Nonlinear Differential Equations
Associated with Degenerate Tangent Numbers

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Abstract
In this paper, we study nonlinear differential equations arising from the generating functions of degenerate tangent numbers. We give explicit identities for the degenerate tangent numbers.

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1 Introduction
The generalization and multivariable forms of special functions in mathematical physics have made significant progress in recent years. The special polynomials of more than one variable provide new means of analysis for the solutions of a wide class of differential equations often encountered in physical problems. Very recently, many mathematicians have studied in the area of the Euler numbers, Bernoulli numbers, tangent numbers, degenerate Euler numbers, degenerate Bernoulli numbers, and degenerate tangent numbers (see [1, 2, 3, 6, 7, 8, 9, 10]). In [1], L. Carlitz introduced the degenerate Bernoulli polynomials. In [4], we introduced the definition of the degenerate tangent numbers and polynomials and we have given some formulae of those numbers and polynomials related to the generalized falling factorial and Stirling
numbers. The degenerate tangent numbers $T_{n,\lambda}$ are defined by the generating function:

$$\frac{2}{(1 + \lambda t)^{2/\lambda} + 1} = \sum_{n=0}^{\infty} \frac{T_{n,\lambda}}{n!} t^n. \quad (1.1)$$

In [5], degenerate tangent numbers of higher order, $T_{n,\lambda}^{(k)}$ are defined by means of the following generating function

$$\left(\frac{2}{(1 + \lambda t)^{2/\lambda} + 1}\right)^k = \sum_{n=0}^{\infty} \frac{T_{n,\lambda}^{(k)}}{n!} t^n. \quad (1.2)$$

The first few of them are

$T_{0,\lambda}^{(k)} = 1, \quad T_{1,\lambda}^{(k)} = -k, \quad T_{2,\lambda}^{(k)} = -k + \lambda k + k^2,$

$T_{3,\lambda}^{(k)} = 3\lambda k - 2\lambda^2 k + 3k^2 - 3\lambda k^2 - k^3,$

$T_{4,\lambda}^{(k)} = 2k - 11\lambda^2 k + 6\lambda^3 k + 3k^2 - 18\lambda k^2 + 11\lambda^2 k^2 - 6k^3 + 6\lambda k^3 + k^4$

$T_{5,\lambda}^{(k)} = -20\lambda k + 50\lambda^3 k - 24\lambda^4 k - 10k^2 - 30\lambda k^2 + 105\lambda^2 k^2 - 50\lambda^3 k^2 - 15k^3$

$\quad + 60\lambda k^3 - 35\lambda^2 k^3 + 10k^4 - 10\lambda k^4 - k^5.$

We recall that the classical Stirling numbers of the first kind $S_1(n, k)$ and $S_2(n, k)$ are defined by the relations (see [10])

$$(x)_n = \sum_{k=0}^{n} S_1(n, k)x^k \text{ and } x^n = \sum_{k=0}^{n} S_2(n, k)(x)_k,$$

respectively. Here $(x)_n = x(x-1) \cdots (x-n+1)$ denotes the falling factorial polynomial of order $n$. We also have

$$\sum_{n=m}^{\infty} S_2(n, m) \frac{t^n}{n!} = \frac{(e^t - 1)^m}{m!} \text{ and } \sum_{n=m}^{\infty} S_1(n, m) \frac{t^n}{n!} = \frac{(\log(1 + t))^m}{m!}. \quad (1.3)$$

The generalized falling factorial $(x|\lambda)_n$ with increment $\lambda$ is defined by

$$(x|\lambda)_n = \prod_{k=0}^{n-1} (x - \lambda k) \quad (1.4)$$

for positive integer $n$, with the convention $(x|\lambda)_0 = 1$. We also need the binomial theorem: for a variable $x,$

$$(1 + \lambda t)^{x/\lambda} = \sum_{n=0}^{\infty} \frac{(x|\lambda)_n}{n!} t^n. \quad (1.5)$$
Many mathematicians have studied in the area of the linear and nonlinear differential equations arising from the generating functions of special polynomials in order to give explicit identities for special polynomials (see [2, 8, 9]). In this paper, we study nonlinear differential equations arising from the generating functions of degenerate tangent numbers. We give explicit identities for the degenerate tangent numbers.

