On the Quasi-inverse of a Non-square Matrix: An Infinite Solution

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Abstract

A non-square matrix $A$ is quasi-invertible if there exists $A^*$ different from the pseudo-inverse matrix such that either $AA^*=I$ or $A^*A = I$ such that $A^*$ provides a solution to the linear system $AX = B$. We describe $A^*$ as the quasi-inverse of $A$. This paper presents the necessary and sufficient conditions for the infinitude in the existence of quasi-inverses of a non-square matrix. It also exhibits the properties of a quasi-inverse matrix. Moreover, this provides an alternative way of solving linear system with many solutions by employing the quasi-inverse of a matrix.
1 Introduction

Consider solving a linear system \( AX = B \) when the number of unknowns exceeds the number of equations or the number of equations exceeds the number of unknowns. For instance, if the number of unknowns is one more than the number of equations, then the system has infinitely many solutions if they exist. On the other hand, if the number of equations is one more than the number of unknowns then the system has a unique solution if it exists. In dealing with these problems we usually employ the traditional elimination method. Now, the solution of the system can be expressed as \( X = A' B \) where \( A' \) is the inverse of \( A \). But note that the coefficient matrix of the linear system given above is non-square, so maybe by a mere coincident or by an accident, we are actually computing the inverse of a non-square matrix. Since the solution of the linear system mentioned above need not be unique, the matrix that satisfies the condition need not also to be unique. This enlightens the way to revisit the discussion of finding the inverse of a non-square matrix.

Not so long ago, the study of finding the inverse of a non-square matrix has become an interest of mathematicians such as Moore\[3\] in 1920 and Stoer\[6\] in 2002. This gives rise to the concept of pseudo-inverse matrix which was first explored by E.H. Moore\[3\]. A pseudo-inverse \( A^+ \) of matrix \( A \) is a generalization of the inverse matrix, widely known as the Moore-Penrose pseudo-inverse, which was independently described by E. H. Moore, Arne Bierhammar\[1\] in 1951 and Roger Penrose\[4\] in 1955. Since the matrix we consider is not necessarily square, the pseudo-inverse matrix is one-sided. That is, \( A^+ \) is a right inverse of an \( m \times n \) matrix \( A \) if \( m < n \) and left inverse otherwise. When referring to a matrix, the term pseudo-inverse, without further specification, is often used to indicate the Moore-Penrose pseudo-inverse.

Even though the pseudo-inverse is widely accepted as an inverse of a non-square matrix, it does not provide a solution to the linear system \( AX = B \) when matrix \( A \) is of order \( m \times n \) when \( m < n \) because if \( A^+ \) is the right inverse of \( A \) then we have \( AA^+X = BA^+ \) which yields \( IX = BA^+ \) implies that \( X = BA^+ \) but \( BA^+ \) is not defined. Due to this shortcoming, the researchers introduced a different one-sided inverse of a non-square matrix called the “quasi-inverse” matrix in order to develop a new method of solving a linear system whose coefficient matrix is non-square. In this case, the non-square matrix whose quasi-inverse exists is described as “quasi-invertible” matrix. For the basic terminologies not stated here, the reader is advised to refer to the book of Kolman and Hill\[2\].
2 Basic Concepts

Definition 2.1 A linear system is a collection of \( m \) linear equations with \( n \) unknowns \( x_1, x_2, \ldots, x_n \),

\[
\begin{align*}
  a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
  a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
  &\vdots \\
  a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m 
\end{align*}
\]

Definition 2.2 An \( m \times n \) matrix \( A = [a_{ij}] \) of order \( mn \) is a rectangular array of numbers having \( m \) rows and \( n \) columns

\[
A = \begin{bmatrix}
  a_{11} & a_{12} & \ldots & a_{1n} \\
  a_{21} & a_{22} & \ldots & a_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{m1} & a_{m2} & \ldots & a_{mn}
\end{bmatrix}.
\]

The row vectors of \( A \) are \((a_{11} \ a_{12} \ \ldots \ a_{1n}), (a_{21} \ a_{22} \ \ldots \ a_{2n}), \ldots, (a_{m1} \ a_{m2} \ \ldots \ a_{mn})\)

while the column vectors are \( \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \ldots, \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} \). If \( m = n \), then we say that \( A \) is a square matrix of order \( n \). If \( m \neq n \), we say that \( A \) is a non-square matrix. Moreover, if the row vectors of \( A \) are independent then we say that \( A \) has independent rows. The independent columns of \( A \) are defined analogously.

Definition 2.3 If \( A \) is an \( m \times n \) matrix, then the matrix \( B = [a_{pk}] \) obtained by deleting a row/s or column/s or both is called a submatrix of \( A \).

Definition 2.4 Let \( A \) be an \( m \times n \) matrix where \( n \geq m + 2 \). The \( m \times (m + 1) \) matrix

\[
B = \begin{bmatrix}
  a_{11} & a_{12} & \ldots & a_{1m} & a_{1p} \\
  a_{21} & a_{22} & \ldots & a_{2m} & a_{2p} \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  a_{m1} & a_{m2} & \ldots & a_{mm} & a_{mp}
\end{bmatrix}
\]

where the \((m + 1)^{th}\) column entries \( a_{ip} = \sum_{t=m+1}^{n} a_{it} \) is called the diminished matrix of \( A \).
Definition 2.5 If \( A = [a_{ij}] \) is an \( m \times n \) matrix, then the \( n \times m \) matrix 
\[ A^T = [a_{ij}^T] \]
where \( a_{ij}^T = a_{ji} \) is called the **transpose** of \( A \).

