On U-BG-Filter of a U-BG-BH-Algebra

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Abstract

In this paper, we introduce the notion of U-BG-filter in U-BG-BH-algebra and observed that every filter of a U-BG-BH-algebra is a U-BG-filter. A necessary and sufficient condition is derived for every $U - BG$-filter of $U - BG - BH$-algebra to become a filter. Some properties of $U - BG$-filter are studied with respect to homomorphism, Cartesian products and quotient $U - BG - BH$-algebra.

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Keywords: BH-algebra, BG-algebra, U-BG-BH-algebra, U-BG-filter, filter, Cartesian products and quotient $U - BG - BH$-algebra

1 Introduction

The notion of U-BG-BH-algebra was introduced and extensively studied by H.H. Abbass and L.S. Mahdi ([3]), in 2014. This class of U-BG-BH-algebra was introduced as a combination of the classes of BH-algebra and BG-algebra. In 1980, E.Y. Deeba ([6]) introduced the notion of filters and in the setting of bounded implicative BCK-algebra constructed quotient algebra via a filter. Also E.Y. Deeba and A.B. Thaheem ([7]) studied afilters in BCK-algebra in 1990. In
1991 C.S.Hoo ([8]) was presented the filters in BCI-algebra. In 1996, J.Meng ([10]) introduced the notion of BCK-filter in BCK-algebra. In 2012 H.H.Abass and H.A.Dahham ([1]) discussed the concept of completely closed filter of a BH-algebra, and completely closed filter with respect to an element of BH-algebra. In this paper, the notion of U-BG-filter of U-BG-BH-algebra is introduced.

2 Preliminary Notes

In this section, some basic concepts about a BG-algebra, BH-algebra, U-BG-BH-algebra, filter, U-BG-filter, subalgebra, normal subset and quotient U-BG-BH-algebra are given.

**Definition 2.1.** ([9]) A BG-algebra is a non-empty set $X$ with a constant $0$ and a binary operation $\ast$ satisfying the following axioms: for all $x, y, z \in X$:

(I) $x \ast x = 0$,

(II) $x \ast 0 = x$,

(III) $(x \ast y) \ast (0 \ast y) = 0$,

**Lemma 2.2.** ([9]) Let $(X, \ast, 0)$ be a BG-algebra. Then

(i) The right cancellation law holds in $X$, i.e. $x \ast y = z \ast y$ implies $x = z$,

(ii) $0 \ast (0 \ast x) = x, \forall x \in X$.

(iii) If $x \ast y = 0$, then $x = y, \forall x, y \in X$.

(iv) If $0 \ast x = 0 \ast y$, then $x = y, \forall x, y \in X$.

(v) $(x \ast (0 \ast y)) \ast y = x, \forall x \in X$.

**Definition 2.3.** ([11]) A BH-algebra is a nonempty set $X$ with a constant $0$ and a binary operation “$\ast$” satisfying the following conditions:

(I) $x \ast x = 0, \forall x \in X$.

(II) $x \ast y = 0$ and $y \ast x = 0$ imply $x = y, \forall x, y \in X$.

(III) $x \ast 0 = x, \forall x \in X$.

**Proposition 2.4.** ([9]) Every BG-algebra is a BH-algebra.

**Definition 2.5.** ([3]) A U-BG-BH-algebra is defined to be a BH-algebra $X$ in which there exists a proper subset $U$ of $X$, such that:
(U1) $0 \in U$, $|U| \geq 2$.

(U2) $U$ is a BG-algebra.

**Definition 2.6.** ([5]) A nonempty subset $S$ of a BH-algebra $X$ is called a BH-subalgebra or subalgebra if $x \ast y \in S$, $\forall x, y \in S$.

**Definition 2.7.** ([1]) Let $X$ be a BH-algebra, a nonempty subset $N$ of $X$ is said to be normal of $X$ if $(x \ast a) \ast (y \ast b) \in N$ for any $x \ast y$ and $a \ast b \in N$, $\forall x, y, a, b \in X$.