2 Nonlinear differential equations associated with degenerate tangent numbers

In this section, we study nonlinear differential equations arising from the generating functions of degenerate tangent numbers. Let

\[ F = F(t, \lambda) = \frac{2}{(1 + \lambda t)^{2/\lambda} + 1} = \sum_{n=0}^{\infty} \frac{T_{n,\lambda} t^n}{n!}. \]  

(2.1)

Then, by (2.1), we have

\[ F^{(1)} = \frac{d}{dt} F(t, \lambda) = \frac{d}{dt} \left( \frac{2}{(1 + \lambda t)^{2/\lambda} + 1} \right) \]
\[ = -\frac{2}{1 + \lambda t} \left( \frac{2}{(1 + \lambda t)^{2/\lambda} + 1} \right) + \frac{1}{1 + \lambda t} \left( \frac{2}{(1 + \lambda t)^{2/\lambda} + 1} \right)^2 \]  

(2.2)

By (2.2), we have

\[ F^2 = 2F + (1 + \lambda t)F^{(1)}. \]  

(2.3)

Taking the derivative with respect to \( t \) in (2.3), we obtain

\[ 2FF^{(1)} = 2F^{(1)} + \lambda F^{(1)} + (1 + \lambda t)F^{(2)}. \]
\[ = (\lambda + 2)F^{(1)} + (1 + \lambda t)F^{(2)} \]  

(2.4)

From (2.2), (2.3), and (2.4), we have

\[ 2F^3 = 8F + (\lambda + 6)(1 + \lambda t)F^{(1)} + (1 + \lambda t)^2 F^{(2)}. \]

Continuing this process, we can guess that

\[ N! F^{N+1} = \sum_{i=0}^{N} a_i(N, \lambda)(1 + \lambda t)^i F^{(i)}, \]  

\[ (N = 0, 1, 2, \ldots), \]  

(2.5)
where $F^{(i)} = \left( \frac{d}{dt} \right)^i F(t, \lambda)$. Differentiating (2.5) with respect to $t$, we have

\begin{equation}
(N + 1)!F^N F^{(1)} = \sum_{i=0}^{N} i\lambda a_i(N, \lambda)(1 + \lambda t)^{i-1}F^{(i)} + \sum_{i=0}^{N} a_i(N, \lambda)(1 + \lambda t)^iF^{(i+1)}
\end{equation}

and

\begin{equation}
(N + 1)!F^N F^{(1)} = (N + 1)!F^N \left( \frac{-2F + F^2}{1 + \lambda t} \right) = (N + 1)! \left( \frac{F^{N+2} - 2F^{N+1}}{1 + \lambda t} \right).
\end{equation}

By (2.5), (2.6), and (2.7), we have

\begin{equation}
(N + 1)!F^{N+2} = 2(N + 1)!F^{N+1} + \sum_{i=0}^{N} \lambda i a_i(N, \lambda)(1 + \lambda t)^{i-1}F^{(i)} + \sum_{i=0}^{N} a_i(N, \lambda)(1 + \lambda t)^iF^{(i+1)}
\end{equation}

\begin{equation}
= 2(N + 1) \sum_{i=0}^{N} a_i(N, \lambda)(1 + \lambda t)^{i}F^{(i)} + \sum_{i=0}^{N} \lambda i a_i(N, \lambda)(1 + \lambda t)^{i}F^{(i)}
\end{equation}

\begin{equation}
+ \sum_{i=0}^{N} a_i(N, \lambda)(1 + \lambda t)^{i+1}F^{(i+1)}
\end{equation}

\begin{equation}
= 2(N + 1) \sum_{i=0}^{N} a_i(N, \lambda)(1 + \lambda t)^{i}F^{(i)} + \sum_{i=0}^{N} \lambda i a_i(N, \lambda)(1 + \lambda t)^{i}F^{(i)}
\end{equation}

\begin{equation}
+ \sum_{i=1}^{N+1} a_{i-1}(N, \lambda)(1 + \lambda t)^{i}F^{(i)}.
\end{equation}

