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An Equilibrium Cooperative Optimization Solution for Two-Person Non-transferable Utility Games

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Abstract

In this paper, we provide an equilibrium cooperative optimization solution to two-person cooperative games in which the payoffs or utility are non-transferable. In particular, a strategic scheme is designed to determine the cooperative weight for joint optimization leading to the equilibrium point on the Pareto optimal frontier. A cooperative optimization solution with a non-cooperative game theoretic foundation is obtained. It is significantly different from existing cooperative solutions like the Nash bargaining solution, the Kalai proportional bargaining solution the Kalai-Smorodinsky bargaining solution.

Keywords: Cooperative games, bargaining solution, Nash program, cooperative optimization

1 Introduction

Reaching a cooperative solution in games with nontransferable payoffs constitutes one of the most difficult problems in game theory. Nash (1950), Kalai (1977) and Kalai and Smorodinsky (1975) presented bargaining solutions based on axioms or principles that one would like the cooperative solution to have. Advancing from the axiomatic approach, the Nash (1953) Program -- which intended to bridge the gap between cooperative and non-cooperative approaches –

proposed the adoption of non-cooperative game aspects in formulating a cooperative solution. For instance, by proposing non-cooperative games that specify the details of negotiation for terms of cooperation, the Nash program supplements the axiomatic approach with the possibility for players to seek their individual interests in the process of cooperation. The cooperative solution then could be understood as the outcome of a series of strategic problems facing individual players. In sum, the Nash program proposes a "non-cooperative game theoretic foundation" for a cooperative solution.

In this paper, we consider two-person cooperative games in which the payoffs or utility are non-transferable. The players agree to achieve group optimality with cooperative optimization. Since the Pareto frontier is generally not a singleton, players would have to decide on which point on the frontier they would choose. A strategic scheme is designed to determine the equilibrium point on the Pareto optimal frontier so that each player is doing the best given the other player is doing his best. Equilibrium payoff weights are obtained for cooperative optimization. The strategic equilibrium for cooperative optimization is a novel cooperative solution for two-person Non-transferrable utility games. It is significantly different from existing cooperative solutions like the Nash (1950) bargaining solution, the Kalai (1977) proportional bargaining solution the Kalai-Smorodinsky (1975) bargaining solution.

The paper is organized as follows. Section 2 presents the basic settings of a cooperative optimization game. An equilibrium cooperative optimization solution is derived in Section 3. Section 4 examines the properties of the solution and provides an illustrative example. Concluding remarks are given in Section 5.

2. Basic Settings of a Cooperative Optimization Game

Consider a two-player noncooperative game of complete information. Player $i \in \{1,2\}$ seeks to maximize his payoff function $u^i(x_1, x_2)$, $x_i \in X_i \subset R$ where x_i is the control of player i. The payoff function $u^i(x_1, x_2)$ for i∈{1,2}, are continuously differentiable in x_1 and x_2. A noncooperative Nash equilibrium (x_1^N, x_2^N) satisfies the conditions that:

$$x_1^N \equiv \arg\max_{x_1 \in X_1} \left\{ u^1(x_1, x_2^N) \right\} \quad \text{and}$$

$$x_2^N \equiv \arg\max_{x_2 \in X_2} \left\{ u^2(x_1^N, x_2) \right\}. \tag{2.1}$$

For simplicity of exposition it is assumed that an agreed upon Nash equilibrium is adopted if there are more than one Nash equilibria. The outcome of the game is denoted as $\{u^1(x_1^N, x_2^N), u^2(x_1^N, x_2^N)\} \equiv \{\underline{u}^1, \underline{u}^2\}$.

It is well known that in general noncooperative Nash equilibrium outcome are not Pareto efficient. In the absence of a Pareto efficient noncooperative Nash equilibrium, players have incentives to cooperate and coordinate their actions in such a way that a Pareto efficient outcome is achieved. Achieving group optimality is the starting point of almost all cooperative games. In a two-person cooperative game with non-transferable payoffs the Pareto optimal boundary can be obtained by solving the optimization problem (see Dockner and Jorgensen (1984), Leitmann (1974) and Yeung and Petrosyan (2005 and 2015)):

$$\max_{x_1,x_2}\{\alpha_i u^i(x_1,x_2)+\alpha_j u^j(x_1,x_2)\} \tag{2.2}$$

with the weights $\alpha_i > 0, \alpha_j > 0$ and $\alpha_i + \alpha_j = 1$, for $i, j \in \{1,2\}$ and $i \neq j$.