**Definition 2.8.** ([2]) A BH-algebra $X$ is callmed medial if $x \ast (x \ast y) = y$, $\forall x, y \in X$.

**Definition 2.9.** A filter of a BH-algebra $X$ is a non-empty subset $F$ of $X$ such that:

(F1) If $x \in F$, and $y \in F$, then $y \ast (y \ast x) \in F$ and $x \ast (x \ast y) \in F$.

(F2) If $x \in F$ and $x \ast y = 0$ then $y \in F$.

**Remark 2.10.** Let $(X, \ast_X, 0_X)$ and $(Y, \ast_Y, 0_Y)$ be BH-algebra. A mapping $f : X \rightarrow Y$ is called a Homomorphism if $f(x \ast_X y) = f(x) \ast_Y f(y)$ for any $x, y \in X$. A homomorphism $f$ is called a monomorphism (resp., epimorphism) if it injective (resp., surjective). A bijective homomorphism is called an isomorphism. Two BH-algebra $X$ and $Y$ are said to be isomorphic, written $X \cong Y$, if there exists an isomorphism $f : X \rightarrow Y$. For any homomorphism $f : X \rightarrow Y$, the set $\{ x \in X : f(x) = 0_Y \}$ is called the kernel of $f$, denoted by Ker($f$), the set $\{ f(x) : x \in X \}$ is called image of $f$, denoted by Im($f$). Notice that $f(0_X) = 0_Y$. ([9]) Let $(X, \ast, 0)$ be a BG-algebra and let $N$ be a normal subalgebra of $X$. Define a relation $\sim_N$ on $X$ by $x \sim_N y$ if and only if $x \ast y \in N$, where $x, y \in X$. Then it is easy to show $\sim_N$ is an equivalence relation on $X$. Denote the equivalence class containing $x$ by $[x]_N$, i.e. $[x]_N = \{ y \in X : x \sim_N y \}$ and let $X/N = \{ [x]_N : x \in X \}$. If $\ast'$ denoted on $X/N$ by $[x]_N \ast' [y]_N = [x \ast y]_N$. Then $(X/N, \ast', [0]_N)$ is a BG-algebra and it is called quotient bg-algebra of $X$ by $N$. The authors in ([1]) generalized this concept to BH-algebra to obtain $(X/N, \ast', [0]_N)$ quotient BH-algebra of $X$ by $N$.

**Remark 2.11.** ([9]) Let $(X, \ast, 0)$ be a BG-algebra and let $N$ be a normal subalgebra of $X$. Define a relation $\sim_N$ on $X$ by $x \sim_N y$ if and only if $x \ast y \in N$, where $x, y \in X$. Then it is easy to show $\sim_N$ is an equivalence relation on $X$. Denote the equivalence class containing $x$ by $[x]_N$, i.e. $[x]_N = \{ y \in X : x \sim_N y \}$ and let $X/N = \{ [x]_N : x \in X \}$. If $\ast'$ denoted on $X/N$ by $[x]_N \ast' [y]_N = [x \ast y]_N$. Then $(X/N, \ast', [0]_N)$ is a BG-algebra and it is called quotient bg-algebra of $X$ by $N$. The authors in ([1]) generalized this concept to BH-algebra to obtain $(X/N, \ast', [0]_N)$ quotient BH-algebra of $X$ by $N$.

**Remark 2.12.** Let $\{ (X_i, \ast_{X_i}, 0_{X_i}) : i \in \lambda \}$ be a family of $U_i - BG - BH$-algebra. Define the cartesian product of all $X_i, i \in \lambda$ to be the structure $\prod_{i \in \lambda} X_i = (\prod_{i \in \lambda} X_i, \circlearrowright, (0_{X_i}))$, where $\prod_{i \in \lambda} X_i$ is the set of tuples $\{ (x_i) : \forall i \in \lambda$ and $x_i \in X_i \}$, and whose binary operation $\circlearrowright$ is give by $(x_i) \circlearrowright (y_i) = (x_i \ast_{X_i} y_i), \forall i \in \lambda$ and $x_i, y_i \in X_i$. Note that the binary operation $\circlearrowright$ is componentwise.
3 Main Results

In this section, we introduce the concepts of a $U - BG$ - filter of a $U - BG$ - algebra. Also, we study some properties of it with examples.

