Hermite-Hadamard Type Inequalities for Fractional Integrals Operators

Loredana Ciurdariu

Department of Mathematics
"Politehnica" University of Timisoara
P-ta. Victoriei, No.2, 300006-Timisoara, Romania

Abstract

Several Hermite-Hadamard type inequalities will be given in this paper for n-time differentiable functions whose n-time derivative in absolute value satisfy different kind of convexities via Riemann-Liouville fractional integral operators.

Mathematics Subject Classification: 26A33, 26D10, 26D15

Keywords: Hermite-Hadamard inequality, convex functions, Holder inequality, Riemann-Liouville fractional integral, fractional integral operator, power mean inequality

1. Introduction

The inequality of Hermite-Hadamard type has been considered very useful in mathematical analysis being very intensely studied, extended and generalized in many directions by many authors, see [24, 7, 6, 10, 1, 14, 18, 25, 12] and the references therein.

Many papers study the Riemann-Liouville fractionals integrals and give new and interesant generalizations of Hermite-Hadamard type inequalities using these kind of integrals, see for instance [9, 8, 10, 11, 12, 19, 16, 18, 14, 24, 25, 26, 27, 28, 21, 30].

We will begin now by recalling the classical definition for the convex functions and then the definitions for other kind of convexities.
Definition 1. A function \( f : I \subset \mathbb{R} \to \mathbb{R} \) is said to be convex on an interval \( I \) if the inequality
\[
\tag{1} f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)
\]
holds for all \( x, y \in I \) and \( t \in [0,1] \). The function \( f \) is said to be concave on \( I \) if the inequality (1) takes place in reversed direction.

It is necessary to recall below also the definition of fractionals integrals, see [9, 11, 10, 19, 20, 26] and then the definition of fractional integral operators. For other type of convexity see also [22, 17].

Definition 2. A function \( f : [a,b] \to \mathbb{R} \) is said to be quasi-convex on \([a,b]\) if
\[
f(tx + (1-t)y) \leq \sup \{ f(x), f(y) \}
\]
holds for all \( x, y \in [a,b] \) and \( t \in [0,1] \).

Definition 3. A function \( f : I \to \mathbb{R} \) is said to be P-convex on \([a,b]\) if it is nonnegative and for all \( x, y \in I \) and \( \lambda \in [9,1] \)
\[
f(tx + (1-t)y) \leq f(x) + f(y).
\]

Definition 4. A function \( f : I \subset \mathbb{R}^+ \to \mathbb{R}^+ \) is said to be s-convex in the first sense on an interval \( I \) if the inequality
\[
f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y)
\]
holds for all \( x, y \in I \), \( t \in [0,1] \) and for some fixed \( s \in (0,1] \).

Definition 5. A function \( f : I \subset \mathbb{R}^+ \to \mathbb{R}^+ \) is said to be s-convex in the second sense on an interval \( I \) if the inequality
\[
f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y)
\]
holds for all \( x, y \in I \), \( t \in [0,1] \) and for some fixed \( s \in (0,1] \).

Definition 6. A function \( f : I \subset \mathbb{R}^+ \to \mathbb{R}^+ \) is said to be s-Godunova-Levin functions of second kind on an interval \( I \) if the inequality
\[
f(tx + (1-t)y) \leq \frac{1}{t^s} f(x) + \frac{1}{(1-t)^s} f(y)
\]
holds for all \( x, y \in I \), \( t \in (0,1) \) and for some fixed \( s \in [0,1] \).

It is easy to see that for \( s = 0 \) s-Godunova-Levin functions of second kind are functions P-convex.

The classical Hermite-Hadamard’s inequality for convex functions is
(2) \[ f \left( \frac{a + b}{2} \right) \leq \frac{1}{b - a} \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{2}. \]

Moreover, if the function \( f \) is concave then the inequality (2) hold in reversed direction.