The optimization problem (2.2) can be specified alternatively as:

$$\max_{x_1,x_2}\{u^i(x_1,x_2)+\frac{\alpha_j}{\alpha_i}u^j(x_1,x_2)\} \equiv \max_{x_1,x_2}\{u^i(x_1,x_2)+\omega_i\, u^j(x_1,x_2)\}. \tag{2.3}$$

Along the Pareto optimality boundary, individual rationality must be maintained so that the players' cooperative payoffs cannot be below their noncooperative Nash equilibrium levels. Hence restrictions on the ranges of α_i and α_j must be identified to guarantee individual rationality. Consider the case when player j is allowed to secured at a payoff level $\hat{u}^j \geq \underline{u}^j$, player i's payoff under cooperation can be obtained by solving the constrained optimization problem:

$$\max_{x_1,x_2} u^i(x_1,x_2) \text{ subject to } u^j(x_1,x_2)-\hat{u}^j \geq 0, \tag{2.4}$$

for $i, j \in \{1,2\}$ and $i \neq j$.

The corresponding Lagrange problem to (2.4) can be expressed as:

$$\max_{x_1,x_2,\lambda_i} L^i_{\hat{u}^j} = \max_{x_1,x_2,\lambda_i}\{u^i(x_1,x_2)+\lambda_i[u^j(x_1,x_2)-\hat{u}^j]\} \tag{2.5}$$

where λ_i is the Lagrange multiplier.

We use $x_1(\hat{u}^j)$, $x_2(\hat{u}^j)$ and $\lambda_i(\hat{u}^j)$ to denote the solution to (2.5) where $\hat{u}^j$ is player j's payoff. Since individual rationality has to be satisfied so that the players' cooperative payoffs must not be less than their noncooperative Nash equilibrium levels. Therefore, player i's most preferred position is $\hat{u}^j = \underline{u}^j$, which yields:

$u^i[x_1(\underline{u}^j),x_2(\underline{u}^j)] \equiv \bar{u}^i$ and $u^j[x_1(\underline{u}^j),x_2(\underline{u}^j)] = \underline{u}^j$, for $i \in \{1,2\}$ and $i \neq j$.

Moreover, player i's least preferred position is $\hat{u}^j = \bar{u}^j$, which yields:

$$u^i[x_1(\bar{u}^j),x_2(\bar{u}^j)] = \underline{u}^i \text{ and } u^j[x_1(\bar{u}^j),x_2(\bar{u}^j)] = \bar{u}^j. \tag{2.6}$$

Remark 2.1. Consider the problem:

$$\max_{x_1,x_2} u^j(x_1,x_2) \text{ subject to } u^i(x_1,x_2)=\hat{u}^i, \tag{2.7}$$

where $\hat{u}^i$ equals the maximized payoff $u^i[x_1(\hat{u}^j),x_2(\hat{u}^j)]$ in the constrained maximization problem (2.4).

The Lagrange multiplier $\lambda_i(\hat{u}^j)$ in problem (2.4) equals the reciprocal of the Lagrange multiplier $\lambda_j(\hat{u}^i)$ in (2.7). ■

Remark 2.2. If the corresponding Pareto optimal boundary is convex, the function $\lambda_i(\hat{u}^j)$ is a monotonically increasing, that is

$$\frac{d\lambda_i(\hat{u}^j)}{d\,\hat{u}^j}>0, \text{ for } i\in\{1,2\} \text{ and } i\neq j. \quad ■$$

Given the results in Remark 2.2, the range of λ_i must be within $[\lambda_i(\underline{u}^j),\,\lambda_i(\overline{u}^j)]$ for individual rationality to hold.

