AB-Ideals of AB-Algebras

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Abstract

The aim of this paper is to introduce and study new algebraic structure, called AB-algebra and investigate some of its properties. We give the notion of AB-ideal of AB-algebra and quotient AB-Algebras, several theorems, properties are stated and proved.

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Introduction

W.A. Dudek introduced new of abstract algebras: is called BCC-algebras as a generalization of the concept of set-theoretic difference and propositional calculus and studied some important properties in ([6], [8]). In [5], S.S. Ahn and H.S. Kim have introduced the notion of QS-algebras. A.T. Hameed and et al [1] introduced the notions of QS-ideal and fuzzy QS-ideal of QS-algebra. A.T. Hameed introduce new of abstract algebras: is called KUS-algebras and define its ideals as KUS-algebras in ([2]). In this paper, we introduce new of abstract algebras: is called AB-algebras which is a generalization of the concept BCC-algebra and define its subalgebras and ideals of AB-algebra. We study quotient of AB-algebra and describe its properties.

1. Preliminaries

In this section, certain definitions, known results and examples that will be used in the sequel are described.
**Definition 1.1.** ([7], [8]) A **BCC-algebra** is a nonempty set $X$ with a constant $(0)$ and a binary operation $(*)$ satisfying the following axioms: for all $x, y, z \in X$.

(i) \[ ((x*y) *(z*y)) *(x*z) = 0, \]
(ii) \[ 0 *x = 0, \]
(iii) \[ x *0 = x, \]
(iv) \[ x*x = 0, \]
(v) \[ x*y = 0 \text{ and } y*x = 0 \Rightarrow x=y. \]

We can define a binary relation $(\leq)$ by putting $x \leq y$ if and only if $x*y=0$.

**Definition 1.2.** ([9]) A nonempty subset $S$ of a BCC-algebra $X$ is called a **subalgebra** of $X$ if $x*y \in S$ whenever $x, y \in S$.

**Definition 1.3.** ([6]) Let $X$ be a BCC-algebra. $I$ is called a BCC-ideal of $X$ if it satisfies the following conditions:

(i) \[ 0 \in I, \]
(ii) \[ (x*y)*z \in I \text{ and } y \in I \text{ imply } x*z \in I. \]

**Definition 1.4.** ([5]) A **QS-algebra** is a nonempty set $X$ with constant $(0)$ and a binary operation $(*)$ the following axioms: for all $x, y, z \in X$.

(QS0): \[ (z*y) * (z*x) = x*y, \]
(QS2): \[ x * 0 = x, \]
(QS3): \[ x * x = 0, \]
(QS4): \[ (x*y) * z = (x*z) * y. \]

We can define a binary relation $(\leq)$ by putting $x \leq y$ if and only if $x*y=0$.

**Definition 1.5([5]).** Let $X$ be a QS-algebra and let $S$ be a nonempty subset of $X$. $S$ is called a **QS-subalgebra** of $X$ if $x*y \in S$ whenever $x \in S$ and $y \in S$.

**Definition 1.6.** ([1]) A nonempty subset $I$ of a QS-algebra $X$ is called a **QS-ideal** of a QS-algebra $X$ if it satisfies: for $x, y, z \in X$,

(IQS1) \( 0 \in I, \)
(IQS2) \( (z*y) \in I \text{ and } (x*y) \in I \text{ imply } (z*x) \in I. \)

**Definition 1.7.** ([2], [4]) A **KUS-algebra** is a nonempty set $X$ with a constant $(0)$ and a binary operation $(*)$ satisfying the following axioms: for any $x, y, z \in X$.

1. \( (z*y)*(z*x)=y*x \)
2. \( 0*x=x \)
3. \( x*x=0 \)
4. \( x*(y*z)=y*(x*z) \)

In $X$, we define a binary relation $(\leq)$ by: $x \leq y$ if and only if $y*x=0$. 
Definition 1.8. ([2], [4]) Let X be a KUS-algebra and let S be a nonempty subset of X. S is called a **KUS-subalgebra** of X if \( x \ast y \in S \) whenever \( x, y \in S \).