Now replacing $N$ by $N + 1$ in (2.5), we find

\begin{equation}
(N + 1)!F^{N+2} = \sum_{i=0}^{N+1} a_i(N + 1, \lambda)(1 + \lambda t)^{i}F^{(i)}.
\end{equation}
By (2.8) and (2.9), we have

\[
\sum_{i=0}^{N+1} a_i (N+1, \lambda)(1 + \lambda t)^i F^{(i)}
\]

\[
= 2(N+1) \sum_{i=0}^{N} a_i (N, \lambda)(1 + \lambda t)^i F^{(i)} + \sum_{i=0}^{N} \lambda i a_i (N, \lambda)(1 + \lambda t)^i F^{(i)}
\]

\[
+ \sum_{i=1}^{N+1} a_{i-1} (N, \lambda)(1 + \lambda t)^i F^{(i)}.
\]  

(2.10)

Comparing the coefficients on both sides of (2.10), we obtain

\[
2(N+1)a_0(N, \lambda) = a_0(N+1, \lambda), \quad a_{N+1}(N+1, \lambda) = a_N(N, \lambda),
\]  

(2.11)

and

\[
a_i(N+1, \lambda) = 2 \left( N + 1 + \frac{\lambda i}{2} \right) a_i(N, \lambda) + a_{i-1}(N, \lambda), \quad (1 \leq i \leq N).
\]  

(2.12)

In addition, by (2.5), we have

\[
F = a_0(0, \lambda) F,
\]  

(2.13)

which gives

\[
a_0(0, \lambda) = 1.
\]  

(2.14)

It is not difficult to show that

\[
F^2 = a_0(1, \lambda) F + a_1(1, \lambda)(1 + \lambda t)F^{(1)} = 2F + (1 + \lambda t)F^{(1)}.
\]  

(2.15)

Thus, by (2.15), we also find

\[
a_0(1, \lambda) = 2, \quad a_1(1, \lambda) = 1.
\]  

(2.16)

From (2.11), we note that

\[
a_0(N+1, \lambda) = 2(N+1)a_0(N, \lambda) = \cdots = 2^{N+1}(N+1)!,
\]  

(2.17)

and

\[
a_{N+1}(N+1, \lambda) = a_N(N, \lambda) = a_{N-1}(N-1, \lambda) = \cdots = 1.
\]  

(2.18)

For \( i = 1, 2, 3 \) in (2.11), we have

\[
a_1(N+1, \lambda) = \sum_{k=0}^{N} 2^k \left( N + 1 + \frac{\lambda}{2} \times 1 \right)_k a_0(N-k, \lambda),
\]
\[ a_2(N + 1, \lambda) = \sum_{k=0}^{N-1} 2^k \left( N + 1 + \frac{\lambda}{2} \times 2 \right)_k a_1(N - k, \lambda), \]

and

\[ a_3(N + 1, \lambda) = \sum_{k=0}^{N-2} 2^k \left( N + 1 + \frac{\lambda}{2} \times 3 \right)_k a_2(N - k, \lambda), \]

where \((x)_k = x(x - 1) \cdots (x - k + 1)\) denotes the falling factorial polynomial of order \(k\). Continuing this process, we can deduce that, for \(1 \leq i \leq N\),

\[ a_i(N + 1, \lambda) = \sum_{k=0}^{N-i+1} 2^k \left( N + 1 + \frac{\lambda}{2} \times i \right)_k a_{i-1}(N - k, \lambda). \quad (2.19) \]

