On Two New Classes of Fibonacci and Lucas Reciprocal Sums with Subscripts in Arithmetic Progression

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Abstract

In this paper expressions are derived for two new classes of Fibonacci and Lucas number reciprocal sums with subscripts in arithmetic progression. We give expressions for both finite and infinite sums. The series that are studied are similar to that of Backstrom [3], Popov [10] and Melham [9], and the results complete the discoveries from [5].

Mathematics Subject Classification: 11B39, 11Y60

Keywords: Fibonacci number, Lucas number, Reciprocal sum

1 Introduction

Let $F_n$ and $L_n$ ($n \geq 0$) denote the Fibonacci and Lucas numbers, respectively. In his paper from 1981, Backstrom [3] develops formulas for series of the form

$$\sum_{i=0}^{\infty} \frac{1}{F_{ai+b} + c}$$

and

$$\sum_{i=0}^{\infty} \frac{1}{L_{ai+b} + c}$$

for certain integer values of $a, b$ and $c$. His initial study was carried on by several authors and results of similar kind were established among others by

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Popov ([10], [11]), André-Jeannin ([1]), and Zhao ([12]). Reciprocal Fibonacci-Lucas series of a more general nature are studied in [2], [6], [7] and [8]. Two other special forms of these series have been considered recently in papers by Melham [9] and Frontczak [5]. Melham derives results for

\[ \sum_{i=0}^{\infty} \frac{L_{2ai+b}}{(F_{2ai+b} + c)^2} \quad \text{and} \quad \sum_{i=0}^{\infty} \frac{F_{2ai+b}}{(L_{2ai+b} + c)^2}, \]

for certain values of \(a, b\) and \(c\), whereas Frontczak examines series of the form

\[ \sum_{i=0}^{\infty} \frac{F_{2pi+m}}{(F_{2pi+q} + n)(F_{2p(i+1)+q} + n)}, \quad \sum_{i=0}^{\infty} \frac{L_{2pi+m}}{(F_{2pi+q} + n)(F_{2p(i+1)+q} + n)}, \]

for integer values of \(p, m\) and \(q\) and reals \(n\). The aim of the current paper is to evaluate in closed-form two new classes of reciprocal Fibonacci-Lucas sums. The first class consists of the following types of finite and infinite sums:

\[ \sum_{i=1}^{\infty} \frac{F_{pi+q}}{(F_{p(i+1)+q} + n)(F_{p(i-1)+q} + n)}, \quad \sum_{i=1}^{\infty} \frac{L_{pi+q}}{(F_{p(i+1)+q} + n)(F_{p(i-1)+q} + n)}, \]

for integer values of \(p\) and \(q\) and non-negative reals \(n\). We show that these sums are special cases of even more general series which we solve in closed-form here. In the second part of the paper, we also solve the problem for another class of reciprocal series, which is closely related to the above one.

2 Main Result

The following Lemma will be used later.

**Lemma 2.1** Let \(u\) and \(v\) be integers such that \(u + v\) and \(u - v\) have the same parity. Then

\[ F_{u+v} - F_{u-v} = \begin{cases} L_u F_v & \text{if } v \text{ is even} \\ F_u L_v & \text{if } v \text{ is odd} \end{cases} \]  \hspace{1cm} (1)

and

\[ L_{u+v} - L_{u-v} = \begin{cases} 5F_u F_v & \text{if } v \text{ is even} \\ L_u L_v & \text{if } v \text{ is odd}. \end{cases} \]  \hspace{1cm} (2)
PROOF: Both identities are well-known. They can be proved easily using the Binet forms (see for instance [4]).

Our main results for the first class of sums are contained in two theorems.

**Theorem 2.2** Let $p$ and $q$ be integers and $n, a \geq 0$ be real numbers. For $p$ even, define the series $S_1(N, p, q, n, a)$ as

$$S_1(N, p, q, n, a) = \sum_{i=1}^{N} \frac{L_{p(i+1)+q}((F_{p(i+1)+q} + n)(F_{p(i-1)+q} + n) - a^2)}{((F_{p(i+1)+q} + n)(F_{p(i-1)+q} + n) + a^2)^2 + a^2 F_p^2 L_p^2}. \quad (3)$$