Definition 1.9. ([3]) A nonempty subset I of a KUS-algebra X is called a **KUS-ideal** of X if it satisfies: for \( x, y, z \in X \),

\[(I\text{kus}_1) \quad 0 \in I,
(I\text{kus}_2) \quad z \ast y \in I \text{ and } y \ast x \in I \text{ imply } z \ast x \in I.\]

2. On AB-algebras

In this section, we introduce new notions called AB-algebras and define its AB-subalgebras and AB-ideals of AB-algebra and give several characteristics.

Definition 2.1. An **AB-algebra** is a nonempty set X with a constant (0) and a binary operation (\( \ast \)) satisfying the following axioms: for all \( x, y, z \in X \).

\[
\begin{align*}
AB_1) & \quad ( (x \ast y) \ast (z \ast y) ) \ast (x \ast z) = 0, \\
AB_2) & \quad 0 \ast x = 0, \\
AB_3) & \quad x \ast 0 = x.
\end{align*}
\]

In X we can define a binary relation (\( \leq \)) by: \( x \leq y \) if and only if, \( x \ast y = 0 \).

Example 2.2. Let \( X = \{0, a, b, c\} \) in which (\( \ast \)) is defined by the following table:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>a</td>
<td>a</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>a</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>c</td>
<td>c</td>
<td>b</td>
<td>b</td>
<td>0</td>
</tr>
</tbody>
</table>

It is easy to show that \( (X; \ast, 0) \) is an AB-algebra.

Example 2.3. Let \( X = \{0, 1, 2, 3, 4\} \) in which (\( \ast \)) is defined by the following table:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
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<tr>
<td>1</td>
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<td>0</td>
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</tr>
<tr>
<td>2</td>
<td>2</td>
<td>2</td>
<td>0</td>
<td>0</td>
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<tr>
<td>3</td>
<td>3</td>
<td>3</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>3</td>
<td>4</td>
<td>3</td>
<td>0</td>
</tr>
</tbody>
</table>

It is easy to show that \( (X; \ast, 0) \) is an AB-algebra.

Remark 2.4. \( (X; \ast, 0) \) is an AB-algebra if and only if, it satisfies that: for all \( x, y, z \in X \),...
(i'): \((x * y) * (z * y) \leq x * z\),
(ii'): \(x * 0 = x\)
(iii'): \(x \leq x\),

**Proposition 2.5.** In any AB-algebra \((X; *, 0)\), the following properties holds: for all \(x, y, z \in X\);
1) \(x * x = 0\),
2) \((x * y) * y = 0\),
3) \(x * y = 0\) implies that \(0 * x = 0 * y\),
4) \(x * 0 = y * 0\) implies that \(x = y\).

**Proof.**
1) Put \(y = z = 0\) in \((AB_1)\), we get \(((x * 0) * (0 * 0)) * (x * 0) = 0\), and by \((AB_2)\) follows that \(x * x = 0\).
2) Put \(z = 0\) in \((AB_1)\), \(((x * y) * (0 * y)) * (x * 0) = 0\), implies that \(((x * y) * 0) * y = 0\), and by \((AB_3)\) follows that \((x * y) * y = 0\).
3) It is clear.
4) It is clear.

**Proposition 2.6.** In any AB-algebra \((X; *, 0)\), the following properties holds: for all \(x, y, z \in X\);

a) \(x \leq y\) implies that \(x * z \leq y * z\),
b) \(x \leq y\) implies that \(z * y \leq z * x\)

**Proof.**
a) Since by \((AB_1)\), \(((x * z) * (y * z)) * (x * y) = 0\), but \(x \leq y\) implies \((x * y) = 0\), then \(((x * z) * (y * z)) * 0 = 0\). By \((AB_2)\), we get \((x * z) * (y * z) = 0\), i.e; \(x * z \leq y * z\).
b) Since \(x \leq y\) implies \((x * y) = 0\), by \((AB_1)\), we obtain \(((z * x) * (x * x)) * (z * y) = 0\), then \(((z * x) * 0) * (z * y) = 0\). By \((AB_2)\), we get \((z * x) * (z * y) = 0\), i.e; \(z * x \leq z * y\).