Definition 2.6 Let \( S = \{1, 2, \ldots, n\} \) arranged in ascending order. A rearrangement \( j_1j_2\ldots j_n \) of the elements of \( S \) is called a **permutation** of \( S \) and is denoted by \( S_n \). The number of permutations of \( S \) is given by \( n! \). A permutation \( j_1j_2\ldots j_n \) is said to have an **inversion** if larger integer \( j_r \) precedes a smaller one \( j_s \). A permutation is called **even** or **odd** according to whether the total number of inversions is even or odd. It should be noted that there are \( n!/2 \) even as well as odd permutations.

Definition 2.7 Let \( A \) be an \( n \times n \) matrix. The **determinant** of \( A \), written \(|A|\) is given by
\[
|A| = \sum (\pm) a_{1j_1}a_{2j_2}\ldots a_{nj_n}
\]
where the summation ranges over all permutations \( j_1j_2\ldots j_n \) of the set \( S = \{1, 2, \ldots, n\} \). The sign + or − depends on whether the permutation \( j_1j_2\ldots j_n \) is even or odd.

Theorem 2.8 Let \( A \) and \( B \) be \( m \times n \) matrices. The following hold:
1. \(|A| = |A^T|\)
2. \(|AB| = |A||B|\).

Definition 2.9 Let \( A \) be an \( n \times n \) matrix. Let \( M_{ij} \) be the \((n-1) \times (n-1)\) submatrix of \( A \) obtained by deleting the \( i \)th row and \( j \)th column of \( A \). The determinant \(|M_{ij}|\) is called the **minor** of \( a_{ij} \). The cofactor \( \langle A_{ij} \rangle \) of \( a_{ij} \) is defined as 
\[
\langle A_{ij} \rangle = (-1)^{i+j}|M_{ij}|
\]

Definition 2.10 Let \( A \) be a square matrix of order \( n \). The **adjoint** of \( A \), denoted by \( \text{adj}A \) is an \( n \times n \) matrix whose \( i,j \)th element is the cofactor \( \langle A_{ji} \rangle \) of \( a_{ji} \). That is,
\[
\text{adj}A = \begin{bmatrix} \langle A_{11} \rangle & \langle A_{21} \rangle & \ldots & \langle A_{n1} \rangle \\ \langle A_{12} \rangle & \langle A_{22} \rangle & \ldots & \langle A_{n2} \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle A_{1n} \rangle & \langle A_{2n} \rangle & \ldots & \langle A_{nn} \rangle \end{bmatrix}
\]

Theorem 2.11 If \( A \) is an \( n \times n \) matrix and \(|A| \neq 0\), then \( A \) is invertible and
\[
A^{-1} = \frac{1}{|A|}(\text{adj}A) = \begin{bmatrix} \langle A_{11} \rangle & \langle A_{21} \rangle & \ldots & \langle A_{n1} \rangle \\ \langle A_{12} \rangle & \langle A_{22} \rangle & \ldots & \langle A_{n2} \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle A_{1n} \rangle & \langle A_{2n} \rangle & \ldots & \langle A_{nn} \rangle \end{bmatrix}
\]
**Theorem 2.12** Let $A$ be an $n \times n$ matrix. Then $A$ is nonsingular if and only if the linear system $AX = B$ has a unique solution for every $n \times 1$ matrix $B$. The solution is given by $X = A^{-1}B$.

**Definition 2.13** [5] Let $A$ be an $m \times n$ matrix. If $A^T A$ is invertible then, the left pseudo-inverse $A^+_L$ of $A$ is defined as

$$A^+_L = (A^T A)^{-1} A^T$$

and $A^+_L A = I$ or $E$.

If $AA^T$ is invertible then, the right pseudo-inverse $A^+_R$ is defined as

$$A^+_R = A^T (AA^T)^{-1}$$

and $AA^+_R = I$ or $E$.

**Remark 2.14** We use the notation $A^+$ to mean $A^+_R$ or $A^+_L$.

**Theorem 2.15** [6] The following are the properties of a pseudo-inverse matrix $A^+$ of matrix $A$.

ii. If $A$ is invertible, then $A^+ = A^{-1}$.

iii. The pseudo-inverse of a zero matrix is its transpose.

iv. $(A^+)^T = A$.

v. $(\alpha A)^+ = \alpha^{-1} A^+$, $\alpha \in \mathbb{R}, \alpha \neq 0$.

**Definition 2.16** Let $A$ be an $m \times n$ matrix. The one-sided inverse $A^*$, not necessarily equal to $A^+$, such that $AA^* = I$ or $A^* A = I$ where $A^*$ can provide non-unique solutions to the linear system $AX = B$ is called a quasi-inverse of $A$. The solution is given by $X = A^* B$. We describe matrix $A$ as quasi-invertible.

### 3 Quasi-inverse of a Non-square Matrix

**Lemma 3.1** Let $A'$ be an invertible matrix and let $A$ be a matrix obtained from $A'$ by adding a rightmost column or a bottom row. Then $A$ is a quasi-invertible and there exists matrix $B$ with a row of $r$ such that either $AB = I$ or $BA = I$.