**Definition 7.** Let \( f \in L[a, b] \). The Riemann-Liouville integrals \( J_{a+}^\alpha f \) and \( J_{b-}^\alpha f \) of order \( \alpha > 0 \) with \( \alpha \geq 0 \) are defined by

\[
J_{a+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x - t)^{\alpha-1}f(t)dt, \ x > a
\]

and

\[
J_{b-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t - x)^{\alpha-1}f(t)dt, \ x < b,
\]

respectively, where \( \Gamma(\alpha) \) is the Gamma function defined by \( \Gamma(\alpha) = \int_0^\infty e^{-t}t^{\alpha-1}dt \) and \( J^0_{a+} f(x) = J^0_{b-} f(x) = f(x) \).

It is well-known that the beta function is defined when \( a, b > 0 \) by

\[
R(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a + b)} = \int_0^1 t^{a-1}(1 - t)^{b-1}dt.
\]

The following class of functions defined formally by

\[
\mathcal{F}_{\rho, \lambda}^\sigma(x) = \sum_{k=0}^\infty \frac{\sigma(k)}{\Gamma(\rho k + \lambda)} x^k \quad (\rho, \lambda > 0; \ |x| < R),
\]

where the coefficients \( \sigma(k), \ (k \in \mathbb{N} = \mathbb{N} \cup \{0\}) \) is a bounded sequence of positive real numbers and \( \mathbb{R} \) is the set of real numbers, as in [21], was introduced in [29] and was used for giving in [3] the following left-sided and right-sided fractional integral operators from below:

\[
(J_{\rho, \lambda,a+;w}^\sigma \varphi)(x) = \int_a^x (x - t)^{\lambda-1}\mathcal{F}_{\rho, \lambda}^\sigma[w(x - t)^\rho]\varphi(t)dt, \quad (x > a > 0),
\]

and

\[
(J_{\rho, \lambda,b-;w}^\sigma \varphi)(x) = \int_x^b (t - x)^{\lambda-1}\mathcal{F}_{\rho, \lambda}^\sigma[w(t - x)^\rho]\varphi(t)dt, \quad (0 < x < b),
\]

where \( \rho, \lambda > 0, \ w \in \mathbb{R} \) and \( \varphi(t) \) is such that the integral on the right side exists. There are new integral inequalities for this operator, see [21, 3, 30] and references therein.

It is important to mention that for example the classical Riemann-Liouville fractional integrals \( J_{a+}^\alpha \) and \( J_{b-}^\alpha \) of order \( \alpha \) were obtained by setting \( \lambda = \alpha, \ sigma(0) = 1 \) and \( w = 0 \) in previous integrals.
In this paper, two new identities are given and then some applications, like Hermite-Hadamard type inequalities for functions whose the n-time derivative in absolute value of certain powers satisfies different type of convexities via Riemann-Liouville fractional integral operators are established.

2. Main results

The following result is a generalization of Lemma 1 from [5] for fractional integral operators for functions n-time differentiable.

**Lemma 1.** Let \( f : [a, b] \to \mathbb{R} \) be an n-time differentiable mapping on \((a, b)\) with \(0 < a < b\), \(\lambda > n - 1\), \(x \in (a, b)\) and \(t \in [0, 1]\). If \( f^{(n)} \in L[a, b] \) then the following equality for generalized fractional integrals holds:

\[
\int_0^1 t^\lambda F_{\rho,\lambda+1}^\sigma[(x-a)^\rho t^\rho]f^{(n)}(tx + (1-t)a)\,dt + \\
+ \int_0^1 (1-t)^\lambda F_{\rho,\lambda+1}^\sigma[(b-x)^\rho(1-t)^\rho]f^{(n)}(tb + (1-t)x)\,dt = \\
= \sum_{k=1}^n (-1)^{k-1} \frac{1}{(x-a)^k} F_{\rho,\lambda-k+2}^\sigma[(x-a)^\rho] - \frac{1}{(x-a)^k} F_{\rho,\lambda-k+2}^\sigma[(b-x)^\rho] \right) f^{(n-k)}(x) + \\
+ \frac{(-1)^n}{(x-a)\lambda+1} \left( J_{\rho,\lambda-n+1,x^-,w}^\sigma f \right)(a) + \frac{1}{(x-a)^\lambda+1} \left( J_{\rho,\lambda-n+1,x^+,w}^\sigma f \right)(b).
\]