**Proposition 2.7.** In any AB-algebra \((X; *, 0)\), the following properties holds: for all \(x, y, z \in X\);

a) \(x*(y*z) = y*(x*z)\)
b) \((y*x) * (0 * x) = y\),
c) \(x = (x * 0) * 0\).

**Proof.**
a) By (iii) and proposition (2.5(2)), \(x = x * 0 = x * (x * z) * z\) it follows that \(x * (y * z) \leq [x * (x * z) * z] * (y * z)\). By the transitivity of \((\leq)\) gives \(x * (y * z) \leq y * (x * z)\).
And we replacing \(x\) by \(y\) and \(y\) by \(x\), we obtain \(y * (x * z) \leq x * (y * z)\). By the anti-symmetry of \((\leq)\), thus \(x * (y * z) = y * (x * z)\).
b) \((y \ast x) \ast (0 \ast x) = y\), \((y \ast x) \ast (y \ast x) = (y \ast x) \ast (0 \ast x) = (y \ast 0) = y\).

c) \(x \ast 0 = (x \ast 0) \ast 0 = (x \ast 0) \ast (0 \ast 0) = ((x \ast 0) \ast 0) \ast 0\), then by proposition (1.5(4)), \(x = (x \ast 0) \ast 0\).

**Definition 2.8.** A nonempty subset \(S\) of an AB-algebra \(X\) is called an **AB-subalgebra** of \(X\) if \(x \ast y \in S\), whenever \(x, y \in S\).

**Definition 2.9.** A nonempty subset \(I\) of an AB-algebra \(X\) is called an **AB-ideal** of \(X\) if it satisfies the following conditions: for all \(x, y, z \in X\).

\[
\begin{align*}
I_{AB1} & : 0 \in I; \\
I_{AB2} & : x \ast (y \ast z) \in I \text{ and } y \in I \text{ imply } x \ast z \in I.
\end{align*}
\]

**Example 2.10.** Let \(X = \{0, 1, 2, 3, 4, 5\}\) in which \((\ast)\) is defined by the following table:

<table>
<thead>
<tr>
<th>(\ast)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
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<td>1</td>
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<td>3</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>0</td>
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<tr>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>0</td>
</tr>
</tbody>
</table>

Then \((X; \ast, 0)\) is an AB-algebra. It is easy to show that \(I_1 = \{0, 1, 2, 3, 4\}\), \(I_2 = \{0, 1, 2\}\) and \(I_3 = \{0, 1, 2\}\) are AB-ideals of \(X\).

**Proposition 2.11.** Every AB-ideal of AB-algebra \(X\) is an AB-subalgebra.

**Proof.** Let \(I\) be an AB-ideal of AB-algebra \(X\), for all \(x, y \in X\) and let \(x, y \in I\), then \((x \ast 0) \ast y \in I\) and \(0 \in I\). Hence by (AB3), \(x \ast y \in I\) Therefore \(I\) is an AB-subalgebra. \(\Diamond\)

**Proposition 2.12.** If \(I\) is an AB-ideal of AB-algebra \(X\) and \(J\) is an AB-ideal of \(I\), then \(J\) is an AB-ideal of \(X\).

**Proof.** Since \(J\) is an AB-ideal of \(I\), then \(0 \in J\). Let \(x, y, z \in X\) such that \(((x \ast y) \ast z) \in J\) and \(y \in J\). It follows that \(((x \ast y) \ast z) \in I\) and \(y \in I\). By assumption, \(I\) is an AB-ideal of \(X\), so \(x \ast z \in I\) and \(y \in J\). From \(J\) is an AB-ideal of \(I\), so \(x \ast z \in J\). Therefore, \(J\) is an AB-ideal of \(X\). \(\Diamond\)

**Theorem 2.13.** Let \(\{I_j : j \in J\}\) be a family of AB-ideals of an AB-algebra \(X\). Then \(\bigcap_{j \in J} I_j\) is an AB-ideal of \(X\).