*Proof:* Let $A' = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$ be an invertible matrix and let

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & a_{1(n+1)} \\ a_{21} & a_{22} & \cdots & a_{2n} & a_{2(n+1)} \\ \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & a_{n(n+1)} \end{bmatrix} \quad \text{or} \quad A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \\ a_{(n+1)1} & a_{(n+1)2} & \cdots & a_{(n+1)n} \end{bmatrix}.$$
Then $A$ is an $n \times m$ or an $m \times n$ matrix where $m = n + 1$. Since $A'$ is invertible, $A'^{-1}$ exists and by Theorem 2.11,

$$A'^{-1} = \frac{1}{|A'|} \text{adj} A'$$

$$= \begin{bmatrix}
\frac{\langle A'_{11} \rangle}{|A'|} & \frac{\langle A'_{21} \rangle}{|A'|} & \ldots & \frac{\langle A'_{n1} \rangle}{|A'|} \\
\frac{\langle A'_{12} \rangle}{|A'|} & \frac{\langle A'_{22} \rangle}{|A'|} & \ldots & \frac{\langle A'_{n2} \rangle}{|A'|} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\langle A'_{1n} \rangle}{|A'|} & \frac{\langle A'_{2n} \rangle}{|A'|} & \ldots & \frac{\langle A'_{nn} \rangle}{|A'|}
\end{bmatrix}.$$ 

Consider an $m \times n$ or $n \times m$ matrix $B = \begin{bmatrix} A'^{-1} \\ 0 \end{bmatrix}$ or $B = \begin{bmatrix} A'^{-1} & 0 \end{bmatrix}$. If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1m} \\
a_{21} & a_{22} & a_{23} & \ldots & a_{2m} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & a_{n3} & \ldots & a_{nm} \\
a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}$ and $B = \begin{bmatrix} A'^{-1} \\ 0 \end{bmatrix}$, then $AB = I$. If $A =$ and $B = \begin{bmatrix} A'^{-1} & 0 \end{bmatrix}$, then $BA = I$. Thus, $B$ satisfies the requirement as a quasi-inverse of $A$. Since $B$ consists of a row or column of zeros, we can replace this row or column by any real number $r$ and the effect is that the entries $\frac{\langle A'_{ij} \rangle}{|A'|}$ of $A'^{-1}$ will be added or subtracted by $d_{jr}r$. That is, we can have

$$B = \begin{bmatrix}
\frac{\langle A'_{11} \rangle + d_{11}r}{|A'|} & \frac{\langle A'_{21} \rangle + d_{21}r}{|A'|} & \ldots & \frac{\langle A'_{n1} \rangle + d_{n1}r}{|A'|} \\
\frac{\langle A'_{12} \rangle + d_{12}r}{|A'|} & \frac{\langle A'_{22} \rangle + d_{22}r}{|A'|} & \ldots & \frac{\langle A'_{n2} \rangle + d_{n2}r}{|A'|} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\langle A'_{1n} \rangle + d_{1n}r}{|A'|} & \frac{\langle A'_{2n} \rangle + d_{2n}r}{|A'|} & \ldots & \frac{\langle A'_{nn} \rangle + d_{nn}r}{|A'|}
\end{bmatrix}$$

or

$$B = \begin{bmatrix}
\frac{\langle A'_{11} \rangle + d_{11}r}{|A'|} & \frac{\langle A'_{21} \rangle + d_{21}r}{|A'|} & \ldots & \frac{\langle A'_{n1} \rangle + d_{n1}r}{|A'|} \\
\frac{\langle A'_{12} \rangle + d_{12}r}{|A'|} & \frac{\langle A'_{22} \rangle + d_{22}r}{|A'|} & \ldots & \frac{\langle A'_{n2} \rangle + d_{n2}r}{|A'|} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\langle A'_{1n} \rangle + d_{1n}r}{|A'|} & \frac{\langle A'_{2n} \rangle + d_{2n}r}{|A'|} & \ldots & \frac{\langle A'_{nn} \rangle + d_{nn}r}{|A'|}
\end{bmatrix}$$

and we can obtain $AB = I$ or $BA = I$. Therefore, $A$ is quasi-invertible. \qed
Theorem 3.2 Let $A$ be an $m \times n$ quasi-invertible matrix where $m = n - 1$ and let $A_j$ be a square submatrix of $A$ whose order is $m$ and is obtained by deleting the $j^{th}$ column of $A$. Then the quasi-inverse of $A$ is an $n \times m$ matrix $A^* = [b_{ji}]$ such that

$$ b_{ji} = \begin{cases} 
\frac{\langle A_{ij} \rangle + |A_i|}{|A_j|}, & \text{if } j \text{ is odd} \\
\frac{\langle A_{ij} \rangle - |A_i|}{|A_j|}, & \text{if } j \text{ is even}
\end{cases} $$

when the number of rows is even and

$$ b_{ji} = \begin{cases} 
\frac{\langle A_{ij} \rangle - |A_i|}{|A_j|}, & \text{if } j \text{ is odd} \\
\frac{\langle A_{ij} \rangle + |A_i|}{|A_j|}, & \text{if } j \text{ is even}
\end{cases} $$

when the number of rows is odd

where in both cases, $1 \leq j \leq m$ and $b_{ni} = r$ for $i = 1, 2, \ldots, m$, $\langle A_{ij} \rangle$ is the cofactor of $a_{ij}$ of the invertible submatrix $A_n$ obtained from $A$ by deleting the $n^{th}$ column. That is,

$$ A^* = \begin{bmatrix}
\langle A_{11} \rangle \pm |A_1| & \langle A_{12} \rangle \pm |A_2| & \cdots & \langle A_{1m} \rangle \pm |A_m| \\
\langle A_{21} \rangle \pm |A_1| & \langle A_{22} \rangle \pm |A_2| & \cdots & \langle A_{2m} \rangle \pm |A_m| \\
\vdots & \vdots & \ddots & \vdots \\
\langle A_{m1} \rangle \pm |A_1| & \langle A_{m2} \rangle \pm |A_2| & \cdots & \langle A_{mm} \rangle \pm |A_m| \\
r & r & \cdots & r
\end{bmatrix}. $$