Note that, here the matrix \(a_i(j, \lambda)_{0 \leq i, j \leq N+1}\) is given by

\[
\begin{pmatrix}
1 & 2 & 2!2^2 & 3!2^3 & \cdots & (N + 1)!2^{N+1} \\
0 & 1 & . & . & \cdots & . \\
0 & 0 & 1 & . & \cdots & . \\
0 & 0 & 0 & 1 & \cdots & . \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 1
\end{pmatrix}
\]

Now, we give explicit expressions for \(a_i(N+1, \lambda)\). By (2.17), (2.18), and (2.19), we have

\[ a_1(N + 1, \lambda) = \sum_{k_1=0}^{N} 2^{k_1} \left( N + 1 + \frac{\lambda}{2} \right)_k a_0(N - k_1, \lambda) \\
= \sum_{k_1=0}^{N} 2^N(N - k_1)! \left( N + 1 + \frac{\lambda}{2} \right)_k, \]

\[ a_2(N + 1, \lambda) \\
= \sum_{k_2=0}^{N-1} 2^{k_2} \left( N + 1 + \frac{\lambda}{2} \times 2 \right)_k a_1(N - k_2, \lambda) \\
= \sum_{k_2=0}^{N-1} \sum_{k_1=0}^{N-k_2-1} 2^{N-1}(N - k_2 - k_1 - 1)! \left( N + 1 + \frac{\lambda}{2} \times 2 \right)_k \left( N - k_2 + \frac{\lambda}{2} \right)_k, \]
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and

\[ a_3(N + 1, \lambda) = \sum_{k_3=0}^{N-2} 2^{k_3} \left( N + 1 + \frac{\lambda}{2} \times 3 \right) a_2(N - k_3, \lambda) \]

\[ = \sum_{k_3=0}^{N-2} \sum_{k_2=0}^{N-k_3-2} \sum_{k_1=0}^{N-k_3-k_2-2} 2^{N-2}(N - k_3 - k_2 - k_1 - 2)! \]

\[ \times \left( N + 1 + \frac{\lambda}{2} \times 3 \right)_{k_3} \left( N - k_3 + \frac{\lambda}{2} \times 2 \right)_{k_2} \left( N - k_3 - k_2 - 1 + \frac{\lambda}{2} \right)_{k_1}. \]

Continuing this process, we have

\[ a_i(N + 1, \lambda) = \sum_{k_i=0}^{N-i+1} \sum_{k_{i-1}=0}^{N-k_i-i+1} \cdots \sum_{k_1=0}^{N-k_{i-1} \cdots k_2 - i + 1} 2^{N-i+1} \]

\[ \times (N - k_i - k_{i-1} - \cdots - k_2 - k_1 - i + 1)! \]

\[ \times \left( N + 1 + \frac{\lambda}{2} \times i \right)_{k_i} \left( N - k_i + \frac{\lambda}{2} \times (i - 1) \right)_{k_{i-1}} \]

\[ \times \left( N - k_i - k_{i-1} - 1 + \frac{\lambda}{2} \times (i - 2) \right)_{k_{i-2}} \]

\[ \times \left( N - k_i - k_{i-1} - k_{i-2} - 2 + \frac{\lambda}{2} \times (i - 3) \right)_{k_{i-3}} \]

\[ \cdots \]

\[ \times \left( N - k_i - k_{i-1} - k_{i-2} - \cdots - k_2 - i + 2 + \frac{\lambda}{2} \right)_{k_1}. \]

(2.20)

Therefore, by (2.20), we obtain the following theorem.