**Proof.** Let \(\{I_j : j \in J\}\) be a family of AB-ideals of \(X\). It is obvious that \(\bigcap_{j \in J} I_j \subseteq X\). Since \(0 \in I_j\), for all \(j \in J\), it follows that \(0 \in \bigcap_{j \in J} I_j\).
Let \((x \ast y) \ast z \in \bigcap_{j \in J} I_j\) and \(y \in \bigcap_{j \in J} I_j\). We get that \((x \ast y) \ast z \in I_j\) and \(y \in I_j\), for all \(j \in J\), then \(x \ast z \in I_j\), for all \(j \in J\). Because \(I_j\) is an \(AB\)-ideal of \(X\). So \(x \ast z \in \bigcap_{j \in J} I_j\), proving our theorem. □

**Corollary 2.14.** Let \(\{I_j; j \in J\}\) be a family of \(AB\)-subalgebras of an \(AB\)-algebra \(X\). Then \(\bigcap_{j \in J} I_j\) is an \(AB\)-subalgebra of \(X\).

**Proof.** By Proposition (2.12) and Theorem (2.13). □

**Theorem 2.15.** Let \(\{J_i; i \in N\}\) be a family of \(AB\)-ideals of \(AB\)-algebra \(X\) where \(J_i \subseteq J_{i+1}\), for all \(n \in N\). Then \(\bigcup_{n=1}^{\infty} J_n\) is an \(AB\)-ideal of \(X\).

**Proof.** Let \(\{J_i; i \in N\}\) be a family of \(AB\)-ideals of \(X\). It can be proved easily that \(\bigcup_{n=1}^{\infty} J_n \subseteq X\). Since \(J_i\) is an \(AB\)-ideal of \(X\) for all \(i\), so \(1 \in \bigcup_{n=1}^{\infty} J_n\).

Let \(((x \ast y) \ast z) \in \bigcup_{n=1}^{\infty} J_n\) and \(y \in \bigcup_{n=1}^{\infty} J_n\). It follows that \(((x \ast y) \ast z) \in J_i\) for some \(j \in N\) and \(y \in J_k\) for some \(k \in N\). Furthermore, let \(J_j \subseteq J_k\).

Hence \(((x \ast y) \ast z) \in J_k\) and \(y \in J_k\). By assumption, \(J_k\) is an \(AB\)-ideal of \(X\), it follows that \(x \ast z \in J_k\). Therefore, \(x \ast z \in \bigcup_{n=1}^{\infty} J_n\), proving that \(\bigcup_{n=1}^{\infty} J_n\) is an \(AB\)-ideal of \(X\). □

**Corollary 2.16.** Let \(\{J_i; i \in N\}\) be a family of \(AB\)-subalgebras of an \(AB\)-algebra \(X\) where \(J_i \subseteq J_{i+1}\), for all \(n \in N\). Then \(\bigcup_{n=1}^{\infty} J_n\) is an \(AB\)-subalgebras of \(X\).

**Proof.** By Proposition (2.12) and Theorem (2.156). □

### 3. Quotient \(AB\)-Algebras

In this section, we describe congruence on \(AB\)-algebras and we study quotient of \(AB\)-algebra and describe its properties.

**Definition 3.1.** Let \(I\) be an \(AB\)-ideal of \(AB\)-algebra \(X\). Define a relation \((\sim)\) on \(X\) by: \(x \sim y\) if and only if \(x \ast y \in I\) and \(y \ast x \in I\), as define a relation \((\sim)\) in ([2],[4]).

**Theorem 3.2.** If \(I\) is an \(AB\)-ideal of \(AB\)-algebra \(X\), then the relation \((\sim)\) is an equivalence relation on \(X\).

**Proof.** Let \(I\) be an \(AB\)-ideal of \(X\) and \(x, y, z \in X\). By proposition (2.5(1)), \(x \ast x = 0\) and assumption, \(x \ast x \in I\). That is, \(x \sim x\). Hence \((\sim)\) is reflexive.

Next, suppose that \(x \sim y\). It follows that \(x \ast y \in I\) and \(y \ast x \in I\). Then \(y \sim x\), so \((\sim)\) is symmetric.

