Applied Mathematical Sciences, Vol. 11, 2017, no. 25, 1239 - 1246
HIKARI Ltd, www.m-hikari.com
https://doi.org/10.12988/ams.2017.74119

Existence of Weak Solution for a $(n+1) \times (n+1)$ Nonsymmetric System of Keyfitz-Kranzer Type

Juan C. Hernández, César A. Gómez and Hernán Garzón

Department of Mathematics
Universidad Nacional de Colombia, Bogotá, D.C., Colombia

Abstract

We establish a theorem on the existence of weak solution for the Cauchy problem associated with a $(n+1) \times (n+1)$ nonsymmetric system of Keyfitz-Kranzer type with specific source terms and bounded measurable initial data.

Mathematics Subject Classification: 35D30, 35D40

Keywords: System of Keyfitz-Kranzer type, Cauchy problem, existence, weak solution

1 Introduction

A $n \times n$ system of Keyfitz-Kranzer type is a $n \times n$ system of partial differential equations of the following form

$$(u_i)_t + \big(u_i \phi(u_1, \ldots, u_n)\big)_x = 0, \quad i = 1, \ldots, n. \tag{1}$$

This type of system was first introduced for two equations by Barbara L. Keyfitz and Herbert C. Kranzer in [2] as a model of an elastic string in the plane. Systems of the form (1) appear in areas as elasticity theory [2] and magnetohydrodynamics [3], an example of a nonsymmetric system of Keyfitz-Kranzer type is the known system of two equations proposed by A. Aw and

M. Rascle in [1] for traffic flow

$$\begin{cases} \rho_t + \big(\rho v\big)_x = 0 \\ (\rho u)_t + \big(\rho u v\big)_x = 0, \end{cases} \tag{2}$$

here ρ and $v = u - p(\rho)$ denote, respectively, the density and the velocity of cars on the roadway and $p(\rho)$ is a smooth strictly increasing function. An improved version of the model (2) includes a relaxation term (a source term) in the second equation [4], [5], [6].

We shall be concerned with a $(n+1) \times (n+1)$ nonsymmetric system of Keyfitz-Kranzer type with specific source terms. Such a system is

$$\begin{cases} \rho_t + \Big(\rho\big(\psi(\rho, u_1, \dots, u_n) - p(\rho)\big)\Big)_x = 0 \\ (\rho u_i)_t + \Big(\rho u_i\big(\psi(\rho, u_1, \dots, u_n) - p(\rho)\big)\Big)_x + \rho h_i(u_i) = 0, \quad i = 1, \dots, n. \end{cases} \tag{3}$$

When $n = 1$, $u_1 = u$, $\psi(\rho, u_1) = u$ and $h_1(u_1) = 0$, we see that the system (2) is a particular case of (3). We will consider the Cauchy problem for the system (3) with bounded measurable initial data

$$\big(\rho(x,0), u_1(x,0), \dots, u_n(x,0)\big) = \big(\rho_0(x), u_{10}(x), \dots, u_{n0}(x)\big), \quad \rho_0(x) \geq 0. \tag{4}$$

The Cauchy problem (3)-(4) without source terms is studied in [7]. By adding some reasonable conditions on the initial data $u_{i0}(x)$ and the functions $h_i(u_i)$ an adaptation of the scheme given in [7] allows us to show the existence of weak solution.

2 Existence of Viscosity Solutions

We consider here the system (3) with viscosity terms; namely

$$\begin{cases} \rho^\epsilon_t + \Big(\rho^\epsilon\big(\psi(u^\epsilon_1, \dots, u^\epsilon_n) - p(\rho^\epsilon)\big)\Big)_x = \epsilon \rho^\epsilon_{xx} \\ (\rho^\epsilon u^\epsilon_i)_t + \Big(\rho^\epsilon u^\epsilon_i\big(\psi(u^\epsilon_1, \dots, u^\epsilon_n) - p(\rho^\epsilon)\big)\Big)_x + \rho^\epsilon h_i(u^\epsilon_i) = \epsilon(\rho^\epsilon u^\epsilon_i)_{xx},\ i = 1, \dots, n, \end{cases} \tag{5}$$

with initial data

$$\big(\rho^\epsilon(x,0), u^\epsilon_1(x,0), \dots, u^\epsilon_n(x,0)\big) = \big(\rho_0(x) + \epsilon, u_{10}(x), \dots, u_{n0}(x)\big), \tag{6}$$

where $\rho_0(x), u_{10}(x), \dots, u_{n0}(x)$ are given by (4).

