelihood function can be determined as

$$\ell((\tilde{x}, \tilde{z})|\theta, p) = f(\tilde{x}_1, \tilde{z}_1; \theta, p) f(\tilde{x}_2, \tilde{z}_2; \theta, p) \ldots f(\tilde{x}_t, \tilde{z}_t; \theta, p) \propto p^{t \tilde{x}} (1 - p)^{t(\tilde{x} - \tilde{z})} \theta^{t \tilde{x}} e^{-t \theta}, \quad (3)$$

where $\tilde{x} = \frac{1}{t} \sum_{i=1}^{t} \tilde{x}_i$ and $\tilde{z} = \frac{1}{t} \sum_{i=1}^{t} \tilde{z}_i$.

We need to determining the marginal distribution of $(X, Z)$ which is given by

$$f(x, z) = \int_0^\infty \int_0^1 f(x, z; \theta, p) \pi(\theta, p) \, dp \, d\theta = \frac{B(\alpha + x, \beta + x - z)}{B(\alpha, \beta)} \frac{\lambda}{(\lambda + 1)^x} \quad (4)$$

The likelihood function of the marginal distribution $(X, Z)$ with the sample $(\tilde{x}, \tilde{z}) = (\tilde{x}_1, \tilde{z}_1), (\tilde{x}_2, \tilde{z}_2), \ldots, (\tilde{x}_t, \tilde{z}_t)$, is given by

$$\ln \left( \ell(t; (\tilde{x}, \tilde{z})) \right) = \ln \left[ \left( \frac{\lambda}{B(\alpha, \beta)} \right)^t \frac{(\lambda + 1)^{-\tilde{x}+1} \prod_{i=1}^t B(\alpha + \tilde{x}_i, \beta + \tilde{x}_i - \tilde{z}_i, \tilde{x}_i)}{\tilde{z}_i!(\tilde{x}_i - \tilde{z}_i)!} \right]$$

$$= t \ln \lambda - \ln B(\alpha, \beta) - (\tilde{x} + 1) \ln(\lambda + 1) + \sum_{i=1}^t \left[ \ln B(\alpha + \tilde{x}_i, \beta + \tilde{x}_i - \tilde{z}_i) + \ln \tilde{x}_i! - \ln \tilde{z}_i! (\tilde{x}_i - \tilde{z}_i)! \right]. \quad (5)$$

The claim size that is above critical value and below critical value can be expressed with the following functions

$$g(x, z) = p_l z + p_s (x - z), \quad (6)$$

with $p_l$ is the weight for the number of claim with claim size above the critical value and $p_s$ is weight for the number of claim with claim size below the critical value and we assume that $p_l > p_s$. Thus, the risk premium can be expressed as

$$P(v) = \sum_{x=0}^{\infty} \sum_{z=0}^{x} g(x, z) f(x, z; \theta, p) = (p_l - p_s) p + p_s \theta. \quad (7)$$
Then the premiums prior given by
\[
\text{Premium} = \int_0^\infty \int_0^\infty P(v)\pi(v)dv = \frac{p_1\alpha + p_2\beta}{\lambda(\alpha + \beta)}.
\]  
(8)

Joint posterior distribution which is obtained by multiplying the prior distribution with the likelihood function
\[
\pi^*(v) = \pi^*((\theta, p) | \tilde{x}, \tilde{z}) = \pi(\theta, p) \ell((\tilde{x}, \tilde{z}) | \theta, p)
\]
\[
\propto e^{-\lambda \theta} p^{x-1}(1-p)^{\beta-1} p^{1\cdot(1-(\tilde{x}-\tilde{z})\theta)tx \cdot e^{-t\theta}}.
\]  
(9)

Determining the posterior premiums by \( P^* = \int_0^\infty \int_0^1 P(v)\pi^*(v)dv \). As a result, it becomes necessary determine the constants obtained from
\[
\int_0^\infty \int_0^1 \pi^*((\theta, p) | \tilde{x}, \tilde{z}) = \int_0^\infty \int_0^1 \pi(\theta, p) \ell((\tilde{x}, \tilde{z}) | \theta, p)
\]
\[
\propto \int_0^\infty \int_0^1 e^{-\lambda \theta} p^{x-1}(1-p)^{\beta-1} p^{1\cdot(1-(\tilde{x}-\tilde{z})\theta)tx \cdot e^{-t\theta}} dp d\theta
\]
\[
\propto B(\alpha + z, \beta + x - z) \frac{x^1}{(\lambda + t)^{x+1}}.
\]  
(10)

where \( x = t\tilde{x} \) and \( z = t\tilde{z} \).