**Proof.** As in [21], we compute first

\[
\int_0^1 t^\lambda F_{\rho,\lambda+1}^\sigma[(x-a)^\rho t^\rho]f''(tx + (1-t)a)\,dt
\]

and then we will prove by induction that

\[
I_1 = \int_0^1 t^\lambda F_{\rho,\lambda+1}^\sigma[(x-a)^\rho t^\rho]f^{(n)}(tx + (1-t)a)\,dt = \\
= \sum_{k=1}^n (-1)^{k-1} \frac{1}{(x-a)^k} f^{(n-k)}(x) F_{\rho,\lambda-k+2}^\sigma[(x-a)^\rho] + \frac{(-1)^n}{(x-a)^\lambda+1} \left( J_{\rho,\lambda-n+1,x^-,w}^\sigma f \right)(a).
\]

Integrating by parts and then changing variables with \( u = tx + (1-t)a \) we get

\[
\int_0^1 t^\lambda F_{\rho,\lambda+1}^\sigma[(x-a)^\rho t^\rho]f''(tx + (1-t)a)\,dt = \\
= F_{\rho,\lambda+1}^\sigma[(x-a)^\rho] \frac{f'(x)}{x-a} - \frac{f(x)}{(x-a)^2} F_{\rho,\lambda}^\sigma[(x-a)^\rho] + \\
+ \frac{1}{(x-a)^2} \int_0^1 t^{\lambda-2} F_{\rho,\lambda-1}^\sigma[(x-a)^\rho t^\rho]f(tx + (1-t)a)\,dt
\]
or

\[
\int_0^1 t^\lambda F_{\rho,\lambda+1}^\sigma[(x-a)^\rho t^\rho]f''(tx + (1-t)a)\,dt = \
\]
\[ \mathcal{F}_{\rho, \lambda+1}^\sigma[(x-a)^\rho] \frac{f'(x)}{x-a} - \frac{f(x)}{(x-a)^2} \mathcal{F}_{\rho, \lambda}^\sigma[(x-a)^\rho] + \frac{1}{(x-a)^{\lambda+1}} \left( \mathcal{J}_{\rho, \lambda-1, x^-; \omega f}^\sigma (a) \right) . \]

Analogously, by using the same method we get:

\[ \int_0^1 (1-t)^\lambda \mathcal{F}_{\rho, \lambda+1}^\sigma[(b-x)^\rho(1-t)^\rho] f''(tb + (1-t)x) dt = -\mathcal{F}_{\rho, \lambda+1}^\sigma[(b-x)^\rho] \frac{f'(x)}{b-x} - \frac{f(x)}{(b-x)^2} \mathcal{F}_{\rho, \lambda}^\sigma[(b-x)^\rho] + \frac{1}{(b-x)^{\lambda+1}} \left( \mathcal{J}_{\rho, \lambda-1, x^-; \omega f}^\sigma (b) \right) . \]

or by substitution \( u = tb + (1-t)x, \)

\[ \int_0^1 (1-t)^\lambda \mathcal{F}_{\rho, \lambda+1}^\sigma[(b-x)^\rho(1-t)^\rho] f''(tb + (1-t)x) dt = -\mathcal{F}_{\rho, \lambda+1}^\sigma[(b-x)^\rho] \frac{f'(x)}{b-x} - \frac{f(x)}{(b-x)^2} \mathcal{F}_{\rho, \lambda}^\sigma[(b-x)^\rho] + \frac{1}{(b-x)^{\lambda+1}} \left( \mathcal{J}_{\rho, \lambda-1, x^-; \omega f}^\sigma (b) \right) . \]

Therefore by induction we have,

\[ I_2 = -\sum_{k=1}^n f^{(n-k)}(x) \frac{1}{(b-x)^{k}} \mathcal{F}_{\rho, \lambda-k+2}^\sigma[(b-x)^\rho] + \frac{1}{(b-x)^{\lambda+1}} \left( \mathcal{J}_{\rho, \lambda-n+1, x^-; \omega f}^\sigma (b) \right) . \]

Now summing \( I_1 \) and \( I_2 \) we obtain the desired equality.

\[ \square \]

Using this lemma we obtain the following result for \( n \)-time differentiable functions whose absolute value is convex via fractional integral operator.