Proof: Let $A$ be an $m \times n$. Since $A$ is quasi-invertible, by Lemma 3.1, there exists a matrix $B$ with a row of $r$ such that $AB = I$. Let $A_j$ be a square submatrix of $A$ of order $m$ obtained by deleting the $j^{th}$ column of $A$. Then $A_n$ is a submatrix of $A$ obtained by deleting the $n^{th}$ column which is invertible, and it follows that $|A_n| \neq 0$. So, let $A^* = [b_{ji}]$ be an $n \times m$ matrix where $b_{ni} = r$ for all $i = 1, 2, \ldots, m$ such that $AA^* = I$. Then we have

$$ AA^* = \begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{bmatrix} \begin{bmatrix}
b_{11} & b_{12} & \cdots & b_{1m} \\
b_{21} & b_{22} & \cdots & b_{2m} \\
\vdots & \vdots & \ddots & \vdots \\
b_{m1} & b_{m2} & \cdots & b_{mm}
\end{bmatrix} = \begin{bmatrix}1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{bmatrix}. $$

Simplifying, we obtain the linear system

$$ a_{11}b_{11} + a_{12}b_{21} + \cdots + a_{1m}b_{m1} + a_{1n}r = 1 $$

$$ a_{11}b_{12} + a_{12}b_{22} + \cdots + a_{1m}b_{m2} + a_{1n}r = 0 $$

$$ \vdots $$

$$ a_{11}b_{1m} + a_{12}b_{2m} + \cdots + a_{1m}b_{mm} + a_{1n}r = 0; $$

$$ a_{21}b_{11} + a_{22}b_{21} + \cdots + a_{2m}b_{m1} + a_{2n}r = 0 $$

$$ \vdots $$

$$ a_{nm}b_{11} + a_{nm}b_{21} + \cdots + a_{nm}b_{m1} + a_{nm}r = 0. $$

$$ a_{11}b_{11} + a_{12}b_{21} + \cdots + a_{1m}b_{m1} + a_{1n}r = 1 $$

$$ a_{11}b_{12} + a_{12}b_{22} + \cdots + a_{1m}b_{m2} + a_{1n}r = 0 $$

$$ \vdots $$

$$ a_{11}b_{1m} + a_{12}b_{2m} + \cdots + a_{1m}b_{mm} + a_{1n}r = 0; $$

$$ a_{21}b_{11} + a_{22}b_{21} + \cdots + a_{2m}b_{m1} + a_{2n}r = 0 $$

$$ \vdots $$

$$ a_{nm}b_{11} + a_{nm}b_{21} + \cdots + a_{nm}b_{m1} + a_{nm}r = 0. $$
\[ a_{21}b_{12} + a_{22}b_{22} + \cdots + a_{2m}b_{mm} + a_{2n}r = 1 \]
\[
\vdots
\]
\[ a_{1m}b_{11} + a_{2m}b_{21} + \cdots + a_{mm}b_{m1} + a_{mn}r = 0 \]
\[
\vdots
\]
\[ a_{m1}b_{1m} + a_{m2}b_{2m} + \cdots + a_{mm}b_{mm} + a_{mn}r = 1. \]

We prove here the case when the number of rows is even and the case when the number of rows is odd can be proved similarly. Solving for \( b_{ji} \) by elimination, we obtain

\[ b_{11} = \frac{\sum(\pm)a_{1k_1}a_{2k_2}a_{3k_3} \cdots a_{mk_m} + (\sum(\pm)a_{1k_1}a_{2k_2}a_{3k_3} \cdots a_{nk_n})^r}{\sum(\pm)a_{1k_1}a_{2k_2}a_{3k_3} \cdots a_{mk_m}} \]

(where the summation ranges over all permutations of \( k_2, k_3, \ldots, k_m \) of the set \( \{2, 3, \ldots, m\} \), \( k_2, k_3, \ldots, k_n \) of the set \( \{2, 3, \ldots, n\} \) and \( k_1, k_2, \ldots, k_m \) of the set \( \{1, 2, \ldots, m\} \) definition 2.6)

\[ = \frac{(-1)^2|M_{11}| - |A_1|^r}{|A_n|} = \frac{(A_{11}) + |A_1|^r}{|A_n|}, \]  

(definition 2.7 and 2.9)

\[ b_{12} = \frac{\sum(\pm)a_{1k_1}a_{2k_2}a_{3k_3} \cdots a_{mk_m} + (\sum(\pm)a_{1k_1}a_{2k_2}a_{3k_3} \cdots a_{nk_n})^r}{\sum(\pm)a_{1k_1}a_{2k_2}a_{3k_3} \cdots a_{mk_m}} \]

(where the summation ranges over all permutations of \( k_2, k_3, \ldots, k_m \) of the set \( \{2, 3, \ldots, m\} \))

\[ = \frac{(-1)^3|M_{21}| - |A_1|^r}{|A_n|} = \frac{(A_{21}) + |A_1|^r}{|A_n|}, \]

\[ \vdots \]

\[ b_{1m} = \frac{\sum(\pm)a_{1k_1}a_{2k_2}a_{3k_3} \cdots a_{mk_m} + (\sum(\pm)a_{1k_1}a_{2k_2}a_{3k_3} \cdots a_{mk_m})^r}{\sum(\pm)a_{1k_1}a_{2k_2}a_{3k_3} \cdots a_{mk_m}} \]

(where the summation ranges over all permutations of \( k_2, k_3, \ldots, k_m \) of the set \( \{2, 3, \ldots, m\} \))

\[ = \frac{(-1)^{m+1}|M_{11}| - |A_1|^r}{|A_n|} = \frac{(A_{11}) + |A_1|^r}{|A_n|}, \]

\[ b_{21} = \frac{\sum(\pm)a_{1k_1}a_{2k_2}a_{3k_3} \cdots a_{mk_m} - (\sum(\pm)a_{1k_1}a_{2k_2}a_{3k_3} \cdots a_{nk_n})^r}{\sum(\pm)a_{1k_1}a_{2k_2}a_{3k_3} \cdots a_{mk_m}} \]