**Definition 3.1.** A non-empty subset $F$ of a $U - BG - BH$ - algebra $X$ is called a $U - BG$ - filter of $X$, if it satisfies $(F_1)$ and

$(F_3)$ If $x \in F$ and $x \ast y = 0$ then $y \in F$, $\forall y \in U$.

**Example 3.2.** Consider the $U - BG - BH$ - algebra $(X; \ast, 0)$, where $X = \{0, 1, 2, 3\}$ and $\ast$ is the binary operation define by the following table:

<table>
<thead>
<tr>
<th>$\ast$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>0</td>
</tr>
</tbody>
</table>

and $U = \{0, 1, 2\}$. The subset $F = \{1, 2\}$ is $U$-BG-filter, but the subset $F = \{1, 3\}$ is not a $U$-BG-filter of $X$, since $3 \ast (3 \ast 1) = 0 \notin F$.

**Remark 3.3.** If $X$ is a $U$-BG-BH-algebra. Then $\{0\}$ and $X$ are a $U$-BG-filter of $X$, called trivial $U$-B-filters of $X$. A $U$-BG-filter $F$ of $X$ is called a proper $U$-BG-filter of $X$ if $F \neq X$.

**Theorem 3.4.** Let $X$ be a $U$-BG-BH-algebra and $S$ is a subalgebra of $X$, satisfies the right cancellation low in $X$. Then $S$ is a $U$-BG-filter of $X$.

**Proof.** (i) Let $x, y \in S$, then $x \ast y \in N$ and $y \ast x \in N$, using Definition(2.6). So $y \ast (y \ast x) \in S$ and $x \ast (x \ast y) \in S$.

(ii) Let $x \in S, x \ast y = 0, y \in U$, then $x \ast y = y \ast y$, [by Definition(2.1)(I)]. We obtain $x = y$, [by using the right cancellation low], so $y \in S$. Therefore $S$ is a $U$-BG-filter of $X$.

**Proposition 3.5.** Let $X$ be a $U$-BG-BH-algebra. Then every filter of $X$ is a $U$-BG-filter of $X$.

**Proof.** Is obvious. [Since $U \subseteq X$ and $F$ is a filter of $X$].

**Remark 3.6.** The convers of proposition (3.5) is not correct in general as in the following example. Consider the $U$-BG-BH-algebra $X$ in example(3.2). The subset $F = \{1, 2\}$ is a $U - BG$ - filter of $X$, but it is not a filter since $1 \in F$ and $1 \ast 3 = 0$ but $3 \notin F$. 
Theorem 3.7. Let \( X \) be a medail U-BG-BH-algebra. Then every a non-empty subset \( A \) of \( X \) is a U-BG-filter of \( X \).

Proof. Let \( A \) be a non-empty subset of \( X \).

(i) Let \( x, y \in A \). Then \( x = y^*(y*x) \) [By Definition(2.8)]. Thus \( y^*(y*x) \in A \). Similarly, \( x^*(x*y) \in A \).

(ii) Let \( x \in A, x*y = 0, y \in U \). Then \( y = x^*(x*y) \) [By Definition(2.8)], imply that \( y = x^*0 \), then \( y = x \) [By definition(2.3)(III)], so \( y \in A \). Therefore, \( A \) is a U-BG-filter of \( X \).

Theorem 3.8. Let \( X \) be a U-BG-BH-algebra, and \( F \) be a U-BG-filter of \( X \) such that \( x*y \neq 0, \forall y \notin F \) and \( x \in F \). Then \( F \) is a filter of \( X \).