**Theorem 1.** Let \( f : [a, b] \to \mathbb{R} \) be an \( n \)-time differentiable mapping on \( (a, b) \) with \( 0 < a < b, \lambda > n-1, x \in (a, b) \) and \( t \in [0, 1] \). If \( f^{(n)} \in L[a, b] \) and \( |f^{(n)}| \) is convex on \( (a, b) \) then the following inequality for generalized fractional integral operators takes place:

\[ \left| \sum_{k=1}^n (-1)^{k-1} f^{(n-k)}(x) \left\{ \frac{\mathcal{F}_{\rho, \lambda-k+2}^\sigma[(x-a)^\rho]}{(x-a)^k} - \frac{\mathcal{F}_{\rho, \lambda-k+2}^\sigma[(b-x)^\rho]}{(b-x)^k} \right\} \right| + \left| \frac{(-1)^n}{(x-a)^{\lambda+1}} \left( \mathcal{J}_{\rho, \lambda-n+1, x^-; \omega f}^\sigma (a) + \frac{1}{(b-x)^{\lambda+1}} \left( \mathcal{J}_{\rho, \lambda-n+1, x^-; \omega f}^\sigma (b) \right) \right) \right| \leq \mathcal{F}_{\rho, \lambda+1}^\sigma[w(x-a)^\rho] \left( \frac{|f^{(n)}(x)|}{\lambda + 2} + \frac{|f^{(n)}(a)|}{(\lambda + 1)(\lambda + 2)} \right) + \mathcal{F}_{\rho, \lambda+1}^\sigma[w(a-x)^\rho] \left( \frac{|f^{(n)}(b)|}{\lambda + 2} + \frac{|f^{(n)}(b)|}{(\lambda + 1)(\lambda + 2)} \right) . \]
Using the properties of modulus, Lemma 1 and that \(|f^{(n)}|\) is convex function we get:

\[
\sum_{k=1}^{n} (-1)^{k-1} f^{(n-k)}(x) \left\{ \frac{\mathcal{F}_\rho^\sigma \lambda-k+2[(x-a)^\rho]}{(x-a)^k} - \frac{\mathcal{F}_\rho^\sigma \lambda-k+2[(b-x)^\rho]}{(b-x)^k} \right\} + \frac{(-1)^n}{(x-a)^{\lambda+1}} (\mathcal{J}_\rho^\sigma \lambda-n+1,x:-\omega f)(a) + \frac{1}{(b-x)^{\lambda+1}} (\mathcal{J}_\rho^\sigma \lambda-n+1,x:+\omega f)(b) = \]

\[
= \int_0^1 t^\lambda \mathcal{F}_\rho^\sigma \lambda+1[(x-a)^\rho t^\rho]f^{(n)}(tx + (1-t)a)dt + \int_0^1 (1-t)^\lambda \mathcal{F}_\rho^\sigma \lambda+1[(b-x)^\rho(1-t)^\rho]f^{(n)}(tb + (1-t)x)dt \leq \sum_{k=0}^\infty \sigma(k) w^k (x-a)^{\rho k} \left( \int_0^1 t^\lambda+1 dt + \int_0^1 t^\lambda(1-t)dt \right) + \sum_{k=0}^\infty \sigma(k) w^k (b-x)^{\rho k} \left( \int_0^1 (1-t)^\lambda t dt + \int_0^1 (1-t)\lambda+1 dt \right).
\]

From here by easily calculus we get the desired inequality.

}\]

By this lemma we also obtain the following result for \(n\)-time differentiable functions whose absolute value is \(s\)-convex in the second sense via fractional integral operator.