Now, consider the optimization problem

$$\max_{x_1,x_2}\{u^i(x_1,x_2)+\omega_i\,u^j(x_1,x_2)\}. \tag{2.8}$$

The optimization problem (2.8) is analogous to the Lagrange problem (2.5) when the level of $\hat{u}^j$ yields $\lambda_i(\hat{u}^j)=\omega_i$. Since the range of λ_i must be restricted within $[\lambda_i(\underline{u}^j),\lambda_i(\overline{u}^j)]$ so $\omega_i\in[\underline{\omega}_i,\overline{\omega}_i]=[\lambda_i(\underline{u}^j),\lambda_i(\overline{u}^j)]$. Player i's most preferred ω_i is $\underline{\omega}_i=\lambda_i(\underline{u}^j)$ and his least preferred is $\overline{\omega}_i=\lambda_i(\overline{u}^j)$, for $i\in\{1,2\}$ and $i\neq j$.

Since the Pareto frontier is generally not a singleton, players would have to decide on which point on the frontier they would choose through an agreement on the choice of $\omega_i=\alpha_j/\alpha_i$ in (2.8).

3. An Equilibrium Cooperative Optimization Solution

In this section we present a solution to the cooperative optimization game provided in Section 2. To do this the players first have to determine a commonly agreed weight ω_i in (2.8) for cooperative optimization. Since player i prefers to make an offer ω_i^* as close to $\underline{\omega}_i$ as possible and as far away from $\overline{\omega}_i$ as possible, for $i\in\{1,2\}$, the players have to make concessions on their preferred offers of $\omega_i^*\in[\underline{\omega}_i,\overline{\omega}_i]$ and $\omega_j^*\in[\underline{\omega}_j,\overline{\omega}_j]$.

A comparable measure on the degree of concessions has to be established. Since the degrees of concession are related to the players' least preferred and most preferred positions, the measure must take into consideration of these. To construct the measure we first consider the following ratios – with the first one involving the player's least preferred weight and the second involving the most preferred weight.

(i) The ratio $\omega_i^* / \overline{\omega}_i$ which yields player i's offered weight as a percentage of his least preferred weight. The higher its value reflects the higher the extent of concession.

(ii) The ratio $\underline{\omega}_i / \omega_i^*$ which yields player i's most preferred weight as a percentage of his offered weight. The higher its value reflects the lower the extent of concession.

We combine the ratios in (i) and (ii) to form the term $\dfrac{\omega_i^* / \overline{\omega}_i}{\underline{\omega}_i / \omega_i^*}$ which exhibits higher extent of concession as its value increases. Since $\omega_i^* / \overline{\omega}_i$ and $\underline{\omega}_i / \omega_i^*$ are proportions, the ratio $\dfrac{\omega_i^* / \overline{\omega}_i}{\underline{\omega}_i / \omega_i^*}$ and $\dfrac{\omega_j^* / \overline{\omega}_j}{\underline{\omega}_j / \omega_j^*}$ are in comparable scales. Using $\dfrac{\omega_i^* / \overline{\omega}_i}{\underline{\omega}_i / \omega_i^*}$ as a measuring index of the degree of concession, a decision rule on the choice of the weight in cooperative optimization can formulated as:

Rule 3.1.

(i) If the index $\dfrac{\omega_i^* / \overline{\omega}_i}{\underline{\omega}_i / \omega_i^*}$ offered by player i is higher than $\dfrac{\omega_j^* / \overline{\omega}_j}{\underline{\omega}_j / \omega_j^*}$ offered by player j, player i's offer ω_i^* will stand, for. The joint payoff to be maximized becomes

$$\max_{x_1,x_2}\{u^i(x_1,x_2)+\omega_i^* u^j(x_1,x_2)\}.$$

(ii) In the case where the indices $\dfrac{\omega_i^* / \overline{\omega}_i}{\underline{\omega}_i / \omega_i^*}$ and $\dfrac{\omega_j^* / \overline{\omega}_j}{\underline{\omega}_j / \omega_j^*}$ are equal and $\omega_i^* = 1/\omega_j^*$, both ω_i^* and ω_j^* will stand. The joint weighted payoff to be maximized becomes

$$\max_{x_1,x_2}\{u^i(x_1,x_2)+\omega_i^* u^j(x_1,x_2)\} \text{ or } \max_{x_1,x_2}\{u^j(x_1,x_2)+\omega_j^* u^i(x_1,x_2)\}.$$