Finally, let \(x \sim y\) and \(y \sim z\). Then \(x \ast y, y \ast x, y \ast z, z \ast y \in I\) and \(((x \ast y) \ast (z \ast y)) \ast (x \ast z) = 0 \in I\) and \(z \ast y \in I\), then \((x \ast y) \ast (x \ast z) \in I\), and since \(x \ast y \in I\), so \(x \ast z \in I\). Similarly, \(z \ast x \in I\). Thus \((\sim)\) is transitive. Therefore \((\sim)\) is an equivalence relation. □
Lemma 3.3. Let I be an AB-ideal of AB-algebra X. For any x, y, u, v ∈ X, if u ∼ v and x ∼ y, then u ∗ x ∼ v ∗ y.

Proof. Assume that u ∼ v and x ∼ y, for any x, y, u, v ∈ X, then u ∗ v, v ∗ u, x ∗ y, y ∗ x ∈ I and by (AB1), we see that (((v ∗ u) ∗ (x ∗ u)) ∗ (v ∗ x)) = 0 and (u ∗ v) ∗ (x ∗ v)) ∗ (u ∗ v) = 0. From assumption and I is an ideal of X, these imply that (u ∗ x) ∗ (v ∗ x) ∈ I and (v ∗ x) ∗ (u ∗ x) ∈ I. This shows that u ∗ x ∼ v ∗ x…(1)

On the other hand, by proposition (2.5), we have that ((x ∗ y) ∗ (v ∗ y)) ∗ (x ∗ v) = 0 and (((y ∗ x) ∗ (v ∗ x)) ∗ (y ∗ v) = 0. From assumption and I is an AB-ideal of X, these imply that (x ∗ y) ∗ (x ∗ v) ∈ I and (y ∗ x) ∗ (y ∗ v) ∈ I. Thus v ∗ x ∼ v ∗ y…(2)

From (1) and (2), u ∗ x ∼ v ∗ y. This completes the proof. □

Corollary 3.4. If I is an AB-ideal of AB-algebra X, then the relation (∼) is a congruence relation on X

Proof. By Theorem (3.2) and Lemma (3.3). □

Definition 3.5. Let I be an AB-ideal of AB-algebra X. Given x ∈ X, the equivalence class [x]I of x is defined as the set of all element of X that are equivalent to x, that is, [x]I = {y ∈ X : x ∼ y}.

We define the set X/I = {[x] : x ∈ X} and a binary operation (◦) on X/I by [x]◦ [y] = [x ∗ y].

Theorem 3.6. If I is an AB-ideal of AB-algebra X with X/I = {[x] : x ∈ X}, then the binary operation ◦ is a mapping from X/I × X/I to X/I.

Proof. Let [x1], [x2], [y1], [y2] ∈ X/I such that [x1] = [x2] and [y1] = [y2]. It follows that x1 ∼ x2 and y1 ∼ y2. By Lemma (3.3), x1 ∗ y1 ∼ x2 ∗ y2, proving that [x1 ∗ y1] = [x2 ∗ y2]. □

Theorem 3.7. If I is an AB-ideal of AB-algebra X, then (X/I; ◦, [0]) is an AB-algebra. Moreover, the set X/I is called the quotient AB-algebra.

Proof. It is clear that [0]◦ [x] = [0 ∗ x] = [x] and [x]◦ [0] = [x ∗ 0] = [0].

Let [x], [y], [z] ∈ X/I Then ((([x]◦ [y]) ◦ ([z]◦ [y])) ◦ ([x]◦ [z])) = ([x ∗ y]◦ [z ∗ y]) ◦ ([x ∗ z]) = ([x ∗ y]◦ ([z ∗ y])) ◦ ([x ∗ z]) = ([x ∗ y]◦ ([z ∗ y])) ◦ ([x ∗ z]) = [0]I .

Therefore, (X/I; ◦, [0]) is an AB-algebra. □

Example 3.8. Let X = {0, 1, 2, 3} be a set with the following table:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
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<tr>
<td>1</td>
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<td>2</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>2</td>
<td>2</td>
<td>0</td>
</tr>
</tbody>
</table>
Then \((X; *, 0)\) is an AB-algebra. It is easy to show that \(I = \{0, 1\}\) is an AB-ideal of \(X\). We can get that \(X/I = \{[0], [1]\}\), and \([1] = [2] = \{1, 2\}\). Let \((\circ)\) be defined on \(X/I\) by:

\[
\begin{array}{ccc}
0 & 0 & 0 \\
0 & 1 & 1 \\
1 & 1 & 0
\end{array}
\]

Then \((X/I; \circ, [0])\) is an AB-algebra.