Proof. Let \( F \) be a U-BG-filter of \( X \) such that \( y \in X \) and \( x \in F \),

(i) Let \( x, y \in F \), then \( y^*(y*x), x^*(x*y) \in F \) [By definition(3.1)(F1)],

(ii) Let \( x \in F, x*y = 0 \). Then we have two cases. **Cases(I):** If \( y \in U \), then \( y \in F \) [By definition(3.1)(F3)]. **Cases(II):** If \( y \notin U \) then either \( y \notin F \) or \( y \in F \). Suppose \( y \notin F \), then \( x*y \neq 0 \), this a contradiction. Thus \( y \in F \).

Therefore \( F \) is a filter of \( X \).

Proposition 3.9. Let \( X \) be a U-BG-BH-algebra and let \( \{F_i, i \in \lambda\} \) be a family of U-BG-filters of \( X \). Then \( \bigcap_{i \in \lambda} F_i \) is a U-BG-filter of \( X \).

Proof. Let \( \{F_i, i \in \lambda\} \) be a family of U-BG-filters of \( X \). To prove \( \bigcap_{i \in \lambda} F_i \) is a U-BG-filter of \( X \).

(i) If \( x, y \in \bigcap_{i \in \lambda} F_i \), then \( x, y \in F_i, \forall i \in \lambda \). Hence \( y^*(y*x), x^*(x*y) \in F_i \) [since \( F_i \) is a U-BG-filter of \( X \), \( \forall i \in \lambda \), by definition(3.1)(F1)]. Then \( y^*(y*x), x^*(x*y) \in \bigcap_{i \in \lambda} F_i \).

(ii) Let \( x \in \bigcap_{i \in \lambda} F_i \) such that \( x*y = 0, y \in U \). Then \( x \in F_i, \forall i \in \lambda \). Thus \( y \in F_i \), [Since \( F_i \) is a U-BG-filter of \( X \), \( \forall i \in \lambda \), by definition(3.1)(F3)]. Therefore, \( \bigcap_{i \in \lambda} F_i \) is a \( U-BG \) - filter of \( X \).

Remark 3.10. The union of \( U-BG \) - filters of U-BG-BH-algebra may be not a \( U-BG \) - filter as in the following example.

Example 3.11. Consider the U-BG-BH-algebra \( X = \{0,1,2,3,4\} \) with binary operation \(^*\) defined by the following table:
where \( U = \{0,1,2\} \). \( F_1 = \{0,4\} \) and \( F_2 = \{0,3\} \) are two \( U - BG \) filters of \( X \). The union of the \( U - BG \) filters is not a \( U-BG \)-filter of \( X \). Since \( 3,4 \in F_1 \cup F_2 \), but \( 3 \ast (3 \ast 4) = 2 \notin F_1 \cup F_2 \).

**Proposition 3.12.** Let \( X \) be a \( U-BG-BH \)-filter and let \( \{F_i, i \in \lambda\} \) be a chain of \( U-BG \)-filters of \( X \). Then \( \bigcup_{i \in \lambda} F_i \) is a \( U-BG \)-filter of \( X \).

**Proof.** Let \( \{F_i, i \in \lambda\} \) be a chain of \( U-BG \)-filters of \( X \). To prove \( \bigcup_{i \in \lambda} F_i \) is a \( U-BG \)-filter of \( X \).

(i) If \( x, y \in \bigcup_{i \in \lambda} F_i \), \( \forall i \in \lambda \), then there exist \( F_j, F_k \in \{F_i\}_{i \in \lambda} \) such that \( x \in F_j \) and \( y \in F_k \). So, either \( F_j \subseteq F_k \) or \( F_k \subseteq F_j \). If \( F_j \subseteq F_k \), then \( x \in F_k \) and \( y \in F_k \), we have \( y \ast (y \ast x) \in F_k \) and \( x \ast (x \ast y) \in F_k \) [since \( F_k \) is a \( U-BG \)-filter of \( X \), \( \forall i \in \lambda \), by definition(3.1)(F1)]. Similarly, if \( F_k \subseteq F_j \). Then \( y \ast (y \ast x), x \ast (x \ast y) \in \bigcup_{i \in \lambda} F_i \).