(iii) In the case where the indices $\dfrac{\omega_i^* / \overline{\omega}_i}{\underline{\omega}_i / \omega_i^*}$ and $\dfrac{\omega_j^* / \overline{\omega}_j}{\underline{\omega}_j / \omega_j^*}$ are equal but $\omega_i^* \neq 1/\omega_j^*$, renegotiation will be made again until (i) or (ii) is reached. ■

Now, consider again the optimization problem

$$\max_{x_1,x_2}\{u^i(x_1,x_2)+\omega_i\, u^j(x_1,x_2)\}.$$

We use $x_1^{\omega_i}$ and $x_2^{\omega_i}$ to denote the solution to the problem. The payoff to player i can be obtained as $u^i(x_1^{\omega_i}, x_2^{\omega_i})$ and the payoff to player j can be obtained as $u^j(x_1^{\omega_i}, x_2^{\omega_i})$.

The problem facing player i can be formulated as:

$$\max_{\omega_i}\left\{ u^i(x_1^{\omega_i}, x_2^{\omega_i}) = \begin{cases} u^i(x_1^{\omega_i}, x_2^{\omega_i}) & \text{if} \quad \dfrac{\omega_i / \overline{\omega}_i}{\underline{\omega}_i / \omega_i^*} \geq \dfrac{\omega_j / \overline{\omega}_j}{\underline{\omega}_j / \omega_j^*} \\ u^i(x_1^{\omega_j}, x_2^{\omega_j}) & \quad\;\;\; \dfrac{\omega_i / \overline{\omega}_i}{\underline{\omega}_i / \omega_i} < \dfrac{\omega_j / \overline{\omega}_j}{\underline{\omega}_j / \omega_j} \end{cases} \right\}. \tag{3.1}$$

An equilibrium solution to the cooperative maximization game problem in Section 2 can be obtained as:

Theorem 3.1. There exists an equilibrium cooperative optimization solution in which player i offers $\omega_i^* = \sqrt{\lambda_i(\underline{u}^j)\lambda_i(\overline{u}^j)}$ and player j offers $\omega_j^* = \sqrt{\lambda_j(\underline{u}^i)\lambda_j(\overline{u}^i)}$, (3.2)

resulting in $\omega_i^* = 1/\omega_j^*$; and the joint payoff to be maximized being

$$\max_{x_1,x_2}\left\{u^i(x_1,x_2) + \sqrt{\lambda_i(\underline{u}^j)\lambda_i(\overline{u}^j)}\; u^j(x_1,x_2)\right\} \text{ or}$$

$$\max_{x_1,x_2}\left\{u^j(x_1,x_2) + \sqrt{\lambda_j(\underline{u}^i)\lambda_j(\overline{u}^i)}\; u^i(x_1,x_2)\right\}. \tag{3.3}$$

Proof. Invoking Remark 2.1, one can obtain $\lambda_i(\underline{u}^j) = 1/\lambda_j(\overline{u}^i)$ and $\lambda_i(\overline{u}^j) = 1/\lambda_j(\underline{u}^i)$. If player i offers $\omega_i^* = \sqrt{\lambda_i(\underline{u}^j)\lambda_i(\overline{u}^j)}$ and player j offers $\omega_j^* = \sqrt{\lambda_j(\underline{u}^i)\lambda_j(\overline{u}^i)}$, the ratio $\dfrac{\omega_j^* / \overline{\omega}_j}{\underline{\omega}_j / \omega_j^*} = \dfrac{\omega_i^* / \overline{\omega}_i}{\underline{\omega}_i / \omega_i^*} =1$. Moreover, $\omega_i^* = \sqrt{\lambda_i(\underline{u}^j)\lambda_i(\overline{u}^j)} = 1\Big/\sqrt{\lambda_j(\overline{u}^i)\big/\lambda_j(\underline{u}^i)} = 1/\omega_j^*$.