\textbf{Theorem 2.} Let \(f : [a, b] \rightarrow \mathbb{R}\) be an \(n\)-time differentiable mapping on \((a, b)\) with \(0 < a < b, \lambda > n-1, x \in (a, b), s \in (0, 1)\) and \(t \in [0, 1]\). If \(f^{(n)} \in L[a, b]\) and \(|f^{(n)}|\) is \(s\)-convex in the second sense on \((a, b)\) then the following inequality for generalized fractional integral operators takes place:

\[
\sum_{k=1}^{n} (-1)^{k-1} f^{(n-k)}(x) \left\{ \frac{\mathcal{F}_\rho^\sigma \lambda-k+2[(x-a)^\rho]}{(x-a)^k} - \frac{\mathcal{F}_\rho^\sigma \lambda-k+2[(b-x)^\rho]}{(b-x)^k} \right\} + \frac{(-1)^n}{(x-a)^{\lambda+1}} (\mathcal{J}_\rho^\sigma \lambda-n+1,x:-\omega f)(a) + \frac{1}{(b-x)^{\lambda+1}} (\mathcal{J}_\rho^\sigma \lambda-n+1,x:+\omega f)(b) \leq \]

\[
\leq \mathcal{F}_\rho^\sigma \lambda+1[w(x-a)^\rho] \left( \frac{|f^{(n)}(x)|}{\lambda + s + 1} + |f^{(n)}(a)|B(\lambda + 1, s + 1) \right) + \mathcal{F}_\rho^\sigma \lambda+1[w(b-x)^\rho] \left( \frac{|f^{(n)}(x)|}{\lambda + s + 1} + |f^{(n)}(b)|B(\lambda + 1, s + 1) \right).
\]

\textit{Proof.} We use the same method as in Theorem 1, but this time we apply the definition of \(s\)-convex function in the second sense. \(\Box\)
Next result is a generalization of Lemma 4 from [4] for fractional integral operators for functions n-time differentiable.

**Lemma 2.** Let \( f : [a, b] \to \mathbb{R} \) be an n-time differentiable mapping on \((a, b)\) with \( 0 < a < b, \lambda > n - 1, x \in (a, b) \) and \( t, r \in [0, 1] \). If \( f^{(n)} \in L[a, b] \) then the following equality for generalized fractional integrals holds:

\[
\begin{align*}
&\int_0^1 t^\lambda \mathcal{F}^\sigma_{\rho,\lambda+1}[(1-r)^\rho(x-a)\rho t^\rho] f^{(n)}(t(a + (1-r)x) + (1-t)a) dt + \\
&+ \int_0^1 (1-t)^\lambda \mathcal{F}^\sigma_{\rho,\lambda+1}[r^\rho(x-a)\rho(1-t)^\rho] f^{(n)}(tx + (1-t)(ra + (1-r)x)) dt + \\
&+ \int_0^1 t^\lambda \mathcal{F}^\sigma_{\rho,\lambda+1}[(1-r)^\rho(b-x)\rho t^\rho] f^{(n)}(tb + (1-t)(rx + (1-r)b)) dt + \\
&+ \int_0^1 (1-t)^\lambda \mathcal{F}^\sigma_{\rho,\lambda+1}[r^\rho(b-x)\rho(1-t)^\rho] f^{(n)}(tb + (1-t)(rx + (1-r)b)) dt = \\
&= \sum_{k=1}^n \frac{(-1)^{k-1}}{(1-r)^k} \frac{f^{(n-k)}(ra + (1-r)x)}{(b-x)^k} \mathcal{F}^\sigma_{\rho,\lambda-k+2}[(1-r)^\rho(x-a)^\rho] + \\
&+ \frac{f^{(n-k)}(rx + (1-r)b)}{(b-x)^k} \mathcal{F}^\sigma_{\rho,\lambda-k+2}[(1-r)^\rho(b-x)^\rho] - \\
&- \sum_{k=1}^n \frac{1}{r^k} \frac{f^{(n-k)}(ra + (1-r)x)}{(a-x)^k} \mathcal{F}^\sigma_{\rho,\lambda-k+2}[r^\rho(x-a)^\rho] + \\
&+ \frac{f^{(n-k)}(rx + (1-r)b)}{(b-x)^k} \mathcal{F}^\sigma_{\rho,\lambda-k+2}[r^\rho(b-x)^\rho] + \\
&+ \frac{1}{r^\lambda+1} (\mathcal{J}^\sigma_{\rho,\lambda-n+1,(ra+(1-r)x)^{-};w} f)(a) + \\
&+ \frac{1}{r^\lambda+1} (\mathcal{J}^\sigma_{\rho,\lambda-n+1,(ra+(1-r)x)^{+};w} f)(x) + \\
&+ \frac{1}{r^\lambda+1} (\mathcal{J}^\sigma_{\rho,\lambda-n+1,(rx+(1-r)b)^{-};w} f)(x) + \\
&+ \frac{1}{r^\lambda+1} (\mathcal{J}^\sigma_{\rho,\lambda-n+1,(rx+(1-r)b)^{+};w} f)(b).
\end{align*}
\]