(where the summation ranges over all permutations of \( k_2, k_3, \ldots, k_m \) of the set \( \{2, 3, \ldots, m\} \) and \( k_1, k_3, \ldots, k_n \) of the set \( \{1, 3, \ldots, n\} \))

\[ = \frac{(-1)^4|M_{12}| - |A_2|^r}{|A_n|} = \frac{(A_{12}) - |A_2|^r}{|A_n|}, \]

\[ b_{22} = \frac{\sum(\pm)a_{1k_1}a_{2k_2}a_{3k_3} \cdots a_{mk_m} - (\sum(\pm)a_{1k_1}a_{2k_2}a_{3k_3} \cdots a_{nk_n})^r}{\sum(\pm)a_{1k_1}a_{2k_2}a_{3k_3} \cdots a_{mk_m}} \]

(where the summation ranges over all permutations of \( k_1, k_2, \ldots, k_m \) of the set \( \{1, 3, \ldots, m\} \))

\[ = \frac{(-1)^4|M_{22}| - |A_2|^r}{|A_n|} = \frac{(A_{22}) - |A_2|^r}{|A_n|}, \]

\[ \vdots \]

\[ b_{2m} = \frac{\sum(\pm)a_{1k_1}a_{2k_2}a_{3k_3} \cdots a_{mk_m} - (\sum(\pm)a_{1k_1}a_{2k_2}a_{3k_3} \cdots a_{nk_n})^r}{\sum(\pm)a_{1k_1}a_{2k_2}a_{3k_3} \cdots a_{mk_m}} \]

(where the summation ranges over all permutations \( k_1, k_3, k_m \) of the set \( \{1, 2, \ldots, m\} \))

\[ = \frac{(-1)^{m+2}|M_{2m}| - |A_2|^r}{|A_n|} = \frac{(A_{2m}) - |A_2|^r}{|A_n|}, \]

\[ \vdots \]
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\[ b_{m1} = \frac{\sum(\pm)a_{k1}a_{k2}\ldots a_{km}+\sum(\pm)a_{1k1}a_{k2}\ldots a_{km}}{\sum(\pm)a_{1k1}a_{k2}\ldots a_{km}} \]
where the summation ranges over all permutations \( k_1, k_2, \ldots, k_m \) of the set \( \{1, 2, \ldots, n-1, n\} \)

\[ b_{m2} = \frac{\sum(\pm)a_{1k1}a_{k2}\ldots a_{km}+\sum(\pm)a_{k1}a_{k2}\ldots a_{kn}}{\sum(\pm)a_{1k1}a_{k2}\ldots a_{kn}} \]
where the summation ranges over all permutations \( k_1, k_3, \ldots, k_m \) of the set \( \{1, 3, \ldots, n-2, n\} \)

\[ b_{mm} = \frac{\sum(\pm)a_{k1}a_{k2}\ldots a_{km}+\sum(\pm)a_{k1}a_{k2}\ldots a_{km}}{\sum(\pm)a_{1k1}a_{k2}\ldots a_{km}} \]
where the summation ranges over all permutations \( k_1, k_2, \ldots, k_m \) of the set \( \{1, 2, \ldots, m\} \)

\[ = (-1)^{2+m|m|A_{mm}+|A_m|}r/|A_m| \]

**Corollary 3.3** If \( A \) is an \( m \times n \) quasi-invertible matrix with \( m = n + 1 \), then the quasi-inverse is an \( n \times m \) matrix

\[ A^* = [b_{ji} \ r] \text{ for } 1 \leq j \leq n \text{ where } \]

\[ b_{ji} = \begin{cases} \frac{\langle A_{ij} \rangle + |A_i|}{|A_m|}, & \text{if } i \text{ is odd} \\ \frac{\langle A_{ij} \rangle - |A_i|}{|A_m|}, & \text{if } i \text{ is even} \end{cases} \]

when the number of columns is even and

\[ b_{ji} = \begin{cases} \frac{\langle A_{ij} \rangle - |A_i|}{|A_m|}, & \text{if } i \text{ is odd} \\ \frac{\langle A_{ij} \rangle + |A_i|}{|A_m|}, & \text{if } i \text{ is even} \end{cases} \]

when the number of columns is odd

\[ b_{mi} = r, \ A_i \text{ is a square submatrix of } A \text{ obtained by deleting the } i^{th} \text{ row, } \langle A_{ij} \rangle \text{ is the cofactor of } a_{ij} \text{ of the invertible submatrix } A_m \text{ obtained from } A \text{ by deleting the } m^{th} \text{ row. That is, } \]

\[ A^* = \begin{bmatrix} \langle A_{11} \rangle + |A_i| & \langle A_{12} \rangle + |A_i| & \ldots & \langle A_{1n} \rangle + |A_i| \\ \langle A_{21} \rangle + |A_i| & \langle A_{22} \rangle + |A_i| & \ldots & \langle A_{2n} \rangle + |A_i| \\ \vdots & \vdots & \ddots & \vdots \\ \langle A_{n1} \rangle + |A_i| & \langle A_{n2} \rangle + |A_i| & \ldots & \langle A_{nn} \rangle + |A_i| \end{bmatrix} r \]

\[ = \frac{\langle A_{11} \rangle + |A_i|}{|A_m|} \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1m} \\ a_{21} & a_{22} & \ldots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix} \]