Lemma 2.1. *We assume that* $p(0) = 0, \lim_{\rho \to 0} \rho p'(\rho) = 0, \lim_{\rho \to +\infty} p(\rho) = -\infty$, $p(\rho) \leq 0$ *and the function* $\rho p(\rho)$ *is strictly concave for positive* ρ *(i.e.*

$2p'(\rho)+\rho p''(\rho)<0$ for $\rho>0$); let $\psi(u_1,\dots,u_n)\in C^2(\mathbb{R}^n)$ be a nonlinear function, nonnegative and convex. If $g_i(\rho,u_i)=\rho h_i(u_i)$, $i=1,\dots,n$ are locally Lipschitz continuous functions and each function $h_i(u_i)$ satisfies the inequalities

$$c_{i_1}u_i+c_{i_2}\le h_i(u_i)\le c_{i_3}u_i+c_{i_4}\,, \tag{7}$$

where c_{i_j}, $i=1,\dots,n$, $j=1,\dots,4$ are constants, then for any $\epsilon>0$ the viscosity solution $(\rho^\epsilon(x,t),u_1^\epsilon(x,t),\dots,u_n^\epsilon(x,t))$ of the Cauchy problem (5)-(6) exists and satisfies

$$\rho^\epsilon\le M(T),\qquad |u_i^\epsilon|\le M(T),\quad i=1,\dots,n,\quad (x,t)\in\mathbb{R}\times[0,T] \tag{8}$$

where $M(T)$ is a positive constant, not dependent on ϵ.

Proof. For each fixed i we proceed as follows. We substitute the first equation of system (5) in the second equation, finding

$$u_{it}+\big(\psi-p(\rho)\big)u_{ix}+h_i(u_i)=\epsilon u_{ixx}+2\epsilon\frac{\rho_x}{\rho}u_{ix}. \tag{9}$$

By appliying the maximum principle to the above equation we get the a-priori estimate of u_i^ϵ given in (8).
We multiply the first equation of system (5) by $p'(\rho)$ and the equation (9) by ψ_{u_i} to obtain

$$p(\rho)_t+\big(\psi-p(\rho)\big)p(\rho)_x+\rho p'(\rho)\big(\psi-p(\rho)\big)_x=\epsilon p(\rho)_{xx}-\epsilon p''(\rho)\rho_x^2, \tag{10}$$

and

$$\psi_{u_i}u_{it}+\big(\psi-p(\rho)\big)\psi_{u_i}u_{ix}+\psi_{u_i}h_i(u_i)=\epsilon\psi_{u_i}u_{ixx}+2\epsilon\frac{\rho_x}{\rho}\psi_{u_i}u_{ix}. \tag{11}$$

From this last equation, we deduce

$$\psi_t+\big(\psi-p(\rho)\big)\psi_x+\sum_{i=1}^n\psi_{u_i}h_i(u_i)=\epsilon\psi_{xx}-\epsilon\sum_{i,j=1}^n\psi_{u_iu_j}u_{ix}u_{jx}+2\epsilon\frac{\rho_x}{\rho}\psi_x. \tag{12}$$

Subtracting the members of (10) from those of (12), we have

$$\begin{aligned}\big(\psi-p(\rho)\big)_t+\big(\psi-p(\rho)-\rho p'(\rho)\big)\big(\psi-p(\rho)\big)_x+\sum_{i=1}^n\psi_{u_i}h_i(u_i)=\epsilon\big(\psi-p(\rho)\big)_{xx}\\ +2\epsilon\frac{\rho_x}{\rho}\big(\psi-p(\rho)\big)_x-\epsilon\sum_{i,j=1}^n\psi_{u_iu_j}u_{ix}u_{jx}+\frac{\epsilon}{\rho}\big(2p'(\rho)+\rho p''(\rho)\big)\rho_x^2.\end{aligned} \tag{13}$$

From (13) and by the assumptions on the functions p and ψ one finds that

$$\big(\psi-p(\rho)\big)_t+\big(\psi-p(\rho)-\rho p'(\rho)\big)\big(\psi-p(\rho)\big)_x-c \leq \epsilon\big(\psi-p(\rho)\big)_{xx}+2\epsilon\frac{\rho_x}{\rho}\big(\psi-p(\rho)\big)_x, \tag{14}$$

where c is a positive constant. Appliying the maximum principle to this inequality we get the estimate $\psi - p(\rho) \leq N(T)$ and therefore $p(\rho) \geq -N(T)$, estimate from which we obtain that $\rho^\epsilon \leq M(T)$. □