According to part (ii) Rule 3.1, the joint payoff maximization problem becomes
$\max_{x_1,x_2}\{u^i(x_1,x_2) + \omega_i^* u^j(x_1,x_2)\}$ or equivalently $\max_{x_1,x_2}\{u^j(x_1,x_2) + \omega_j^* u^i(x_1,x_2)\}$.

Note that a higher ω_i^* would imply a lower payoff to player i. Hence, Player i has no incentive to raise ω_i^* above $\sqrt{\lambda_i(\underline{u}^j)\lambda_i(\overline{u}^j)}$ and win the bid but receive a payoff lower than the payoff he receives while losing with offer $\omega_i^* = \sqrt{\lambda_i(\underline{u}^j)\lambda_i(\bar{u}^j)}$.

On the other hand player i would not lower ω_i^* to a level $\hat{\omega}_i$ which is lower than $\sqrt{\lambda_i(\underline{u}^j)\lambda_i(\bar{u}^j)}$ and expose himself to a potential loss if player j offers a $\hat{\omega}_j < \sqrt{\lambda_j(\bar{u}^i)/\lambda_j(\underline{u}^i)}$ such that $\dfrac{\hat{\omega}_j/\bar{\omega}_j}{\underline{\omega}_j/\hat{\omega}_j} > \dfrac{\hat{\omega}_i/\bar{\omega}_i}{\underline{\omega}_i/\hat{\omega}_i}$.

Hence player i would offer $\omega_i^* = \sqrt{\lambda_i(\underline{u}^j)\lambda_i(\bar{u}^j)}$ and player j would offer $\omega_j^* = \sqrt{\lambda_j(\underline{u}^i)\lambda_j(\bar{u}^i)}$. ■

Note that from Theorem 3.1 player i's solution weight $\omega_i^* = \sqrt{\lambda_i(\underline{u}^j)\lambda_i(\bar{u}^j)}$ is the geometric mean of most preferred Lagrange multiplier and the least preferred. This cooperative solution is a novel solution for two-person Non-transferrable utility games and is significantly different from the cooperative solutions in Nash (1950), Kalai (1977) and Kalai-Smorodinsky (1975).

Finally, to make computation more simple, we invoke the result $\lambda_i(\bar{u}^j) = 1/\lambda_j(\underline{u}^j)$ from Remark 2.1 and express the

$$\max_{x_1,x_2}\left\{u^i(x_1,x_2) + \sqrt{\lambda_i(\underline{u}^j)\lambda_i(\bar{u}^j)}\; u^j(x_1,x_2)\right\} \quad \text{as}$$

$$\max_{x_1,x_2}\left\{u^i(x_1,x_2) + \sqrt{\lambda_i(\underline{u}^j)/\lambda_j(\underline{u}^i)}\; u^j(x_1,x_2)\right. . \tag{3.4}$$

Since given the noncooperative Nash equilibrium outcome $\{\underline{u}^1, \underline{u}^2\}$, the multipliers $\lambda_1(\underline{u}^2)$ and $\lambda_2(\underline{u}^1)$ can be readily obtained by solving the corresponding Lagrange problems.

4. Properties of the Solution and an Illustrative Example

The equilibrium cooperative optimization solution derived in Section 3 satisfies the following properties which are assumed to have to be maintained in the traditional axiomatic approach.

(i) The condition of Pareto optimality -- because it is impossible to increase the payoff of one player without decreasing the payoff of the other.

(ii) The condition of symmetry (in the sense that the labels of the players do not matter) --because switching the labels of the players leaves the cooperative game problem and its solution unchanged (see Roth (1979)).

(iii) Independence of irrelevant alternatives other than the points of minimal expectations. This condition is the counterpart of the Nash (1950) Bargaining solution's property of independence of irrelevant alternatives, as proposed by Roth (1977). It states that the solution of the problem does not change as the set of feasible outcomes is reduced, so long as the point of minimal expectations in the Pareto set remains unchanged and the point originally selected remains feasible. Along the Pareto frontier, the point $(\underline{u}^1, \bar{u}^2)$ is player 1's point of minimal expectation and $(\bar{u}^1, \underline{u}^2)$ is player 2's point of minimal expectation. The concession equilibrium solution of the problem will not change as the set of feasible outcomes is reduced, so long as the points of minimal expectations in the Pareto set remain unchanged and the point originally selected remains feasible.