For $p$ odd, define the series $S_2(N, p, q, n, a)$ as

$$S_2(N, p, q, n, a) = \sum_{i=1}^{N} \frac{F_{p(i+1)+q}((F_{p(i+1)+q} + n)(F_{p(i-1)+q} + n) - a^2)}{((F_{p(i+1)+q} + n)(F_{p(i-1)+q} + n) + a^2)^2 + a^2 L_p^2 F_p^2}. \quad (4)$$

Then

$$S_1(N, p, q, n, a) = \frac{1}{F_p} \left( \frac{F_q + n}{(F_q + n)^2 + n} + \frac{F_{p+q} + n}{(F_{p+q} + n)^2 + n} - \frac{F_{p(N+1)+q} + n}{(F_{p(N+1)+q} + n)^2 + n} - \frac{F_{pN+q} + n}{(F_{pN+q} + n)^2 + n} \right), \quad (5)$$

and

$$S_2(N, p, q, n, a) = \frac{1}{L_p} \left( \frac{F_q + n}{(F_q + n)^2 + n} + \frac{F_{p+q} + n}{(F_{p+q} + n)^2 + n} - \frac{F_{p(N+1)+q} + n}{(F_{p(N+1)+q} + n)^2 + n} - \frac{F_{pN+q} + n}{(F_{pN+q} + n)^2 + n} \right). \quad (6)$$

Especially for $a = 0$ we have the evaluations

$$\sum_{i=1}^{N} \frac{L_{p(i+1)+q}}{(F_{p(i+1)+q} + n)(F_{p(i-1)+q} + n)} = \frac{1}{F_p} \left( \frac{1}{F_q} + \frac{1}{F_{p+q}} - \frac{1}{F_{p(N+1)+q}} - \frac{1}{F_{pN+q}} \right), \quad (7)$$

$$\sum_{i=1}^{\infty} \frac{L_{p(i+1)+q}}{(F_{p(i+1)+q} + n)(F_{p(i-1)+q} + n)} = \frac{1}{F_p} \left( \frac{1}{F_q} + \frac{1}{F_{p+q}} \right), \quad (8)$$

$$\sum_{i=1}^{N} \frac{F_{p(i+1)+q}}{(F_{p(i+1)+q} + n)(F_{p(i-1)+q} + n)} = \frac{1}{L_p} \left( \frac{1}{F_q} + \frac{1}{F_{p+q}} - \frac{1}{F_{p(N+1)+q}} - \frac{1}{F_{pN+q}} \right), \quad (9)$$

and

$$\sum_{i=1}^{\infty} \frac{F_{p(i+1)+q}}{(F_{p(i+1)+q} + n)(F_{p(i-1)+q} + n)} = \frac{1}{L_p} \left( \frac{1}{F_q} + \frac{1}{F_{p+q}} \right). \quad (10)$$

For the last four equations to hold, we need the assumption on $q$ and $n$ that $(q, n) \neq (0, 0)$.
When \( p = 1 \), the special results in (9) and (10) are trivial, since the sums are telescoping sums. The same is true for (7) and (8) when \( p = 2 \). To reveal the telescoping nature, one uses the relation \( L_i = F_{i+2} - F_{i-2}, i \geq 2 \).

The second theorem has an analogous structure:

**Theorem 2.3** Let \( p \) and \( q \) be integers and \( n, a \geq 0 \) be real numbers. For \( p \) even, define the series \( S_3(N, p, q, n, a) \) as

\[
S_3(N, p, q, n, a) = \sum_{i=1}^{N} \frac{F_{p+i}((L_{p(i+1)+q} + n)(L_{p(i-1)+q} + n) - a^2)}{((L_{p(i+1)+q} + n)(L_{p(i-1)+q} + n) + a^2)^2 + 25a^2F_p^2F_{p+i}^2}.
\]  

(11)

For \( p \) odd, define the series \( S_4(N, p, q, n, a) \) as

\[
S_4(N, p, q, n, a) = \sum_{i=1}^{N} \frac{L_{p+i}((L_{p(i+1)+q} + n)(L_{p(i-1)+q} + n) - a^2)}{((L_{p(i+1)+q} + n)(L_{p(i-1)+q} + n) + a^2)^2 + 2a^2L_p^2L_{p+i}^2}.
\]  

(12)

Then

\[
S_3(N, p, q, n, a) = \frac{1}{5F_p} \left( \frac{L_q + n}{(L_q + n)^2 + a^2} + \frac{L_{p+q} + n}{(L_{p+q} + n)^2 + a^2} - \frac{L_{p(N+1)+q} + n}{(L_{p(N+1)+q} + n)^2 + a^2} - \frac{L_{pN+q} + n}{(L_{pN+q} + n)^2 + a^2} \right),
\]  

