On the Nonexistence of Nontrivial Solutions for
Some P-Laplacian Equations

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Abstract
We use the Morawetz multiplier to show the nonexistence of nontrivial solutions of certain decay order for a p-Laplacian equation and a system of coupled p-Laplacian equations in the entire Euclidean space.

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1. Introduction

The study of the theory and the applications of the p-Laplacian equation

\[-\nabla \cdot (|\nabla u|^{p-2} \nabla u) = 0\]

and a more general p-Laplacian equation

\[-\nabla \cdot (|\nabla u|^{p-2} \nabla u) + f(x, u) = 0, \quad (1.1)\]
where $p > 1$, $u = u(x)$, $x \in \mathbb{R}^n$, $n \geq 2$, and $f$ is a continuous function, has been an active research topic. See, for example, Aronsson [1], Chabrowski and Fu [2], Diening, Lindqvist and Kawohl [3], Ma and Zhang [5], Wang [9], and Wu and Yang [10]. One of the interesting questions is about the nonexistence of nontrivial solutions for the equation (1.1). See, for example, Galakhov and Salieva [4], Ôtani [7], and Pucci and Servadei [8].

In this article, we will show that there are no nontrivial solutions of certain decay order for the equation (1.1) and the system of coupled general $p$-Laplacian equations

\[-\nabla \cdot (|\nabla u|^{p-2} \nabla u) + f(x, u) + \left[a(x)u^2 + b(x)v^2\right]u = 0,\]

\[-\nabla \cdot (|\nabla v|^{p-2} \nabla v) + g(x, v) + \left[c(x)u^2 + d(x)v^2\right]v = 0,\]

where $u = u(x)$, $v = v(x)$, $a(x)$, $b(x)$, $c(x)$, and $d(x)$ are real-valued functions, $x \in \mathbb{R}^n$, $n \geq 2$. The method is to use the Morawetz multiplier (Morawetz [6]). The conditions on $f(x, u)$, $g(x, v)$, $a(x)$, $b(x)$, $c(x)$, and $d(x)$ for no nontrivial solutions of certain decay order will be given.

As usual, $x = (x_1, x_2, ..., x_n)$, $\nabla u$ denotes the gradient of $u$, $\nabla \cdot u$ denotes the divergence of $u$, and $r = |x|$. We use the notation $u_r = \partial u/\partial r = (x/r) \cdot \nabla u$ and $\partial_j = \partial/\partial x_j$. $F_t(x, s)$ denotes $\partial F(x, s)/\partial t = (x/r) \cdot \nabla_x F(x, s)$. $C^k(\mathbb{R}^n)$ is the space of functions whose partial derivatives of order up to and including $k$ are continuously differentiable.

Define the following twelve sets of functions which we will use in this article:

\[\mathcal{P}_A^m(\mathbb{R}^n) = \{a \mid a \in C^1(\mathbb{R}^n), \sup_{|x| \geq \rho} (|x|^m |a(x)|) < \infty \text{ for some } \rho > 0 \text{ and } m > 0, \quad \text{and} \quad ra_t(x) - (n-2)a(x) \geq 0\},\]

\[\mathcal{N}_A^m(\mathbb{R}^n) = \{a \mid a \in C^1(\mathbb{R}^n), \sup_{|x| \geq \rho} (|x|^m |a(x)|) < \infty \text{ for some } \rho > 0 \text{ and } m > 0, \quad \text{and} \quad ra_t(x) - (n-2)a(x) \leq 0\}\]

\[\mathcal{P}_B^m(\mathbb{R}^n) = \{b \mid b \in C^1(\mathbb{R}^n), \sup_{|x| \geq \rho} (|x|^m |b(x)|) < \infty \text{ for some } \rho > 0 \text{ and } m > 0, \quad \text{and} \quad 2rb_t(x) - (n-3)b(x) \geq 0\}\]
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\[ \text{NB}_m(\mathbb{R}^n) = \{ b \mid b \in C^1(\mathbb{R}^n), \sup_{|x| \geq \rho} (|x|^{10} |b(x)|) < \infty \text{ for some } \rho > 0 \text{ and } m > 0, \text{ and} \]
\[ 2rb(x) - (n-3)b(x) \leq 0 \}, \]

\[ \text{E}_m(\mathbb{R}^n) = \{ a \mid a \in C^1(\mathbb{R}^n), \sup_{|x| \geq \rho} (|x|^m |a(x)|) < \infty \text{ for some } \rho > 0 \text{ and } m > 0 \}, \]

\[ \text{D}_{h,k}(\mathbb{R}^n) = \{ u \mid u \in C^k(\mathbb{R}^n), \lim_{R \to \infty} \sup_{|x| = R} (|x|^n |D^\alpha u(x)|) = 0, \text{ for all multi-indices } \alpha \text{ and } \beta \in N^0_n \text{ such that } |\alpha| \leq h \text{ and } |\beta| \leq k \}, \]