(ii) Let \( x \in \bigcup_{i \in \lambda} F_i \) such that \( x \ast y = 0, y \in U \). There exists \( j \in \lambda \) such that \( x \in F_j \). Hence \( y \in F_j \), [Since \( F_i \) is a \( U-BG \)-filter of \( X \), \( \forall i \in \lambda \), by definition(3.1)(F3)]. Thus \( y \in \bigcup_{i \in \lambda} F_i \). Therefore, \( \bigcup_{i \in \lambda} F_i \) is a \( U-BG \)-filter of \( X \).

**Proposition 3.13.** Let \( X \) and \( Y \) be \( U-BG-BH \)-algebras and \( f : (X, \ast_X, 0) \rightarrow (Y, \ast_Y, 0_Y) \) be a \( BH \)-homomorphism. Then \( ker(f) \) is a \( U-BG \)-filter of \( X \).

**Proof.** (i) Let \( x, y \in ker(f) \). Then \( f(x) = 0_Y, f(y) = 0_Y, \) so \( f(y \ast_X (y \ast_X x)) = f(y) \ast_Y (f(y) \ast_Y f(x)) = 0_Y. \) Thus \( y \ast_X (y \ast_X x) \in ker(f) \) Similarly, \( x \ast_X (x \ast_X y) \in ker(f) \).

(ii) Let \( x \in ker(f) \) and \( y \in U \) such that \( x \ast_X y = 0_X \). Then \( f(x) = 0_Y \). Now, \( f(x \ast_X y) = f(x) \ast_Y f(y) = f(0_X) = 0_Y \). [By Proposition(2.10)]. So, \( 0_Y \ast_Y f(y) = f(y) \ast_Y f(y) \), [by Definition(2.1)(I)], we obtain \( f(y) = 0_Y \), [by Lemma(2.2)(i)]. Therefore, \( y \in ker(f) \). [By Remark(2.10)]. Then \( ker(f) \) is a \( U-BG \)-filter of \( X \).

**Theorem 3.14.** Let \( f : (X, \ast_X, 0_X) \rightarrow (Y, \ast_Y, 0_Y) \) be a \( U-BG-BH \)-monomorphism, and let \( F \) be a \( U-BG \)-filter of \( X \), such that \( f(U) \) is a \( BG \)-algebra of \( X \). Then \( f(F) \) is a \( f(U) - BG \)-filter of \( Y \).
Proof. Let $F$ be a U-BG-filter of $X$.

(i) Let $x, y \in f(F)$. Then there exist $a, b \in F$ such that $x = f(a), y = f(b)$. Then $y \ast_Y (y \ast_Y x) = f(b) \ast_Y (f(b) \ast_Y f(a)) = f(b) \ast_Y (f(b \ast_X (b \ast_X a))) = f(b \ast_X (b \ast_X a)) \in f(F)$. Since $b \ast_X (b \ast_X a) \in F$, by Definition 3.1. Hence $y \ast_Y (y \ast_Y x) \in f(F)$.

(ii) Let $x \in f(F)$ such that $x \ast_Y y = 0_Y, y \in f(U)$. Then there exist $a \in F$ and $b \in U$ such that $x = f(a)$ and $y = f(b)$. Now, $x \ast_Y y = f(a) \ast_Y f(b) = f(a \ast_X b) = 0_Y = f(0_X)$. Then $a \ast_X b = 0_X$, [since $f$ is an injective]. Thus, $b \in F$, [by definition 3.1]. So, $y = f(b) \in f(F)$. Therefore, $f(F)$ is a U-BG-filter of $X$.

\[\square\]

**Theorem 3.15.** Let $f : (X, \ast_X, 0_X) \to (Y, \ast_Y, 0_Y)$ be a U-BG-BH-epimorphism, such that $f^{-1}(U)$ is a BG-algebra of $X$. If $F$ is a U-BG-filter of $Y$. Then $f^{-1}(F)$ is $f^{-1}(U) - BG$-filter of $X$.

Proof. Let $F$ be a U-BG-filter of $Y$.