(13)

and

\[
S_4(N, p, q, n, a) = \frac{1}{L_p} \left( \frac{L_q + n}{(L_q + n)^2 + a^2} + \frac{L_{p+q} + n}{(L_{p+q} + n)^2 + a^2} - \frac{L_{p(N+1)+q} + n}{(L_{p(N+1)+q} + n)^2 + a^2} - \frac{L_{pN+q} + n}{(L_{pN+q} + n)^2 + a^2} \right).
\]  

(14)

Especially for \( a = 0 \) we have the evaluations

\[
\sum_{i=1}^{N} \frac{F_{p+i}}{(L_{p(i+1)+q} + n)(L_{p(i-1)+q} + n)} = \frac{1}{5F_p} \left( \frac{1}{L_q + n} + \frac{1}{L_{p(q)+n}} - \frac{1}{L_{p(N+1)+q} + n} - \frac{1}{L_{pN+q} + n} \right),
\]  

(15)

\[
\sum_{i=1}^{\infty} \frac{F_{p+i}}{(L_{p(i+1)+q} + n)(L_{p(i-1)+q} + n)} = \frac{1}{5F_p} \left( \frac{1}{L_q + n} + \frac{1}{L_{p(q)+n}} \right);
\]  

(16)

\[
\sum_{i=1}^{N} \frac{L_{p+i}}{(L_{p(i+1)+q} + n)(L_{p(i-1)+q} + n)} = \frac{1}{L_p} \left( \frac{1}{L_q + n} + \frac{1}{L_{p(q)+n}} - \frac{1}{L_{p(N+1)+q} + n} - \frac{1}{L_{pN+q} + n} \right),
\]  

(17)

and

\[
\sum_{i=1}^{\infty} \frac{L_{p+i}}{(L_{p(i+1)+q} + n)(L_{p(i-1)+q} + n)} = \frac{1}{L_p} \left( \frac{1}{L_q + n} + \frac{1}{L_{p(q)+n}} \right).
\]  

(18)
Again, when \( p = 1 \), then (17) and (18) reduce to trivial telescoping sums. We prove both theorems simultaneously.

PROOF: The proof is an application of the following result from [4]:

**Theorem 2.4** Let \( f(x) \) and \( g(x) \) be two real functions. For \( k \geq 1 \) an integer, consider the new function \( h(x) \) defined as

\[
h(x) = \frac{f(g(x + k)) - f(g(x - k))}{1 + f(g(x + k))f(g(x - k))}.
\]

(19)

Let \( \tan^{-1}(x) \) denote the principal branch of the inverse tangent function. If \( f(g(i + k))f(g(i - k)) > -1 \) for all \( i \), then

\[
\sum_{i=1}^{n} \tan^{-1} h(i) = \sum_{m=-k+1}^{k} \tan^{-1} f(g(n + m)) - \sum_{m=-k+1}^{k} \tan^{-1} f(g(m)) \quad (20)
\]

and

\[
\sum_{i=1}^{\infty} \tan^{-1} h(i) = 2k \tan^{-1} f(g(\infty)) - \sum_{m=-k+1}^{k} \tan^{-1} f(g(m)) \quad (21)
\]

We are going to apply this result in case \( k = 1 \). To ensure a concise presentation, we define the generalized Fibonacci numbers \( G_i \) through the relation \( G_{i+1} = G_i + G_{i-1} \) with initial terms \( G_0 \) and \( G_1 \).

For \( a \neq 0 \) define \( f(x) \) as \( f(x) = \frac{a}{x} \). Furthermore, for integers \( p \geq 1 \) and \( q \geq 0 \) and \( n \geq 0 \) a real number define \( g(i) = G_{pi+q} + n \). Then, obviously \( f(g(0)) = \frac{a}{G_q + n}, f(g(1)) = \frac{a}{G_{p+q} + n} \) and \( \lim_{i \to \infty} f(g(i)) = 0 \). It follows that

\[
\sum_{i=1}^{N} \tan^{-1} \left( \frac{ax_{1,i}}{x_{2,i} + a^2} \right) = \tan^{-1} \left( \frac{a}{G_q + n} \right) + \tan^{-1} \left( \frac{a}{G_{p+q} + n} \right)
\]

\[
- \tan^{-1} \left( \frac{a}{G_{p(N+1)+q} + n} \right) - \tan^{-1} \left( \frac{a}{G_{pN+q} + n} \right),
\]

where we have set

\[
x_{1,i} = G_{p(i+1)+q} - G_{p(i-1)+q}, \quad x_{2,i} = (G_{p(i+1)+q} + n)(G_{p(i-1)+q} + n). \quad (22)
\]