**Proof.** We denote

\[
\begin{align*}
I_1 &= \int_0^1 t^\lambda \mathcal{F}^\sigma_{\rho,\lambda+1}[(1-r)^\rho(x-a)\rho t^\rho] f^{(n)}(t(a + (1-r)x) + (1-t)a) dt, \\
I_2 &= \int_0^1 (1-t)^\lambda \mathcal{F}^\sigma_{\rho,\lambda+1}[r^\rho(x-a)\rho(1-t)^\rho] f^{(n)}(tx + (1-t)(ra + (1-r)x)) dt, \\
I_3 &= \int_0^1 t^\lambda \mathcal{F}^\sigma_{\rho,\lambda+1}[(1-r)^\rho(b-x)\rho t^\rho] f^{(n)}(tb + (1-t)(rx + (1-r)b)) dt
\end{align*}
\]
integral operators takes place:

Let $\rho, \sigma, \lambda > n - 1, x \in (a, b)$ and $t, r \in [0, 1]$. If $f^{(n)} \in \mathcal{L}[a, b]$ and $|f^{(n)}|$ is convex on $(a, b)$ then the following inequality for generalized fractional integral operators takes place:

\[
I_4 = \int_0^1 (1 - t)^\lambda \mathcal{F}_{\rho, \lambda + 1}^{\sigma} [r^\rho (b - x)^\rho (1 - t)^\rho] f^{(n)}(tb + (1 - t)(rx + (1 - r)b)) dt.
\]

As in Lemma 1 we prove by induction that

\[
I_1 = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{(1 - r)^k (x - a)^k} f^{(n-k)}(ra + (1 - r)x) \mathcal{F}_{\rho, \lambda - k + 2}^{\sigma} [(1 - r)^\rho (x - a)^\rho] +
\]

\[
+ \frac{(-1)^n}{(1 - r)^{\lambda + 1} (x - a)^{\lambda + 1}} (\mathcal{J}_{\rho, \sigma, \lambda - n + 1}^{\rho, \lambda + 1}; x^{(1 - r)x}; w f)(a)
\]

and then similarly we can find $I_2$, $I_3$ and $I_4$. Therefore we have:

\[
I_2 = - \sum_{k=1}^{n} \frac{1}{r^k (x - a)^k} f^{(n-k)}(ra + (1 - r)x) \mathcal{F}_{\rho, \lambda - k + 2}^{\sigma} [r^\rho (x - a)^\rho] +
\]

\[
+ \frac{(-1)^n}{r^{\lambda + 1} (x - a)^{\lambda + 1}} (\mathcal{J}_{\rho, \sigma, \lambda - n + 1}^{\rho, \lambda + 1}; x^{(1 - r)x}; w f)(x)
\]

Summing now $I_1$, $I_2$ $I_3$ and $I_4$ we find the desired equality.

\[
\square
\]