(iv) Independence of affine transformation of the payoff functions.

Now we consider a numerical example.

Example 4.1.

Consider a two-player noncooperative game of complete information with nontransferable payoffs. Players 1 and 2 seek to maximize their respective payoffs

$$u^1(x_1, x_2) = 10x_1 - x_1^2 - x_2, \text{ and}$$
$$u^2(x_1, x_2) = 16x_2 - x_2^2 - 2x_1. \tag{4.1}$$

A noncooperative Nash equilibrium (x_1^N, x_2^N) can be obtained as:

$$x_1^N = 5 \quad \text{and} \quad x_2^N = 8. \tag{4.2}$$

The outcome of the game is $\{u^1(x_1^N, x_2^N), u^2(x_1^N, x_2^N)\} = \{\underline{u}^1, \underline{u}^2\} = \{17, 54\}$.

The players agree to cooperate. When the payoff of player 2 is to remain the same as that under a Nash equilibrium, player 1's payoff under cooperation can be obtained by solving the problem:

$$\max_{x_1, x_2} (10x_1 - x_1^2 - x_2) \text{ subject to } 16x_2 - x_2^2 - 2x_1 = 54. \tag{4.3}$$

Solving problem (4.3) yields:

$$\lambda_1(\underline{u}^2) = 1/2, \; x_1(\underline{u}^2) = 4.5, \; x_2(\underline{u}^2) = 7, \text{ and } u^1[x_1(\underline{u}^2), x_2(\underline{u}^2)] = 17.75. \tag{4.4}$$

On the other hand, when the payoff of player 1 is to remain the same as that under a Nash equilibrium, player 2's payoff under cooperation can be obtained by solving the problem:

$$\max_{x_1, x_2} (16x_2 - x_2^2 - 2x_1) \text{ subject to } (10x_1 - x_1^2 - x_2) = 17. \tag{4.5}$$

Solving problem (4.3) yields:

$$\lambda_2(\underline{u}^1) = \sqrt[3]{2} = 1.25992105, \; x_1(\underline{u}^1) = 4.206299472, \; x_2(\underline{u}^1) = 7.370039475,$$

and

$$u^2[x_1(\underline{u}^1), x_2(\underline{u}^1)] = 55.19055079. \tag{4.6}$$

Following Theorem 3.1 and (3.4), an equilibrium cooperative optimization solution will arrive with player 1 offering $\omega_1^* = \sqrt{\lambda_1(\underline{u}^2)/\lambda_2(\underline{u}^1)}$ and player 2 offering $\omega_2^* = \sqrt{\lambda_2(\underline{u}^1)/\lambda_1(\underline{u}^2)}$. Both players will jointly maximize the objective:

$$
\begin{aligned}
&(10x_1 - x_1^2 - x_2) + \sqrt{\lambda_1(\underline{u}^2)/\lambda_2(\underline{u}^1)}\,(16x_2 - x_2^2 - 2x_1) \\
&= (10x_1 - x_1^2 - x_2) + 0.629960525(16x_2 - x_2^2 - 2x_1). \qquad (4.7)
\end{aligned}
$$

Solving the problem of maximizing (4.7) yields the players' cooperative optimization strategies $x_1^* = 4.370039475$ and $x_2^* = 7.206299474$, and the payoffs under cooperation $u^1(x_1, x_2) = 17.39685026$ and $u^2(x_1, x_2) = 54.62996045$.

5. Concluding Remarks

Reaching a cooperative solution in games with nontransferable payoffs constitutes one of the most difficult problems in game theory. In this paper, we provide a strategic equilibrium for cooperative optimization to two-person cooperative games in which the payoffs or utility are non-transferable. This equilibrium cooperative solution satisfies the axioms of (i) Pareto optimality, (ii) symmetry, (iii) independence of irrelevant alternatives other than the points of minimal expectations, and (iv) independence of affine transformation of the payoffs. Future research along this line of research is expected.

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