\[ \text{F}_m(\mathbb{R}^n) = \{ u \mid \lim_{R \to \infty} \sup_{|x| = R} |F(x, u(x))| = 0 \}, \]

\[ \text{G}_m(\mathbb{R}^n) = \{ v \mid \lim_{R \to \infty} \sup_{|x| = R} |G(x, v(x))| = 0 \}, \]

\[ \text{PF}(\mathbb{R}^n) = \{ u \mid \int_{\mathbb{R}^n} [nF(x, u(x)) + rF_1(x, u(x)) - ((n-1)/2)f(x, u(x))u(x)] \, dx \geq 0 \}, \]

\[ \text{PG}(\mathbb{R}^n) = \{ v \mid \int_{\mathbb{R}^n} [nG(x, v(x)) + rG_1(x, v(x)) - ((n-1)/2)g(x, v(x))v(x)] \, dx \geq 0 \}, \]

\[ \text{NF}(\mathbb{R}^n) = \{ u \mid \int_{\mathbb{R}^n} [nF(x, u(x)) + rF_1(x, u(x)) - ((n-1)/2)f(x, u(x))u(x)] \, dx \leq 0 \}, \]

and

\[ \text{NG}(\mathbb{R}^n) = \{ v \mid \int_{\mathbb{R}^n} [nG(x, v(x)) + rG_1(x, v(x)) - ((n-1)/2)g(x, v(x))v(x)] \, dx \leq 0 \}. \]

**Remark 1.** A function \( u \) is said to be of decay order \( (h, k) \) if and only if \( u \in \text{D}_{h,k}(\mathbb{R}^n) \).

**Remark 2.** \( \text{PA}_m(\mathbb{R}^n), \text{NA}_m(\mathbb{R}^n), \text{PB}_m(\mathbb{R}^n), \) and \( \text{NB}_m(\mathbb{R}^n) \) are all subsets of \( \text{E}_m(\mathbb{R}^n) \).

All the functions are assumed to be real-valued in this article.

### 2. A general \( p \)-Laplacian equation

Consider the general \( p \)-Laplacian equation
\[-\nabla \cdot (|\nabla u|^{p-2} \nabla u) + f(x, u) = 0, \tag{2.1}\]

where \( f \) is a continuous function.

Let \( F(x, u) \) be the antiderivative of \( f(x, u) \) with respect to \( u \) such that \( F(x, 0) = 0 \).

Multiplying Equation (2.1) by \( \zeta(u_r + ((n-1)u/(2r))) \), where \( \zeta \in C^1(\mathbb{R}^n) \) and \( \zeta(x) = \zeta(|x|) = \zeta(r) \), we get

\[0 = (-\nabla \cdot (|\nabla u|^{p-2} \nabla u) + f(x, u)) \zeta(u_r + ((n-1)u/(2r))) = \nabla \cdot Y + Z, \tag{2.2}\]

where

\[Y \text{ depends on } \zeta, 1/r, u, \nabla u, \text{ and } F(x, u)\]

and

\[Z = -((\zeta' - \zeta)|\nabla u|^{p-2} [(u_r)^2 + ((n-1)/(2r)) uu] - [(\zeta'/(p) - (\zeta(2pr))(n+1)p - 2(n-1))]|\nabla u|^{p-2} - (\zeta' + ((n-1)/r)\zeta)F(x, u) + ((n-1)/(2r))\zeta f(x, u)u - \zeta F_r(x, u).\]

**Theorem 1**

Let \( u \) be a \( C^2 \) solution of (2.1).

(a) Assume \( 1 < p < 2n/(n + 1) \). If \( u \in D_n, 1(\mathbb{R}^n) \cap F_n(\mathbb{R}^n) \cap \text{PF}(\mathbb{R}^n) \), then \( u \equiv 0 \).

(b) Assume \( p > 2n/(n + 1) \). If \( u \in D_n, 1(\mathbb{R}^n) \cap F_n(\mathbb{R}^n) \cap \text{NF}(\mathbb{R}^n) \), then \( u \equiv 0 \).