**Proof:** Suppose that \( A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix} \). Since \( A \) is quasi-invertible,

by Lemma 3.1, there exists matrix \( B \) such that \( BA = I \). Let
Consider now $A^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}$ and $B^T = \begin{bmatrix} b_{11} & b_{21} & \cdots & b_{n1} \\ b_{12} & b_{22} & \cdots & b_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ b_{1n} & b_{2n} & \cdots & b_{nn} \end{bmatrix}$. Since $A^T B^T = (BA)^T = I^T = I$, $A^T$ is quasi-invertible and its quasi-inverse is $B^T$. Note that $A^T$ is an $n \times m$ matrix with $n = m - 1$. So the order of $A^T$ satisfies the hypothesis of Theorem 3.2. Thus, by Theorem 3.2, we obtain

$$B^T = \begin{bmatrix} \langle A_{11} \rangle \pm |A_1| r \\ |A_1| \\ \langle A_{12} \rangle \pm |A_2| r \\ |A_2| \\ \vdots & \vdots & \ddots & \vdots \\ |A_{1m}| \\ \langle A_{1n} \rangle \pm |A_n| r \\ |A_n| \\ \langle A_{21} \rangle \pm |A_2| r \\ |A_2| \\ \vdots & \vdots & \ddots & \vdots \\ |A_{m1}| \\ \langle A_{2n} \rangle \pm |A_n| r \\ |A_n| \\ \vdots & \vdots & \ddots & \vdots \\ |A_{mn}| \end{bmatrix}$$

where

$$b_{ij} = \begin{cases} \frac{\langle A_{ji} \rangle + |A_i| r}{|A_m|}, & \text{if } i \text{ is odd} \\ \frac{\langle A_{ji} \rangle - |A_i| r}{|A_m|}, & \text{if } i \text{ is even} \end{cases}$$

when the number of columns is even and

$$b_{ij} = \begin{cases} \frac{\langle A_{ji} \rangle - |A_i| r}{|A_m|}, & \text{if } i \text{ is odd} \\ \frac{\langle A_{ji} \rangle + |A_i| r}{|A_m|}, & \text{if } i \text{ is even} \end{cases}$$

when the number of columns is odd.

Since $(B^T)^T = B$, we have

$$b^T_{ij} = b_{ji} = \begin{cases} \frac{\langle A_{ij} \rangle + |A_i| r}{|A_m|}, & \text{if } i \text{ is odd} \\ \frac{\langle A_{ij} \rangle - |A_i| r}{|A_m|}, & \text{if } i \text{ is even} \end{cases}$$

when the number of columns is even and

$$b^T_{ij} = b_{ji} = \begin{cases} \frac{\langle A_{ij} \rangle - |A_i| r}{|A_m|}, & \text{if } i \text{ is odd} \\ \frac{\langle A_{ij} \rangle + |A_i| r}{|A_m|}, & \text{if } i \text{ is even} \end{cases}$$

when the number of columns is odd.
Let \( B = A^* \). Then \( A^* = \begin{bmatrix} [A_{11}] \pm |A_1|r & \langle A_{21} \rangle \pm |A_2|r & \cdots & \langle A_{n1} \rangle \pm |A_n|r \\ [A_{1m}] & [A_{2m}] & \cdots & [A_{nm}] \\ \vdots & \vdots & \ddots & \vdots \\ [A_{1r}] & [A_{2r}] & \cdots & [A_{rr}] \end{bmatrix} \). □

**Corollary 3.4** Let \( A \) be an \( m \times n \) matrix with \( n \geq m + 2 \). If the diminished matrix of \( A \) is quasi-invertible then \( A \) is quasi-invertible. The quasi-inverse of \( A \) is given by \( A^* = [b_{ji}] \) where \( b_{ji} = 0 \) or \( 1 \) for \( m + 1 \leq j \leq n \) obtained by “extending” \( A^* \) as defined in Theorem 3.2.

**Proof:** Let \( A \) be an \( m \times n \) matrix with \( n \geq m + 2 \) and suppose that the diminished matrix \( B \) of \( A \) is quasi-invertible. By Theorem 3.2, there exists \( B^* \) such that \( BB^* = I \). Now, extend \( B^* \) to an \( n \times m \) matrix \( A^* = [b_{ji}] \) such that \( b_{ji} = 0 \) or \( 1 \) for \( m + 1 \leq j \leq n \). Then we have \( AA^* = I \). □

**Theorem 3.5** Let \( A \) be an \( m \times n \) quasi-invertible matrix. If \( A \) is right invertible with \( A^* \) a right inverse then \( A^T \) is left invertible and \( A^T \) is a left inverse of \( A^T \). Similarly, if \( A \) is left invertible then \( A^T \) is right invertible and \( A^T \) is a right inverse of \( A^T \).

**Proof:** Let \( A \) be an \( m \times n \) quasi-invertible matrix. Suppose that \( A \) is right invertible. By Lemma 3.1 and Theorem 3.2 there exists \( A^* \) such that \( AA^* = I \). Now, \( A^T A^T = (AA^*)^T = I^T = I \) which shows that \( A^T \) is left invertible and \( A^T \) is the left inverse. If \( A \) is left invertible, then there exists \( A^* \) such that \( A^* A = I \). Now, \( A^T A^T = (A^* A)^T = I^T = I \) which shows that \( A^T \) is right invertible and \( A^T \) is the right inverse. □

**Theorem 3.6** An \( m \times n \) quasi-invertible matrix has infinitely many quasi-inverses if and only if \( n = m - 1 \) or \( n = m + 1 \).