Then, posterior premium can be determining as
\[
P^* = \int_0^\infty \int_0^1 P(v)\pi^*(v)dv
\]
\[
\propto \int_0^\infty \int_0^1 [((p_l - p_s)p + p_s)\theta][\theta^x e^{-(\lambda + t)\theta}p^{\alpha + z - 1}(1 - p)^{\beta + x - z - 1}] dp d\theta.
\]

from equation (10), afterwards
\[
P^* = \int_0^\infty \int_0^1 [((p_l - p_s)p + p_s)\theta^{x + 1}e^{-(\lambda + t)\theta} p^{\alpha + z - 1}(1 - p)^{\beta + x - z - 1}
\]
\[
= \frac{1}{\int_0^\infty \int_0^1 \pi^*((\theta, p) | \tilde{x}, \tilde{z})} dp d\theta
\]
\[
= \frac{p_1(\alpha + z) + p_2(\beta + x - z)}{(\alpha + \beta + x)} \frac{x^1}{\lambda + t}.
\]  
(11)

Finally, Bayes premium is the ratio of posterior premiums with prior premium, that is
\[
\frac{P^*}{\text{Premium}} = \left[\frac{p_1(\alpha + z) + p_2(\beta + x - z)}{(\alpha + \beta + x)}\right] \frac{\lambda(\alpha + \beta)}{(\alpha + \beta + x)(\lambda + t)(p_1a + p_2b)}.
\]  
(12)
3 Numerical Experiment

We used data which is available at the website of Faculty of Business and Economics, Macquarie University (Sydney, Australia) and the claim sizes which has been regrouped by Déniz [2] for application of determining premiums. There are 67856 policyholders which are 4333 of them make claim once, 271 twice, 18 three times, and 2 four times.

Table 3.1 shows the number of claims (X) and total number of claims with claim size larger than critical value (Z), assume the critical value is \( \psi = 274 \). Table 3.2 depicts the premiums payment with number of claims (X) and the total number of claims with claim size larger than critical value (Z). We assume \( p_I = 1 \) and \( p_x = 0.5 \). Regarding the model selection criteria, we use the Akaike Information Criterion (AIC), see Akaike [1] for details. We use chi square test for the goodness-of-fit test between observed and expected frequencies.

Table 3.1 The number of claims (x) and the total number of claims with claim size larger than critical value (z). \( \psi =274 \), observed (in bold) and expected data (below)

<table>
<thead>
<tr>
<th>x \ z</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>Total</th>
</tr>
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<tbody>
<tr>
<td>0</td>
<td>63232</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>63232</td>
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<tr>
<td></td>
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<td></td>
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<td></td>
<td>63253.84</td>
</tr>
<tr>
<td>1</td>
<td>815</td>
<td>3518</td>
<td></td>
<td></td>
<td></td>
<td>4333</td>
</tr>
<tr>
<td></td>
<td>817.69</td>
<td>3472.34</td>
<td></td>
<td></td>
<td></td>
<td>4290.03</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
<td>99</td>
<td>166</td>
<td></td>
<td></td>
<td>271</td>
</tr>
<tr>
<td></td>
<td>10.57</td>
<td>89.77</td>
<td>190.62</td>
<td></td>
<td></td>
<td>290.96</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>2</td>
<td>8</td>
<td>8</td>
<td></td>
<td>18</td>
</tr>
<tr>
<td></td>
<td>0.14</td>
<td>1.74</td>
<td>7.39</td>
<td>10.46</td>
<td></td>
<td>19.73</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
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<td>0.002</td>
<td>0.03</td>
<td>0.19</td>
<td>0.54</td>
<td>0.57</td>
<td>1.34</td>
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<td>Total</td>
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<td>9</td>
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<td></td>
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<td>3563.89</td>
<td>198.20</td>
<td>11.00</td>
<td>0.57</td>
<td>67856.00</td>
</tr>
</tbody>
</table>

\[ \alpha = 9031200, \ \beta = 2126720, \ \lambda = 13.7444, \ \text{AIC} = 40750.9, \ \chi^2 = 11.97629754. \]

Table 3.1 describes the classification of observation data (in bold) and the expectation value (below) where x indicates the number of claims and z indicates the total number of claims with claim size larger than a critical value. In addition, we also use standard Pearson’s chi squared test to see the relationship between the number of claims (x) and the total number of claims with claim size larger than critical value (z). We obtained \( \chi^2 \) is 11.97629754, while \( \chi^2 \) tables with 1 for degrees of freedom and 5% error is 3.841. This result shows that there is a relationship between the number of claims (x) and the total number of claims with claim size larger than critical value (z).
Table 3.2 Bonus malus premium between the number of claims \((x)\) and total number of claims with claim size larger than critical value \((z)\), \(\psi = 274\)