(i) Let $x, y \in f^{-1}(F)$. Then $f(x), f(y) \in F$.

So $f(y) \ast_Y (f(y) \ast_Y f(x)) \in F$, [since $F$ is a U-BG-filter of $Y$]. Thus, $f(y) \ast_Y (f(y) \ast_Y f(x)) = f(y \ast_X (y \ast_X x)) \in F$, [since $F$ is a U-BG-filter of $Y$]. Therefore, $y \ast_X (y \ast_X x) \in f^{-1}(F)$. Similarly, $x \ast_X (x \ast_X y) \in f^{-1}(F)$.

(ii) Let $x \in f^{-1}(F)$ such that $x \ast_X y = 0_X, y \in f^{-1}(U)$. Then $f(x) \in F$ and $f(x \ast_X y) = f(x) \ast_Y f(y) = f(0_X) = 0_Y, f(y) \in U$, Hence $f(y) \in F$. Thus $y \in f^{-1}(F)$. Therefore, $f^{-1}(F)$ is a U-BG-filter of $X$.

\[\square\]

**Theorem 3.16.** Let $X$ be a $U-BG-BH-$ algebra, $N$ be a normal subalgebra of $X$ and $U/N$ is a $BG-$ algebra, such that $(X/N, \ast', [0])$ is a $U/N-BG-BH-$ algebra. If $F$ is a U-BG-filter of $X$, then $F/N$ is a $U/N-BG-$filter of $X/N$.

Proof. Let $X$ be a U-BH-algebra, and let $F$ be a U-BG-filter of $X$. To prove $F/N$ is a $U/N-BG-$filter of $X/N$.

(i) Let $[x]_N, [y]_N \in F/N$, then $[y]_N \ast' ([y]_N \ast' [x]_N) = [y]_N \ast' [y \ast x]_N = [y \ast (y \ast x)]_N$, Hence $[y]_N \ast' ([y]_N \ast' [x]_N) \in F/N$ [Since $y \ast (y \ast x) \in F$, $F$ is a U-BG-filter of $X$]. Similarly, $[x]_N \ast' ([x]_N \ast' [y]_N) \in F/N$.

(ii) Let $[x]_N \in F/N$ and $[y]_N \in U$, $[x]_N \ast' [y]_N = [0]_N$.

Since $[x]_N \ast' [y]_N = [0]_N$, then $[x \ast y]_N = [0]_N$, Hence $(x \ast y) \ast 0 \in N$. [By definition 2.11]. So $x \ast y \in N$, then $y \in [x]_N$. We obtain $[y]_N = [x]_N$, then $[y]_N \in F/N$. Therefore, $F/N$ is a $U/N-BG-$filter of $X/N$.

\[\square\]

**Theorem 3.17.** Let $\{X_i, \ast, (0_i) : i \in \lambda\}$ be a family of $U_BG-BH-$algebras. Then $\left(\prod_{i \in \lambda} X_i, \ast, (0_i)\right)$ is a $\prod_{i \in \lambda} U_i - BG - BH-$algebra.
Proof. 1. To prove $(\prod_{i \in \lambda} X_i, \otimes, 0_{X_i})$ is a BH-algebra.

(i) Let $(x_i) \in \prod_{i \in \lambda} X_i, \forall i \in \lambda$, and $x_i \in X_i$. Then $(x_i) \otimes (x_i) = (x_i \otimes x_i) = (0_{X_i})$, [Since $x_i \cdot x_i = 0_{X_i}, \forall i \in \lambda$ and $x_i \in X_i$].

(ii) Let $(x_i), (y_i) \in \prod_{i \in \lambda} X_i, \forall i \in \lambda$, and $x_i, y_i \in X_i$

such that $(x_i) \otimes (y_i) = (0_{X_i})$, and $(y_i) \otimes (x_i) = (0_{X_i})$, then $(x_i \cdot x_i, y_i) = (0_{X_i})$, and $(y_i \cdot x_i, x_i) = (0_{X_i})$. Then $x_i \cdot x_i, y_i = 0_{X_i}$, and $y_i \cdot x_i, x_i = 0_{X_i}$. So, $x_i = y_i, \forall i \in \lambda$, $x_i \in X_i$. Therefore, $(x_i) = (y_i)$.