**Proof:** Let \( A \) be an \( m \times n \) matrix. Since \( A \) is quasi-invertible, by Theorem 3.2 and Lemma 3.1, there exists an \( n \times m \) matrix \( A^* \) such that \( AA^* = I \) or \( A^* A = I \). Note that there are \( n - m \) or \( m - n \) rows of \( r \) depending on whether \( n > m \) or \( n < m \). Suppose that there are infinitely many quasi-inverses of \( A \). Assume that \( n \leq m - 1 \) or \( n \geq m + 1 \). If \( n < m - 1 \) or \( n > m + 1 \) then by Corollary 3.4, the rows \( b_{ji} = r \) for a quasi-inverse matrix hold only for \( r = 1 \) or \( r = 0 \) which means that there are only two quasi-inverses of \( A \) contradicting the assumption. Thus, \( n = m - 1 \) or \( n = m + 1 \).

Conversely, suppose that \( n = m - 1 \) or \( n = m + 1 \) for an \( m \times n \) matrix \( A \). Then the number of rows or columns of \( A \) exceeds only by one. By Theorem 3.2, the row \( b_{ni} = r \) of a quasi-inverse \( A^* \) assumes any real number. Let \( M = \{ A^* : AA^* = I \text{ or } A^* A = I \} \). We have to show that \( |M| = +\infty \). Suppose
that $|M| < +\infty$. Then for any $r' > r$, $r, r' \in R$ such that $b_{ni} = r'$ of $A^*$ does not yield a quasi-inverse. That is, for any $r' > r$, $AA^* \neq I$ which is a contradiction. Since for every $r \in R$ if $b_{ni} = r$ is the last row $A^*$, we always have $AA^* = I$ or $A^*A = I$, therefore we conclude that there are infinitely many quasi-inverses of $A$. □

**Remark 3.7** Let $A^*$ be the quasi-inverse of an $m \times n$ matrix $A$. Then $AA^* = \mathbf{I}_m$ if $m < n$ and $A^*A = \mathbf{I}_n$ if $m > n$.

**Remark 3.8** Let $A$ be an $m \times n$ quasi-invertible matrix and $A^* = [b_{ji}]$ where $b_{ni} = r$ for all $i = 1, 2, \ldots, m$ a quasi-inverse of $A$. If $r = 0$, we call $A^*$ the basic quasi-inverse of $A$.

**Theorem 3.9** Let $A$ be an $m \times n$ quasi-invertible matrix and let $B$ be an invertible matrix of order $m$ if $m < n$ and of order $n$ if $m > n$. Then

$$(BA)^* = A^*B^{-1} \text{ if } m < n$$

and

$$(AB)^* = B^{-1}A^* \text{ if } m > n.$$  

**Proof**: Let $A$ be an $m \times n$ quasi-invertible matrix. Consider the following cases:

Case 1. $m < n$.

Let $B$ be an invertible matrix of order $m$. Then $BA$ is defined. Since $A$ is quasi-invertible, $A^*$ exists and $AA^* = \mathbf{I}$. Now,

$$(BA)(A^*B^{-1}) = B(AA^*)B^{-1}$$
$$= BIB^{-1}$$
$$= BB^{-1} = \mathbf{I}$$

implies that $A^*B^{-1} = (BA)^*$.

Case 2. $m > n$.

Let $B$ be an invertible matrix of order $n$. Then $AB$ is defined. Since $A$ is quasi-invertible, there exists $A^*$ such that $A^*A = \mathbf{I}$. Now,

$$(B^{-1}A^*)(AB) = B^{-1}(A^*A)B$$
$$= B^{-1}IB$$
$$= B^{-1}B = \mathbf{I}$$

implies that $B^{-1}A^* = (AB)^*$.

□

**Theorem 3.10** Let $A$ be a matrix. If $A$ is invertible, then $A^* = A^{-1}$ and hence $A^* = A^+$. 
Proof: Let $A$ be a matrix. Since $A$ is invertible, $A$ is a square matrix. Hence, $A^*$ has no row of $r \in R$. Consequently,

$$
A^* = \begin{bmatrix}
\langle A_{11} \rangle & \langle A_{12} \rangle & \cdots & \langle A_{1n} \rangle \\
\langle A_{21} \rangle & \langle A_{22} \rangle & \cdots & \langle A_{2n} \rangle \\
\vdots & \vdots & \ddots & \vdots \\
\langle A_{m1} \rangle & \langle A_{m2} \rangle & \cdots & \langle A_{mn} \rangle 
\end{bmatrix}.
$$

Thus, $A^* = A^{-1}$. By Theorem 2.15, $A^{-1} = A^*$, therefore, $A^* = A^+$. □

Theorem 3.11 If $A$ is a quasi-invertible matrix then $(cA)^* = c^{-1}A^*$, $0 \neq c \in R$ where last row of $(cA)^*$ consists of $c^{-1}r$.

Proof: Suppose that $A$ is a quasi-invertible matrix. Then $A$ is either left or right invertible. If $A$ is right invertible, then there exists $A^*$ such that $AA^* = I$. Let $0 \neq c \in R$. Then $cA$ is quasi-invertible. Now, $(cA)(cA)^* = (cA)(A^*c^{-1}) = c(AA^*c^{-1}) = c(cIc^{-1}) = cc^{-1}I = I$ which holds only if the last row of $(cA)^*$ consists of $c^{-1}r$. If $A$ is left invertible, then there exists $A^*$ such that $A^*A = I$ and $cA$ is quasi-invertible. Now, $(cA)^*(cA) = (A^*c^{-1})(cA) = A^*(cc^{-1})A = A^*A = I$. Thus, in either case, $(cA)^* = c^{-1}A^*$. □

4 The Linear System $AX = B$

This section discusses some results of finding the solution of the linear system $AX = B$ using the quasi-inverse as well as the pseudo-inverse of an $m \times n$ matrix $A$. We consider the cases when $m = n - 1$ or $m = n + 1$.