Lemma 2.2. *With the hypotheses of lemma 2.1, we assume (6). Then, the following a-priori bounds hold for the diffusion system (5)*

$$\rho^\epsilon(x,t) \geq c(t,\epsilon) > 0, \tag{15}$$

where $c(t,\epsilon)$ *could tend to* 0 *as* $t \to +\infty$ *or* $\epsilon \to 0$.

Proof. Let us consider the first equation in (5), we rewrite this equation as

$$\nu_t - \epsilon\nu_{xx} = -\epsilon(\nu_x)^2 + -\big(\psi - p(\rho)\big)\nu_x + \big(\psi - p(\rho)\big)_x, \tag{16}$$

where $\nu = -\ln\rho$.

Let $k_\epsilon(x,t) = \frac{1}{\sqrt{4\pi\epsilon t}}\exp\big(-\frac{x^2}{4\epsilon t}\big)$ be the heat kerner for $\nu_t - \epsilon\nu_{xx}$, a solution of (16) with initial data $\nu_0^\epsilon(x) = -\ln\big(\rho_0(x)+\epsilon\big)$ satisfies the following integral equation

$$\begin{aligned}&\nu(x,t) = \nu_0^\epsilon(x) * k_\epsilon(x,t)\\ &+\int_0^t\left(-\epsilon\Big(\nu_x+\tfrac{1}{2\epsilon}\big(\psi-p(\rho)\big)\Big)^2+\tfrac{1}{4\epsilon}\big(\psi-p(\rho)\big)^2+\big(\psi-p(\rho)\big)_x\right)*_x k_\epsilon(x,t-s)ds\,,\end{aligned}$$

from which we get

$$\begin{aligned}\nu(x,t) &\leq \nu_0^\epsilon(x) * k_\epsilon(x,t) + \int_0^t\left(\tfrac{1}{4\epsilon}\big(\psi-p(\rho)\big)^2+\big(\psi-p(\rho)\big)_x\right) *_x k_\epsilon(x,t-s)ds\\ &= \nu_0^\epsilon(x) * k_\epsilon(x,t) + \int_0^t \tfrac{1}{4\epsilon}\big(\psi-p(\rho)\big)^2 *_x k_\epsilon(x,t-s)ds\\ &\qquad + \int_0^t \big(\psi-p(\rho)\big) *_x \big(k_\epsilon(x,t-s)\big)_x ds\\ &\leq -\ln\epsilon + \frac{N_1}{\epsilon}t + N_2\sqrt{\frac{t}{\epsilon}}\,.\end{aligned}$$

Thus

$$\rho(x,t) \geq \epsilon\exp-\left(\frac{N_1}{\epsilon}t + N_2\sqrt{\frac{t}{\epsilon}}\right) \geq c(t,\epsilon) > 0.$$

□

3 A L^1 estimate of $u^{\epsilon}_{i\,x}(\cdot,t)$

Lemma 3.1. *If the total variation of $u_{i0}(x) = u_i(x,0)$ is bounded and the functions $h_i(u_i)$ are such that $h_i'(u_i) \geq 0$, then $u_{ix}(\cdot,t)$, $i = 1,\ldots,n$ is bounded in $L^1(\mathbb{R})$, moreover we have*

$$TV\big(u_i(x,t)\big) = \int_{-\infty}^{+\infty} \big|u_{ix}(x,t)\big|\,dx \leq \int_{-\infty}^{+\infty} \big|u_{i0x}(x)\big|\,dx = TV\big(u_{i0}(x)\big), \quad (17)$$

$i = 1,\ldots,n$.