(iii) Let $(x_i) \in \prod_{i \in \lambda} X_i, \forall i \in \lambda$, and $x_i \in X_i$. So, $(x_i) \otimes (0_i) = (x_i \cdot 0_i) = (x_i), [Since x_i \cdot 0_i = x_i, \forall i \in \lambda$ and $x_i \in X_i$, by definition(2.3)(III)].

Therefore, $(\prod_{i \in \lambda} X_i, \otimes, (0_i))$ is a BH-algebra.

2. $|\prod_{i \in \lambda} U_i| \geq 2$, [Since $|U_i| \geq 2$].

3. To prove $\prod_{i \in \lambda} U_i$ is a BG-algebra. Let $(x_i) \in \prod_{i \in \lambda} U_i, \forall i \in \lambda$ and $x_i \in U_i$.

It is clear that (i) $(x_i) \otimes (x_i) = (0_i)$ and (ii) $(x_i) \otimes (0_i) = (x_i), \forall i \in \lambda, x_i \in X_i$. Now, (iii) Let $(x_i), (y_i) \in \prod_{i \in \lambda} X_i, \forall i \in \lambda$, $x_i, y_i \in X_i$. So

$((x_i) \otimes (y_i)) \otimes ((0_i) \otimes (y_i)) = (x_i \cdot X_i, y_i) \otimes (0_i \cdot y_i, y_i) = ((x_i \cdot X_i, y_i) \cdot (0_i \cdot y_i, y_i)) = (x_i)$, [since $U_i$ is a BG-algebra]. So $\prod_{i \in \lambda} U_i$ is a BG-algebra.

Therefore, $(\prod_{i \in \lambda} X_i, \otimes, (0_i))$ is a $\prod_{i \in \lambda} U_i - BG - BH$–algebra.

\[\square\]

Theorem 3.18. Let $(\prod_{i \in \lambda} X_i, \otimes, (0_{X_i}))$ is a $\prod_{i \in \lambda} U_i - BG - BH$–algebra. If $\{F_i : i \in \lambda\}$ be a family of $U_i - BG$–filters of $X_i$. Then $\prod_{i \in \lambda} F_i$ is a $\prod_{i \in \lambda} U_i - BG$ – filter of the product algebra $\prod_{i \in \lambda} X_i$.

Proof. (i) Let $x = (x_i), y = (y_i) \in \prod_{i \in \lambda} F_i, \forall x_i, y_i \in F_i$, and $i \in \lambda$,

$y \otimes (y \otimes x) = (y_i) \otimes ((y_i) \otimes (x_i)) = (y_i \cdot X_i, (y_i \cdot x_i, x_i)) \in \prod_{i \in \lambda} F_i$, [Since $y_i \cdot X_i, (y_i \cdot x_i, x_i) \in F_i$, by Definition(3.1)(F1)],

(ii) Let $(x_i) \in \prod_{i \in \lambda} F_i$, and $(y_i) \in \prod_{i \in \lambda} U_i$ such that $(x_i) \otimes (y_i) = (0_{X_i}), \forall i \in \lambda, x_i, y_i \in X_i$,

Then $x_i \cdot X_i, y_i = (0_{X_i}), y_i \in U_i, \forall i \in \lambda$. 

\[\square\]
So \( x_i \in F_i, x_i \ast y_i = 0_i, y_i \in U_i, \forall i \in \lambda \), Hence \( y_i \in F_i \), [Since \( F_i \) is a \( U_i-BG-filter \) of \( X_i \)], then \( (y_i) \in \prod_{i \in \lambda} F_i \). Therefore, \( \prod_{i \in \lambda} F_i \) is a \( \prod_{i \in \lambda} U_i-BG-filter \) of \( \prod_{i \in \lambda} X_i \).

References


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