**Theorem 3.** Let $f : [a, b] \to \mathbb{R}$ be an $n$-time differentiable mapping on $(a, b)$ with $0 < a < b$, $\lambda > n - 1, x \in (a, b)$ and $t, r \in [0, 1]$. If $f^{(n)} \in \mathcal{L}[a, b]$ and $|f^{(n)}|$ is convex on $(a, b)$ then the following inequality for generalized fractional integral operators takes place:

\[
\left| \sum_{k=1}^{n} \frac{(-1)^{k-1}}{(1 - r)^k} \left( \frac{f^{(n-k)}(ra + (1 - r)x)}{(x - a)^k} \mathcal{F}_{\rho, \lambda - k + 2}^{\sigma} [(1 - r)^\rho (x - a)^\rho] +
\right.ight.
\]

\[
\left. \left. + \frac{f^{(n-k)}(ra + (1 - r)x)}{(b - x)^k} \mathcal{F}_{\rho, \lambda - k + 2}^{\sigma} [(1 - r)^\rho (b - x)^\rho] \right) -
\right.
\]

\[
- \sum_{k=1}^{n} \frac{1}{r^k} \left( \frac{f^{(n-k)}(ra + (1 - r)x)}{(x - a)^k} \mathcal{F}_{\rho, \lambda - k + 2}^{\sigma} [r^\rho (x - a)^\rho] +
\right.
\]

\[
\left. \left. + \frac{f^{(n-k)}(ra + (1 - r)x)}{(b - x)^k} \mathcal{F}_{\rho, \lambda - k + 2}^{\sigma} [r^\rho (b - x)^\rho] \right) +
\right.
\]

\[
+ \frac{(-1)^n}{(1 - r)^{\lambda + 1} (x - a)^{\lambda + 1}} (\mathcal{J}_{\rho, \sigma, \lambda - n + 1}^{\rho, \lambda + 1}; x^{(1 - r)x}; w f)(a)
\]

\[
+ \frac{1}{r^{\lambda + 1} (x - a)^{\lambda + 1}} (\mathcal{J}_{\rho, \sigma, \lambda - n + 1}^{\rho, \lambda + 1}; x^{(1 - r)x}; w f)(x)
\]

\[
+ \frac{(-1)^n}{(1 - r)^{\lambda + 1} (b - x)^{\lambda + 1}} (\mathcal{J}_{\rho, \sigma, \lambda - n + 1}^{\rho, \lambda + 1}; x^{(1 - r)b}; w f)(b)
\]

\[
\leq
\]

\[
\]
\[
\leq \mathcal{F}_{\rho, \lambda}^{\sigma} ((1 - r)^{\rho}(x - a)^{\rho}w) \left( \frac{|f^{(n)}(ra + (1 - r)x)|}{\lambda + 2} + \frac{|f^{(n)}(a)|}{(\lambda + 1)(\lambda + 2)} \right) + \\
+ \mathcal{F}_{\rho, \lambda}^{\sigma} [r^{\rho}(x - a)^{\rho}w] \left( \frac{|f^{(n)}(x)|}{(\lambda + 2)(\lambda + 1)} + \frac{|f^{(n)}(ra + (1 - r)x)|}{\lambda + 2} \right) + \\
+ \mathcal{F}_{\rho, \lambda}^{\sigma} ((1 - r)^{\rho}(b - x)^{\rho}w) \left( \frac{|f^{(n)}(rx + (1 - r)b)|}{\lambda + 2} + \frac{|f^{(n)}(a)|}{(\lambda + 1)(\lambda + 2)} \right) + \\
+ \mathcal{F}_{\rho, \lambda}^{\sigma} [r^{\rho}(b - x)^{\rho}w] \left( \frac{|f^{(n)}(b)|}{(\lambda + 2)(\lambda + 1)} + \frac{|f^{(n)}(rx + (1 - r)b)|}{\lambda + 2} \right).
\]

**Proof.** We use the same method as in Theorem 1, we shall apply Lemma 2 and the definition of the convex functions. \[\square\]

**References**


Received: May 30, 2017; Published: June 28, 2017