Proof. Differentiating equation (9) with respect to x, we have

$$u_{itx} + \Big(\big(\psi - p(\rho)\big)u_{ix}\Big)_x + h_{ix}(u_i) = \epsilon u_{ixxx} + \big(2\epsilon\rho^{-1}\rho_x u_{ix}\big)_x.$$

Let $\theta = u_{ix}$, then from the above equation we obtain

$$\theta_t + \Big(\big(\psi - p(\rho)\big)\theta\Big)_x + h'(u_i)\theta = \epsilon\theta_{xx} + \big(2\epsilon\rho^{-1}\rho_x\theta\big)_x, \quad (18)$$

multiplying (18) by the sequence of smooth functions $g'(\theta,\alpha)$, where α is a parameter, we see that

$$g(\theta,\alpha)_t + \Big(\big(\psi - p(\rho)\big)g(\theta,\alpha)\Big)_x + \big(\psi - p(\rho)\big)_x\big(g'(\theta,\alpha)\theta - g(\theta,\alpha)\big) + h_i'(u_i)g'(\theta,\alpha)\theta$$
$$= \epsilon g(\theta,\alpha)_{xx} - \epsilon g''(\theta,\alpha)\theta_x^2 + \big(2\epsilon\rho^{-1}\rho_x g(\theta,\alpha)\big)_x + \big(2\epsilon\rho^{-1}\rho_x\big)_x\big(g'(\theta,\alpha)\theta - g(\theta,\alpha)\big).$$

we choose $g(\theta,\alpha)$ such that $g''(\theta,\alpha) \geq 0$, $g'(\theta,\alpha) \to \operatorname{sign}\theta$ and $g(\theta,\alpha) \to |\theta|$ as $\alpha \to 0$, so that

$$|\theta|_t + \Big(\big(\psi - p(\rho)\big)|\theta|\Big)_x + h_i'(u_i)|\theta| = \epsilon|\theta|_{xx} - \epsilon g''(\theta,\alpha)\theta_x^2 + \big(2\epsilon\rho^{-1}\rho_x|\theta|\big)_x,$$

this equation yields

$$|\theta|_t + \Big(\big(\psi - p(\rho)\big)|\theta|\Big)_x \leq \epsilon|\theta|_{xx} + \big(2\epsilon\rho^{-1}\rho_x|\theta|\big)_x. \quad (19)$$

Integrating (19) in $\mathbb{R}\times[0,t]$, we obtain (17). □

4 n entropy-entropy flux pairs

Introducing the variables $m_i = \rho u_i$, we rewrite the homogeneous system associated to (3) as

$$\begin{cases} \rho_t + \left(\rho\Big(\psi\big(\rho, \frac{m_1}{\rho},\ldots,\frac{m_n}{\rho}\big) - p(\rho)\Big)\right)_x = 0 \\ m_{it} + \left(\rho\Big(\psi\big(\rho, \frac{m_1}{\rho},\ldots,\frac{m_n}{\rho}\big) - p(\rho)\Big)\right)_x = 0, \quad i = 1,\ldots,n. \end{cases} \quad (20)$$

An entropy $\eta = \eta(\rho, m_1, \cdots, m_n)$ for (20) and its associated entropy flux $q = q(\rho, m_1, \cdots, m_n)$ are functions satisfying $\nabla q = \nabla \eta dF_{(\rho, m_1, \ldots, m_2)}$, where $dF_{(\rho, m_1, \ldots, m_2)}$ is the Jacobian matrix of the flux functions in (20). Let G be a convex function, and

$$\eta_i(\rho, m_1, \ldots, m_n) = \rho G\left(\frac{m_i}{\rho}\right), \tag{21}$$

$$q_i(\rho, m_1, \ldots, m_n) = \rho G\left(\frac{m_i}{\rho}\right)\left(\psi\left(\frac{m_1}{\rho}, \ldots, \frac{m_n}{\rho}\right) - p(\rho)\right), \tag{22}$$

$i = 1, \ldots, n$, then the pairs (η_i, m_i) are covex entropy-entropy flux pairs.

5 H^{-1}_{loc} compactness

Lemma 5.1. *Under the conditions given in the lemmas 2.1 and 3.1 and if $g(\rho)$ is an arbitrary smooth function, it follows that*

$$g(\rho^\epsilon)_t + \left(\int^{\rho^\epsilon} g'(s) f'(s)\, ds + g(\rho^\epsilon)\psi\big(u_1^\epsilon, \ldots, u_n^\epsilon\big)\right)_x \tag{23}$$

is compact in $H^{-1}_{loc}(\mathbb{R} \times \mathbb{R}^+)$, where $f(s) = -sp(s)$.