Theorem 4.1 Let $A$ be an $m \times n$ matrix with $m = n - 1$, $X = [x_1, x_2, \ldots, x_n]^T$, and $B = [c_1, c_2, \ldots, c_m]^T$. If $A$ is quasi-invertible and $x_n = (c_1 + c_2 + \cdots + c_m)r$, then the linear system $AX = B$ has many solutions. Each solution is given by

$$
X = A^*B.
$$

Proof: Let $A$ be an $m \times n$ matrix. Then $AX = B$ is

$$
\begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{22} & a_{23} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
\vdots \\
x_n
\end{bmatrix}
= 
\begin{bmatrix}
c_1 \\
c_2 \\
\vdots \\
c_m
\end{bmatrix}.
$$

Simplifying the left side and equating to the right side, we obtain the linear system

\begin{align*}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= c_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= c_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= c_m
\end{align*}
\[
\begin{align*}
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= c_m. \\
\end{align*}
\]
Solving each variable by elimination and substituting \(x_n = (c_1 + c_2 + \cdots + c_m)r\), we obtain
\[
\begin{align*}
x_1 &= \frac{(A_{11})_{+|A_1|}r}{|A_n|}c_1 + \frac{(A_{21})_{+|A_1|}r}{|A_n|}c_2 + \cdots + \frac{(A_{m1})_{+|A_1|}r}{|A_n|}c_m \\
x_2 &= \frac{(A_{12})_{+|A_1|}r}{|A_n|}c_1 + \frac{(A_{22})_{+|A_1|}r}{|A_n|}c_2 + \cdots + \frac{(A_{m2})_{+|A_1|}r}{|A_n|}c_m \\
& \quad \vdots \\
x_{n-1} &= \frac{(A_{1,n-1})_{+|A_1|}r}{|A_n|}c_1 + \frac{(A_{2,n-1})_{+|A_1|}r}{|A_n|}c_2 + \cdots + \frac{(A_{m,n-1})_{+|A_1|}r}{|A_n|}c_m 
\end{align*}
\]
where \(A_{ij}\) is the adjoint of \(a_{ij}\) and \(A_j\) are submatrices of \(A\) of order \(m\) obtained by deleting the \(j\)th column of \(A\). From Theorem 3.2, we find that
\[
b_{ji} = \begin{cases} 
\frac{(A_{ij})_{+|A_1|}r}{|A_n|}, & \text{if } j \text{ is odd} \\
\frac{(A_{ij})_{-|A_1|}r}{|A_n|}, & \text{if } j \text{ is even}
\end{cases}
\]
when the number of rows is even and
\[
b_{ji} = \begin{cases} 
\frac{(A_{ij})_{-|A_1|}r}{|A_n|}, & \text{if } j \text{ is even} \\
\frac{(A_{ij})_{+|A_1|}r}{|A_n|}, & \text{if } j \text{ is odd}
\end{cases}
\]
when the number of rows is odd with \(1 \leq j \leq n - 1\). Thus,
\[
\begin{align*}
x_1 &= b_{11}c_1 + b_{12}c_2 + \cdots + b_{1m}c_m \\
x_2 &= b_{21}c_1 + b_{22}c_2 + \cdots + b_{2m}c_m \\
& \quad \vdots \\
x_{n-1} &= b_{m1}c_1 + b_{m2}c_2 + \cdots + b_{mm}c_m.
\end{align*}
\]
Hence, \(X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_{11}c_1 + b_{12}c_2 + \cdots + b_{1m}c_m \\ b_{21}c_1 + b_{22}c_2 + \cdots + b_{2m}c_m \\ \vdots \\ b_{m1}c_1 + b_{m2}c_2 + \cdots + b_{mm}c_m \\ r_{c_1} + r_{c_2} + \cdots + r_{c_m} \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mm} \\ r & r & \cdots & r \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{bmatrix}
\)
Therefore, \(X = A^*B\).

**Corollary 4.2** Let \(AX = B\) be a linear system with \(A\) an \(m \times n\) matrix where \(m = n + 1\). If \(A\) is quasi-invertible and the augmented matrix has a row equivalent to zero, then the system has a unique solution given by \(X = A^*B\) where \(A^*\) is a basic quasi-inverse of \(A\).

**Proof:** Let \(AX = B\) be a linear system with \(A\) an \(m \times n\) matrix. Suppose that \(A\) is quasi-invertible. Since \(m = n + 1\), by Lemma 3.1 there exists \(A^*\) such that \(A^*A = I\). Since \([A : B]\) has a row equivalent to zero, we can make this to be the last row. Let \(A^*\) be the basic quasi-inverse of \(A\). Then we have, \(A^*AX = A^*B \Rightarrow IX = A^*B \Rightarrow X = A^*B\). 
\[\square\]
**Theorem 4.3** Let $A$ be an $m \times n$ matrix where $m = n + 1$. If $A^T A$ is invertible and the linear system $AX = B$ has a unique solution, then the solution can be given by

$$X = A_L^+ B$$

where $A_L^+$ is a left pseudo-inverse of $A$.

**Proof:** Let $A$ be an $m \times n$ matrix where $m = n + 1$. Suppose that $A^T A$ is invertible, $(A^T A)^{-1}$ exists and since the linear system $AX = B$ has a solution, we have

$$AX = B \Rightarrow A^T AX = A^T B.$$ 

Multiplying by $(A^T A)^{-1}$, we obtain

$$(A^T A)^{-1}(A^T A)X = (A^T A)^{-1}A^T B \Rightarrow IX = (A^T A)^{-1}A^T B = A_L^+ B.$$ \qed

**References**


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