**Theorem 2.1** For \( N = 0, 1, 2, \ldots, \) the nonlinear functional equation

\[ N! F^{N+1} = \sum_{i=0}^{N} a_i(N, \lambda)(1 + \lambda t)^i F^{(i)} \]

has a solution

\[ F = F(t, \lambda) = \frac{2}{(1 + \lambda t)^{2/\lambda + 1}}, \]
where
\[ a_0(N, \lambda) = N! 2^N, \]
\[ a_{N, \lambda}(N) = 1, \]
\[ a_i(N, \lambda) = \sum_{k_i=0}^{N-i} \sum_{k_{i-1}=0}^{N-k_i} \cdots \sum_{k_1=0}^{N-k_1-k_2-\cdots-k_i} 2^{N-i} \]
\[ \times (N - k_i - k_{i-1} - \cdots - k_2 - k_1 - i)! \]
\[ \times \left( N + \frac{\lambda}{2} \times i \right)^{k_i} \left( N - k_i - 1 + \frac{\lambda}{2} \times (i - 1) \right)^{k_{i-1}} \]
\[ \times \left( N - k_i - k_{i-1} - 2 + \frac{\lambda}{2} \times (i - 2) \right)^{k_{i-2}} \]
\[ \times \left( N - k_i - k_{i-1} - k_{i-2} - 3 + \frac{\lambda}{2} \times (i - 3) \right)^{k_{i-3}} \cdots \]
\[ \times \left( N - k_i - k_{i-1} - k_{i-2} - \cdots - k_2 - i + 1 + \frac{\lambda}{2} \right)^{k_1}. \]

Here is a plot of the surface for this solution. In Figure 1(left), we plot of

![Surface plot](image1)

the surface for this solution. In Figure 1(right), we shows a higher-resolution density plot of the solution. From (1.1) and (1.2), we note that
\[ N! F^{N+1} = N! \left( \frac{2}{(1 + \lambda t)^{2/\lambda} + 1} \right)^{N+1} = N! \sum_{n=0}^{\infty} T_{n, \lambda}^{(N+1)} \frac{t^n}{n!}. \]

(2.21)

From (2.5), we note that
\[ F^{(i)} = \left( \frac{d}{dt} \right)^i F(t, \lambda) = \sum_{l=0}^{\infty} T_{i+l, \lambda} \frac{t^l}{l!}. \]

(2.22)
From Theorem 2.1, (1.5), (2.21), and (2.22), we can derive the following equation:

\[
N! \sum_{n=0}^{\infty} T_{n,\lambda}^{(N+1)} \frac{t^n}{n!} = \sum_{i=0}^{N} a_i(N, \lambda)(1 + \lambda t)^i E^{(i)} \\
= \sum_{i=0}^{N} a_i(N, \lambda) \sum_{k=0}^{\infty} (i)_k \lambda^k \frac{t^k}{k!} \sum_{l=0}^{\infty} T_{i+l,\lambda} t^l \\
= \sum_{n=0}^{\infty} \left( \sum_{i=0}^{N} \sum_{k=0}^{n} \binom{n}{k} a_i(N, \lambda)(i)_k \lambda^k T_{n-k+i,\lambda} \right) \frac{t^n}{n!}.
\]

(2.23)

By comparing the coefficients on both sides of (2.23), we obtain the following theorem.

**Theorem 2.2** For \( k, N = 0, 1, 2, \ldots \), we have

\[
T_{n,\lambda}^{(N+1)} = \frac{1}{N!} \sum_{i=0}^{N} \sum_{k=0}^{n} \binom{n}{k} a_i(N, \lambda)(i)_k \lambda^k T_{n-k+i,\lambda},
\]

where

\[
a_0(N, \lambda) = N! 2^N, \quad a_N(N) = 1,
\]

\[
a_i(N, \lambda) = \sum_{k_i=0}^{N-i} \sum_{k_{i-1}=0}^{N-i-k_i} \cdots \sum_{k_1=0}^{N-i-k_{i-1}} 2^{N-i} \times (N - k_i - k_{i-1} - \cdots - k_2 - k_1 - i)! \\
\times \left( N - k_i - k_{i-1} - \cdots - k_2 - k_1 - i \right) \binom{N-k_i-k_{i-1}-\cdots-k_2-k_1-i+2}{k_1+\lambda/2}.
\]

**References**


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