Proof. The proof is a repetition of the argument used in proving the first half of Lemma 6 in [7]. □

A consequence of the previous lemma is the following result,

Corollary 5.2. *Let $f(s)$ be the function given in lemma 5.1. We assume the hypotheses of the lemmas 2.1 and 3.1. Then*

$$\rho_t^\epsilon + \Big(\rho^\epsilon \psi\big(u_1^\epsilon, \ldots, u_n^\epsilon\big) - \rho^\epsilon p(\rho^\epsilon)\Big)_x, \tag{24}$$

$$f(\rho^\epsilon)_t + \left(\int^{\rho^\epsilon} f'^2(s)\, ds + f(\rho^\epsilon)\psi\big(u_1^\epsilon, \ldots, u_n^\epsilon\big)\right)_x \tag{25}$$

$$\rho_t^\epsilon + \Big(\rho^\epsilon \psi\big(u_1^\epsilon, \ldots, u_n^\epsilon\big) - \rho^\epsilon p(\rho^\epsilon) + u_i^\epsilon\Big)_x, \quad i = 1, \ldots, n, \tag{26}$$

are compact in $H^{-1}_{loc}(\mathbb{R} \times \mathbb{R}^+)$.

Lemma 5.3. *With the hypotheses of the lemmas 2.1 and 3.1, it follows that for each $i = 1, \ldots, n$ we have*

$$(\rho^\epsilon u_i^\epsilon)_t + \Big(\rho^\epsilon u_i^\epsilon\big(\psi\big(u_1^\epsilon, \ldots, u_n^\epsilon\big) - p(\rho^\epsilon)\big)\Big)_x \tag{27}$$

is compact in $H^{-1}_{loc}(\mathbb{R} \times \mathbb{R}^+)$.

Proof. If (η_i, q_i) is an convex entropy-entropy flux pair given by (21)-(22), then multiplying by $\nabla\eta_i$ the system (20) gives

$$\eta_{it} + q_{ix} = \epsilon\eta_{ixx} - \epsilon\rho F''(u_i)u_{ix}^2 - \rho h_i(u_i)G'(u_i). \tag{28}$$

We shall take a strictly convex function $G(u)$ in the equation (28), so we obtain that

$$\epsilon\rho^\epsilon\big(u_{ix}^\epsilon\big)^2, \quad i = 1,\ldots,n, \text{ are bounded in } L^1_{loc}(\mathbb{R}\times\mathbb{R}^+). \tag{29}$$

According to the system (5) we can write

$$(\rho^\epsilon u_i^\epsilon)_t + \Big(\rho^\epsilon u_i^\epsilon\big(\psi\big(u_1^\epsilon,\ldots,u_n^\epsilon\big) - p(\rho^\epsilon)\big)\Big)_x = \epsilon(\rho_x^\epsilon u_i^\epsilon + \rho^\epsilon u_{ix}^\epsilon)_x - \rho^\epsilon h_i(u_i^\epsilon). \tag{30}$$

It follows from the proof of Lemma 5.1 that

$$\epsilon(\rho_x^\epsilon)^2 \text{ is bounded in } L^1_{loc}(\mathbb{R}\times\mathbb{R}^+). \tag{31}$$

The estimates (31) and (29) allows us to show respectively that $\epsilon(\rho_x^\epsilon u_i^\epsilon)_x$ and $\epsilon(\rho^\epsilon u_{ix}^\epsilon)_x$ are compact in $H^{-1}_{loc}(\mathbb{R}\times\mathbb{R}^+)$, with the aid of the Cauchy-Schwarz inequality. The term $\rho h_i(u_i)$ is bounded in $\mathcal{M}(\mathbb{R}\times\mathbb{R}^+)$ (space of Radon measures) since it is bounded in $L^1_{loc}(\mathbb{R}\times\mathbb{R}^+)$. (29) is bounded in $W^{-1,\infty}_{loc}(\mathbb{R}\times\mathbb{R}^+)$. Using now Murat's lemma we see that (27) is compact in $H^{-1}_{loc}(\mathbb{R}\times\mathbb{R}^+)$. □

The preceding lemma admits the next corollary,

Corollary 5.4. *The assumptions in the lemmas 2.1 and 3.1 imply that*

$$(\rho^\epsilon u_i^\epsilon)_t + \Big(\rho^\epsilon u_i^\epsilon\big(\psi\big(u_1^\epsilon,\ldots,u_n^\epsilon\big) - p(\rho^\epsilon)\big) + (u_i^\epsilon)^2\Big)_x \tag{32}$$

is compact in $H^{-1}_{loc}(\mathbb{R}\times\mathbb{R}^+)$.

