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A Note on Global Smoothing Effects for the Unitary Group Associated with the B-O Equation

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Abstract

In this note we show a global smoothing effects for the unitary group associated to B-O type equation.

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1 Introduction

In this work we describe a result on global smoothing effects for the unitary group associated to B-O type equation

$$u_t + \mathcal{H}\Delta u + uu_x = 0, \tag{1}$$

Observe that (1) is a bidimensional extension of the Benjamin-Ono equation

$$\partial_t u + \mathcal{H}\partial_x^2 u + uu_x = 0, \tag{2}$$

which describes certain models in physics about wave propagation in a stratified thin regions . This last equation shares with the KdV equation

$$u_t + u_{xxx} + uu_x = 0 \tag{3}$$

many interesting properties. For example, both possess infinite conservation laws, they have solitary waves as solutions which are stable and behave like soliton (this last is evidenced by the existence of multisoliton type solutions). Also, the local and global well-posedness were proven in the Sobolev spaces context (in low regularity spaces inclusive, see, e.g., [2], [7], [5], [6] and [8]).

The plan of this paper is the following: In Section 2, we present the basic notations and results that we will need. In Section 3, we examine the global smoothing effects.

2 Preliminaries

The following notations will be used through this paper.

1. $\mathcal{S}(\mathbb{R}^2)$ is the Schwartz space.

2. $\mathcal{S}'(\mathbb{R}^2)$ is the space of tempered distributions.

3. For $f \in \mathcal{S}'(\mathbb{R}^2)$, $\widehat{f}$ is the Fourier transform of f and $\check{f}$ is the inverse Fourier transform of f. We recall that

$$\widehat{f}(\xi,\eta) = \frac{1}{2\pi}\int_{\mathbb{R}^2} f(x,y)e^{-i(x\xi+y\eta)}dxdy,$$

   for all $(\xi,\eta) \in \mathbb{R}^2$, when $f \in \mathcal{S}(\mathbb{R}^2)$.

4. $\mathcal{H}^{(x)}$ is the Hilbert transform with respect to the variable x. If $f \in \mathcal{S}(\mathbb{R}^2)$,

$$\mathcal{H}^{(x)}f(x,y) = \mathcal{H}f(x,y) = \sqrt{\frac{2}{\pi}}\left(\text{p.v.}\int_{-\infty}^{\infty}\frac{1}{\xi - x}f(x,\xi)\,d\xi\right).$$

5. If X, Y are Banach spaces, $B(X,Y)$ is the space of all continuous linear operators endowed with the norm:

$$\|T\|_{B(X,Y)} = \sup_{\|x\|=1}\|Tx\|.$$

   If $X = Y$ we simply write $B(X)$.

The following results are part of the important stock of tools that are used in the analysis.

The first of them is given by the following proposition due to Young (its proof can be found in [1]).

Theorem 1.1. *Let $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$, $1 \leq p, q \leq \infty$ with $\frac{1}{p} + \frac{1}{q} \geq 1$. then $f * g \in L^r(\mathbb{R}^n)$, where $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$. Moreover,*

$$\|f * g\|_r \leq \|f\|_p \|g\|_q.$$

Theorem 1.2. *(Riesz-Thorin). Let $p_0 \neq p_1$, $q_0 \neq q_1$. Let T be a bounded linear operator from $L^{p_0}(X, A, \mu)$ to $L^{q_0}(Y, B, \nu)$ with norm M_0 and from $L^{p_1}(X, A, \mu)$ to $L^{q_1}(Y, B, \nu)$ with norm M_1. Then T is bounded from $L^{p_\theta}(X, A, \mu)$ to $L^{q_\theta}(Y, B, \nu)$ with norm M_θ such that*

$$M_\theta \leq M_0^{1-\theta} M_1^\theta$$

with

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \quad \theta \in (0,1).$$

2 Global Smoothing Effects

If $\phi \in \mathcal{S}$ then

$$e^{t(-\mathcal{H}\Delta)}\phi(x,y) = \frac{1}{2\pi}\int e^{i(\operatorname{sgn}(\xi)(\xi^2+\eta^2)t + x\xi + y\eta)}\widehat{\phi}(\xi,\eta)d\xi d\eta = \frac{1}{2\pi} I(t) * \phi(x,y)$$

where $I(t) = (e^{i\operatorname{sgn}(\xi)(\xi^2+\eta^2)t})^\vee$. Describes the solution of the linear problem

$$\begin{cases} \partial_t u = -\mathcal{H}\Delta u, \\ u(0) = \phi(x,y) \end{cases}$$