The first theorem follows by setting \( G = F \), using

\[
F_{p(i+1)+q} - F_{p(i-1)+q} = \left\{ \begin{array}{ll}
L_{pi+q}F_p & \text{if } p \text{ is even} \\
F_{pi+q}L_p & \text{if } p \text{ is odd}
\end{array} \right.
\]

(23)
and differentiating w.r.t. the parameter \( a \).

The second theorem is obtained upon setting \( G = L \), using

\[
L_{p(i+1)+q} - L_{p(i-1)+q} = \begin{cases} 
5 F_{p(i)+q} F_p & \text{if } p \text{ is even} \\
L_{p(i)+q} L_p & \text{if } p \text{ is odd}
\end{cases}
\tag{24}
\]

and differentiating w.r.t. the parameter \( a \).

It is obvious from the proof, that the identities also hold for negative values of \( n \). However, some caution is necessary to avoid singularities. We conclude this section with a short list of examples of infinite series evaluations that were discovered in this paper:

\[
\sum_{i=1}^{\infty} \frac{L_{4i}}{(F_{4i-4} + 1)(F_{4i+4} + 1)} = \frac{5}{12}, \quad \sum_{i=1}^{\infty} \frac{L_{4i+2}}{F_{4i-2} F_{4i+6}} = \frac{3}{8},
\]

\[
\sum_{i=1}^{\infty} \frac{F_{3i+1}}{F_{3i-2} F_{3i+4}} = \frac{1}{3} = \sum_{i=1}^{\infty} \frac{F_{3i}}{(F_{3i-3} + 1)(F_{3i+3} + 1)},
\]

\[
\sum_{i=1}^{\infty} \frac{F_{2i+2}}{(L_{2i} + 2)(L_{2i+4} + 2)} = \frac{14}{225}, \quad \sum_{i=1}^{\infty} \frac{L_{3i+3}}{(L_{3i-1})(L_{3i+6} - 1)} = \frac{5}{51}.
\]

3 A second class of sums

Using the same approach as in the previous section, it is possible to derive formulas for another class of reciprocal series. For \( p, q \geq 1 \) and \( a \) a real number, we define the four series \( S_5 - S_8 \) according to

\[
S_5(N, p, q, a) = \sum_{i=1}^{N} \frac{(L_{p(i)+q} + L_{p(i)})((F_{p(i)+1} + q + F_{p(i)+1})(F_{p(i)-1} + q + F_{p(i)-1}) - a^2)}{((F_{p(i)+1} + q + F_{p(i)+1})(F_{p(i)-1} + q + F_{p(i)-1}) + a^2)^2 + a^2 F_p^2(L_{p(i)+q} + L_{p(i)})^2}
\tag{25}
\]

\( p \) even,

\[
S_6(N, p, q, a) = \sum_{i=1}^{N} \frac{(F_{p(i)+q} + F_{p(i)})(F_{p(i)+1} + q + F_{p(i)+1})(F_{p(i)-1} + q + F_{p(i)-1}) - a^2)}{((F_{p(i)+1} + q + F_{p(i)+1})(F_{p(i)-1} + q + F_{p(i)-1}) + a^2)^2 + a^2 L_p^2(F_{p(i)+q} + F_{p(i)})^2}
\tag{26}
\]

\( p \) odd,

\[
S_7(N, p, q, a) = \sum_{i=1}^{N} \frac{(F_{p(i)+q} + F_{p(i)})(L_{p(i)+1} + q + L_{p(i)+1})(L_{p(i)-1} + q + L_{p(i)-1}) - a^2)}{((L_{p(i)+1} + q + L_{p(i)+1})(L_{p(i)-1} + q + L_{p(i)-1}) + a^2)^2 + 25 a^2 F_p^2(F_{p(i)+q} + F_{p(i)})^2}
\tag{27}
\]
From the previous section. We state the results in one theorem, a proof of which we leave as an exercise:

**Theorem 3.1** We have

\[
S_5(N, p, q, a) = \sum_{i=1}^{N} \frac{(L_{pi+q} + L_{pi})((L_{p(i+1)+q} + L_{p(i+1)})(L_{p(i-1)+q} + L_{p(i-1)}) - a^2)}{((L_{p(i+1)+q} + L_{p(i+1)})(L_{p(i-1)+q} + L_{p(i-1)}) + a^2)^2 + a^2L_p^2(L_{pi+q} + L_{pi})^2}
\]

(28)

For odd.