<table>
<thead>
<tr>
<th>(t)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x=0)</td>
<td>0</td>
<td>1</td>
<td>0.9321776</td>
<td>0.8729707</td>
<td>0.8208356</td>
<td>0.7745768</td>
</tr>
<tr>
<td>(x=1)</td>
<td>0</td>
<td>1.8643552</td>
<td>1.7459414</td>
<td>1.6416713</td>
<td>1.5491535</td>
<td>1.4665073</td>
</tr>
<tr>
<td>(x=2)</td>
<td>0</td>
<td>2.7965327</td>
<td>2.6189119</td>
<td>2.4625068</td>
<td>2.3237301</td>
<td>2.1997608</td>
</tr>
<tr>
<td>(x=3)</td>
<td>0</td>
<td>3.7287101</td>
<td>3.4918824</td>
<td>3.2833421</td>
<td>3.0983067</td>
<td>2.9330143</td>
</tr>
<tr>
<td>(x=4)</td>
<td>0</td>
<td>4.6608875</td>
<td>4.3648528</td>
<td>4.1041775</td>
<td>3.8728832</td>
<td>3.6662677</td>
</tr>
</tbody>
</table>

Table 3.2 shows that new policyholders who join the insurance must pay a premium of 100%, while policyholders with a particular claim and the specific time period have increase or decrease of premium payments. The greater the number of claims with claim size larger than critical value, the higher the premium payment. Similarly, when it is compared with the time with the same claim, the more time the policyholders joining, the smaller the premium to be paid. In addition to grouping the critical value 274, we also use 3755. As it can be seen from Table 3.3 and 3.4, each group data and premium payment has to be paid by the policy holders who has critical value 3755.

Table 3.3 The number of claims \((x)\) and the total number of claims with claim size larger than critical value \((z)\). \(\psi = 3755\), observed (in bold) and expected data (below)

<table>
<thead>
<tr>
<th>(x \times z)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x=0)</td>
<td>63232</td>
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<td>63253.84</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(x=1)</td>
<td>3753</td>
<td>580</td>
<td>4333</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(x=2)</td>
<td>233</td>
<td>29</td>
<td>9</td>
<td>271</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(x=3)</td>
<td>14</td>
<td>4</td>
<td>0</td>
<td>0</td>
<td>18</td>
<td></td>
</tr>
<tr>
<td>(x=4)</td>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>Total</td>
<td>67234</td>
<td>613</td>
<td>9</td>
<td>0</td>
<td>0</td>
<td>67855</td>
</tr>
</tbody>
</table>

\(\alpha = 1623940, \beta = 11082000, \lambda = 13.7444, \text{AIC} = 48341.4, \chi^2 = 30.1306\)
### Comparing bonus malus premiums of two types claim

Table 3.4 Bonus malus premium between the number of claims \((x)\) and total number of claims with claim size larger than critical value \((z)\), \(\psi = 3755\)

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<th>2</th>
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<th>5</th>
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<tbody>
<tr>
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<td>0.9321776</td>
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<td>1.4665073</td>
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<tr>
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<tr>
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<td>3.2833431</td>
<td>3.0983076</td>
<td>2.9330152</td>
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</tr>
<tr>
<td>(x=4)</td>
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<td>3.6662692</td>
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</tr>
</tbody>
</table>

In conclusion, the premium significance between \(\psi = 274\) and \(\psi = 3755\) is not significant.

## 4 Conclusion

Automobile insurance with bonus malus system in the determination of premiums can be determined by looking at the number of claims and the claim sizes. We use a Poisson distribution with parameter \(\theta > 0\) for the number of claims \((X)\) and \(Z = \sum_{i=1}^{x} Z_i\) is the total number of claims with claim size larger than critical value is binomial \((x, p)\). In addition, there are two assumptions as the prior distribution that all policyholders have the same probability of making a claim is assumed exponential prior \((\lambda)\) and the assumption that all policyholders with \(x\) claim have the same probability of making a claim with claim size larger than \(\psi\) assumed beta prior \((\alpha, \beta)\). Finally, the model of premium payment of bonus malus system in this case with Bayesian method is

\[
[p_l (\alpha + z) + p_s (\beta + x - z)](x + 1)\lambda(\alpha + \beta)
\]

\[
(\alpha + \beta + x)(\lambda + t)(p_l \alpha + p_s \beta)
\]

The more often the policyholder submits a claim, the greater the price premium to be paid over the previous period, and vice versa. The alteration of critical value \((\psi)\) does not affect premium payment significantly.
References


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