Lemma 5.5. *If the conditions of the lemmas 2.1 and 3.1 are satisfied, then*

$$\Big(\rho^\epsilon\psi\big(u_1^\epsilon,\ldots,u_n^\epsilon\big)\Big)_t + \Big(\rho^\epsilon\psi^2\big(u_1^\epsilon,\ldots,u_n^\epsilon\big) - \rho^\epsilon p(\rho^\epsilon)\psi\big(u_1^\epsilon,\ldots,u_n^\epsilon\big)\Big)_x \tag{33}$$

is compact in $H^{-1}_{loc}(\mathbb{R}\times\mathbb{R}^+)$.

Proof. Using (12) and the first equation of the system (5), it is possible to write

$$(\rho\psi)_t + \big(\rho\psi^2 - \rho p(\rho)\psi\big)_x = \epsilon(\rho\psi)_{xx} - \epsilon\rho\sum_{i,j=1}^n \psi_{u_iu_j}u_{ix}u_{jx} - \rho\sum_{i=1}^n \psi_{u_i}h_i(u_i). \tag{34}$$

By the Cauchy-Schwarz inequality, we deduce from (31) and (29) that $\epsilon(\rho\psi)_{xx}$ is compact in $H^{-1}_{loc}(\mathbb{R}\times\mathbb{R}^+)$. The other two terms on the right-hand side of (34) are bounded in $L^1_{loc}(\mathbb{R}\times\mathbb{R}^+)$ and hence in $\mathcal{M}(\mathbb{R}\times\mathbb{R}^+)$. $(\rho\psi)_t + \big(\rho\psi^2 - \rho p(\rho)\psi\big)_x$ is bounded in $W^{-1,\infty}_{loc}(\mathbb{R}\times\mathbb{R}^+)$ and by Murat's lemma is compact in $H^{-1}_{loc}(\mathbb{R}\times\mathbb{R}^+)$. □

6 Pointwise convergence

Lemma 6.1. *We assume the hypotheses of the lemmas 2.1 and 3.1. Then (a subsequence of) $\{\rho^\epsilon(x,t)\}$ and (a subsequence of) $\{u_i^\epsilon(x,t)\}$, $i = 1,\dots,n$ converge pointwisely.*

Proof. The proof is a repetition of the argument used in proving the pointwise convergence in [7] (see [7], Theorem 3). □

The pointwise convergence of $\{\rho^\epsilon(x,t)\}$ and $\{u_i^\epsilon(x,t)\}$, $i = 1,\dots,n$, ensures the existence of a weak solution of (3)-(4).

References

[1] A. Aw and M. Rascle, Resurrection of "second order" models of traffic flow, *Indiana. Univ. Math. J.,* **26** (1977), 373-392.

[2] B. Keyfitz and H. Kranzer, A system of non-strictly hyperbolic conservation laws arising in elasticity theory, *Arch. Rat. Mech. Anal.,* **72** (1980), 219-241. https://doi.org/10.1007/bf00281590

[3] F. James, Y.-J. Peng and B. Perthame, Kinetic formulation for chromatography and some other hyperbolic systems, *J. Math. Pure Appl.,* **74** (1995), 367-385.

[4] F. Siebel and W. Mauser, On the fundamental diagram of traffic flow, *SIAM J. Appl. Math.,* **66** (2006), 1150-1162. https://doi.org/10.1137/050627113

[5] M. Rascle, An improved macroscopic model of traffic flow: Derivation and links with the Lighthill-Whitham model, *Math. Comput. Model,* **35** (2002), no. 5-6, 581-590. https://doi.org/10.1016/s0895-7177(02)80022-x

[6] P. Bagnerini, R. M. Colombo and A. Corli, On the role of source terms in continuum traffic flow models, *Math. Comput. Modelling,* **44** (2006), no. 9-10, 917-930. https://doi.org/10.1016/j.mcm.2006.02.019

[7] Y. G. Lu, Existence of global bounded weak solutions to nonsymmetric systems of Keyfitz-Kranzer type, *Arch. Rat. Mech. Anal.,* **261** (2011), 2792-2815. https://doi.org/10.1016/j.jfa.2011.07.008

Received: April 17, 2017; Published: May 3, 2017