Then each of the series allows an evaluation, which is very similar to the one from the previous section. We state the results in one theorem, a proof of which we leave as an exercise:

\[
S_6(N, p, q, a) = \frac{1}{F_p} \left( \frac{F_q}{F_q^2 + a^2} + \frac{F_{p+q} + F_p}{(F_{p+q} + F_p)^2 + a^2} - \frac{F_{p(N+1)+q} + F_{p(N+1)}}{(F_{p(N+1)+q} + F_{p(N+1)})^2 + a^2} - \frac{F_{pN+q} + F_{pN}}{(F_{pN+q} + F_{pN})^2 + a^2} \right),
\]

(29)

\[
S_6(N, p, q, a) = \frac{1}{L_p} \left( \frac{F_q}{F_q^2 + a^2} + \frac{F_{p+q} + F_p}{(F_{p+q} + F_p)^2 + a^2} - \frac{F_{p(N+1)+q} + F_{p(N+1)}}{(F_{p(N+1)+q} + F_{p(N+1)})^2 + a^2} - \frac{F_{pN+q} + F_{pN}}{(F_{pN+q} + F_{pN})^2 + a^2} \right),
\]

(30)

\[
S_7(N, p, q, a) = \frac{1}{5F_p} \left( \frac{L_q + 2}{(L_q + 2)^2 + a^2} + \frac{L_{p+q} + L_p}{(L_{p+q} + L_p)^2 + a^2} - \frac{L_{p(N+1)+q} + L_{p(N+1)}}{(L_{p(N+1)+q} + L_{p(N+1)})^2 + a^2} - \frac{L_{pN+q} + L_{pN}}{(L_{pN+q} + L_{pN})^2 + a^2} \right),
\]

(31)

and

\[
S_8(N, p, q, a) = \frac{1}{L_p} \left( \frac{L_q + 2}{(L_q + 2)^2 + a^2} + \frac{L_{p+q} + L_p}{(L_{p+q} + L_p)^2 + a^2} - \frac{L_{p(N+1)+q} + L_{p(N+1)}}{(L_{p(N+1)+q} + L_{p(N+1)})^2 + a^2} - \frac{L_{pN+q} + L_{pN}}{(L_{pN+q} + L_{pN})^2 + a^2} \right).
\]

(32)

Upon setting \( a = 0 \) and letting \( N \to \infty \) we get

\[
\sum_{i=1}^{\infty} \frac{L_{pi+q} + L_{pi}}{(F_{p(i+1)+q} + F_{p(i+1)})(F_{p(i-1)+q} + F_{p(i-1)})} = \frac{1}{F_p} \left( \frac{1}{F_q} + \frac{1}{F_{p+q} + F_p} \right),
\]

(33)

\[
\sum_{i=1}^{\infty} \frac{F_{pi+q} + F_{pi}}{(F_{p(i+1)+q} + F_{p(i+1)})(F_{p(i-1)+q} + F_{p(i-1)})} = \frac{1}{L_p} \left( \frac{1}{F_q} + \frac{1}{F_{p+q} + F_p} \right),
\]

(34)
\[ \sum_{i=1}^{\infty} \frac{F_{pi+q} + F_{pi}}{(L_{p(i+1)} + L_{p(i+1)}) (L_{p(i-1)+q} + L_{p(i-1)})} = \frac{1}{5F_p} \left( \frac{1}{L_q+2} + \frac{1}{L_{p+q} + L_p} \right), \]  

(35)

and

\[ \sum_{i=1}^{\infty} \frac{L_{pi+q} + L_{pi}}{(L_{p(i+1)} + L_{p(i+1)}) (L_{p(i-1)+q} + L_{p(i-1)})} = \frac{1}{L_p} \left( \frac{1}{L_q+2} + \frac{1}{L_{p+q} + L_p} \right). \]  

(36)

The interesting reader is invited to explore the relationships between the sums \( S_1 - S_4 \) and \( S_5 - S_8 \).

References


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Received: March 19, 2017; Published: April 28, 2017