Lemma 2.1. *For any x, y and $t \neq 0$ real numbers,*

$$I(t)(x,y) = \frac{c}{t} e^{-\frac{i}{4t}(x^2+y^2)} \int_{\frac{x}{\sqrt{t}}}^{\infty} e^{\frac{i}{4}s^2} ds + \frac{\bar{c}}{t} e^{\frac{i}{4t}(x^2+y^2)} \int_{\frac{x}{\sqrt{t}}}^{\infty} e^{-\frac{i}{4}s^2} ds,$$

where $c = (1+i)/2$.

Proof. Is clear that

$$\begin{aligned} 2\pi I(1)(x,y) = & \int_{\mathbb{R}} \int_0^\infty e^{i(\xi^2+\eta^2+x\xi+y\eta)}\, d\xi d\eta + \int_{\mathbb{R}} \int_{-\infty}^0 e^{i(-\xi^2-\eta^2+x\xi+y\eta)}\, d\xi d\eta \\ = & e^{-\frac{i}{4}(x^2+y^2)} \left(\int_{\mathbb{R}} e^{i(\eta+y/2)^2}\, d\eta \right) \left(\int_0^\infty e^{i(\xi+x/2)^2}\, d\xi \right) + \\ & + e^{\frac{i}{4}(x^2+y^2)} \left(\int_{\mathbb{R}} e^{-i(\eta-y/2)^2}\, d\eta \right) \left(\int_{-\infty}^0 e^{-i(\xi-x/2)^2}\, d\xi \right). \end{aligned}$$

A simple change of variable prove the theorem for $t = 1$. Using the homogeneity property of the Fourier transform, the theorem follows for any $t \neq 0$. $\square$

The last lemma implies the following $L^p - L^q$ estimate for the group $e^{t(-\mathcal{H}\Delta)}$.

Proposition 2.2. *For any $f \in L^1 \bigcap L^2$, it has that*

$$|e^{t(-\mathcal{H}\Delta)} f|_{\frac{2}{1-\theta}} \leq c|t|^{-\theta} |f|_{\frac{2}{1+\theta}},$$

for $\theta \in [0,1]$

Proof. We obtain the result by using the Young's inequality for convolution, the lemma above and the Riesz-Thorin Interpolation Theorem. □

The next theorem describes the global smoothing property of the group $\{e^{t(-\mathcal{H}\Delta)}\}_{t=-\infty}^{\infty}$.

Theorem 2.3. *The group $\{e^{t(-\mathcal{H}\Delta)}\}_{t=-\infty}^{\infty}$ satisfies:*

$$\left(\int_{-\infty}^{\infty} \|e^{t(-\mathcal{H}\Delta)} f\|_p^q \right)^{\frac{1}{q}} \leq c\|f\|_2,$$

$$\left(\int_{-\infty}^{\infty} \left\| \int_{-\infty}^{\infty} e^{(t-s)(-\mathcal{H}\Delta)} g(.,s)\, ds \right\|_p^q dt \right)^{\frac{1}{q}} \leq c \left(\int_{-\infty}^{\infty} \|g(.,t)\|_{p'}^{q'}\, dt \right)^{\frac{1}{q'}}$$

and

$$\left\| \int_{-\infty}^{\infty} e^{t(-\mathcal{H}\Delta)} g(.,t)\, dt \right\|_2 \leq c \left(\int_{-\infty}^{\infty} \|g(.,t)\|_{p'}^{q'}\, dt \right)^{\frac{1}{q'}}$$

with $2 \leq p < \infty$, $\frac{2}{q} = 1 - \frac{2}{p'}$ and $\frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1$. where $c = c(p)$ is a constant that depends only on p.

The proof of this result is essentially the same as in [1]

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