**Proposition 3.9.** Let \(X\) be an AB-algebra and \(I, J\) be any sets such that \(I \subseteq J \subseteq X\). Suppose that \(I\) is an AB-ideal of \(X\), then \(J\) is an AB-ideal of \(X\) if and only if \(J/I\) is an AB-ideal of \(X/I\).

**Proof.** Let \(I\) be an AB-ideal of \(X\) with \(I \subseteq J \subseteq X\). Suppose firstly that \(J\) is an AB-ideal of \(X\), then \(J/I = \{[x] : x \in J\}\), where \([x] = \{y \in J : x \sim y\}\), and \(X/I = \{[x] : x \in X\}\), where \([x] = \{y \in X : x \sim y\}\). Obviously, \(J/0 \subseteq X/I\) and \([0]_J \subseteq 0\).

Now, let \([(x \circ y) \circ z] = ([x] \circ [y]) \circ [z] \in J/I\) and \([y]_J \in 0\). Then \([(x \circ y) \circ z] = ([x] \circ [y]) \circ [z] \subseteq J/I\), it follows that \((x \circ y) \circ z \in J\) and \(y \in J\). By assumption, \((x \circ z) \in J\). Accordingly, \([x \circ z] = [x] \circ [z]_J \in J/I\), this shows that \(J/I\) is an AB-ideal of \(X/I\).

On the other hand, suppose that \(J/I\) is an AB-ideal of \(X/I\) and \(I\) is an AB-ideal of \(X\) with \(I \subseteq J \subseteq X\). Thus, \(0 \in J\). Let \((x \circ y) \circ z \in J\) and \(y \in J\). It follows that \((x \circ y) \circ z \in J\), \([y]_J \in J/I\). Since \((x \circ y) \circ z = ([x] \circ [y]) \circ [z]_J\), so \((x \circ y) \circ [z]_J \in J/I\). By hypothesis, \([x \circ z] = [x] \circ [z]_J \in I/I\) implies \(x \circ z \in J\). □

**Corollary 3.10.** Let \(X\) be an AB-algebra and \(I, J\) be any sets such that \(I \subseteq J \subseteq X\). Suppose that \(I\) is an AB-subalgebra of \(X\). Then \(J\) is an AB-subalgebra of \(X\) if and only if \(J/I\) is an AB-subalgebra of \(X/I\).

**Proof.** Similar to that of Proposition (3.9). □

**Corollary 3.11.** Let \(I\) be AB-ideal of AB-algebra \(X\) with \(I \subseteq X\), then \(I\) is an AB-ideal of \(X\).

Next, the basic properties of equivalence classes are considered are as the following theorem.

**Theorem 3.12.** Let \(I\) be an AB-subalgebra of AB-algebra \(X\) and \(a, b \in X\). Then:

1. \([a]_I = 1 \iff a \in I\),
2. \([a]_I = [b]_I\) or \([a]_I \cap [b]_I = \emptyset\).

**Proof.** Let \(I\) be an AB-subalgebra of \(X\) and \(a, b \in X\).

1. It is clear due to the fact that \(a \sim a\) for all \(a \in X\) and \(a \ast a = 0 \in I\), so we get that \(a \in [a]_I = 1\). Conversely, let \(x \in [a]_I\). Then \(x \sim a\), it follows that
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x ⋆ a, a ⋆ x ∈ I. By hypothesis, x ∈ I. Hence, [a] ∈ I. To show that I ⊆ [a] , choose x ∈ I. Since I is an AB-subalgebra of X, we have x ⋆ a, a ⋆ x ∈ I. Thus, x ∼ a, this means that x ∈ [a] and shows that I ⊆ [a] . Consequently, [a] = I.

(2) Assume that [a] ∩ [b] ≠ Ø. Then there is x ∈ [a] ∩ [b] such that x ∈ [a] and x ∈ [b] . It follows that x ∼ a and x ∼ b, so a ∼ b by the symmetric and transitive properties. Thus [a] = [b] . □

References


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