(c) Assume \( p = 2n/(n + 1) \). If \( u \in D_n, 1(\mathbb{R}^n) \cap F_n(\mathbb{R}^n) \), then we have a constraint

\[\int_{\mathbb{R}^n} [nF(x, u) - ((n-1)/2)f(x, u)u + rF_r(x, u)] \, dx = 0.\]

**Proof:**

Let \( R > 0 \). Integrating both sides of (2.2) in \( |x| \leq R \) and using the Divergence theorem, we get

\[\int_{|x|=R} Y \cdot (x/R) \, ds + \int_{|x|<R} Z \, dx = 0.\]

Let \( \zeta(x) = \zeta(|x|) = \zeta(r) = r \). Then we have
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\[
\int_{|x| = R} -R|\nabla u|^{p-2}(\nabla u)(u_r + ((n - 1)u/(2R))) + (x/p)|\nabla u|^p + xF(x, u) \cdot (x/R) \, ds
\]

\[
+ \int_{|x| \leq R} \left\{ \frac{1}{p} - \frac{1}{2p}((n+1)p - 2(n-1))|\nabla u|^p - nF(x, u) + ((n-1)/2)f(x, u)u - rF_r(x, u) \right\} \, dx = 0.
\]

Let \( R \to \infty \). We get

\[
\int_{\mathbb{R}^n} \left\{ \frac{(2n - (n + 1)p)/(2p)|\nabla u|^p - nF(x, u) + ((n-1)/2)f(x, u)u - rF_r(x, u) \right\} \, dx = 0,
\]

since \( u \) is in \( D_{n,1}(\mathbb{R}^n) \cap F_0(\mathbb{R}^n) \). Now we rewrite the above equation as

\[
\int_{\mathbb{R}^n} \left[ nF(x, u) - ((n-1)/2)f(x, u)u + rF_r(x, u) \right] \, dx
\]

(2.3)

For the case of assertion (a), since \( p < 2n/(n + 1) \) and \( u \) is in \( PF(\mathbb{R}^n) \), the left-hand side of the above equation (2.3) is non-positive and the right-hand side of it is non-negative. Therefore, both sides of the equation (2.3) must be zero.

Thus \( \int_{\mathbb{R}^n} \left((2n - (n + 1)p)/(2p)|\nabla u|^p \, dx = 0.\)

Since \( u \) is of decay order \( (n, 1) \), \( u \equiv 0.\)

For the case of assertion (b), the argument is similar to the above argument. In this case, the left-hand side of the equation (2.3) is non-negative and the right-hand side of it is non-positive. Therefore, both sides of the equation (2.3) must be zero.

Thus \( \int_{\mathbb{R}^n} \left((2n - (n + 1)p)/(2p)|\nabla u|^p \, dx = 0.\)

Since \( u \) is of decay order \( (n, 1) \), \( u \equiv 0.\)

Assertion (c) follows from the equation (2.3).
Remark 3. As an example for Theorem 1, we take \( f(x, u) = q(x)|u|^{s-1}u, \ s \geq 1 \). Then \( F(x, u) = (1/(s + 1))q(x)|u|^{s+1} \), where \( q(x) = q(|x|) = q(r) \).

Assume \( \sup_{|x| \geq h} (|x|^m|q(x)|) < \infty \) for some \( h > 0 \) and \( m \geq 0 \).

For \( u \) to be in \( F_n(R^n) \), we need

\[
\lim_{R \to \infty} (\sup_{|x| = R} |F(x, u(x))|) = 0,
\]

that is,

\[
\lim_{R \to \infty} (\sup_{|x| = R} q(|x|)|u(x)|^{s+1}) = 0.
\]

This would be satisfied if

\[
\lim_{R \to \infty} (\sup_{|x| = R} |x|^m|u(x)|^{s+1}) = 0.
\] (2.4)

The above condition (2.4) would be satisfied if \( u \) is of decay order \( (n+m, 0) \).

As for \( u \) to be in \( PF(R^n) \), since

\[
nF(x, u) + rF(x, u) - ((n-1)/2)f(x, u)u = [(n/(s + 1))q+ (r/(s + 1))q- ((n-1)/2)q] |u|^{s+1},
\]

\( u \) would be in \( PF(R^n) \) if \( (n/(s + 1))q+ (r/(s + 1))q- ((n-1)/2)q \geq 0 \).

Thus, if \( rq \geq - ((n - ns + s +1)/2)q \), then \( u \) is in \( PF(R^n) \).

Therefore, if \( u \) is of decay order \( (n + m, 1) \) and \( rq \geq - ((n - ns + s +1)/2)q \), \( u \) satisfies the assumptions of Theorem 1 (a) on \( u \). Similarly, if \( u \) is of decay order \( (n + m, 1) \) and \( rq \leq - ((n - ns + s +1)/2)q \), then \( u \) satisfies the assumptions of Theorem 1(b) on \( u \).

Remark 4. Similar conclusions can be derived for \( f(x, u) = q_1(x)|u|^{a-1}u + q_2(x)|u|^{b-1}u, \) where \( a > b \geq 1 \).

3. A system of coupled general p-Laplacian equations

Consider the system of coupled nonlinear p-Laplacian equations
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\[-\nabla \cdot (|u|^{p-2} \nabla u) + f(x, u) + [a(x)u^2 + b(x)v^2]u = 0, \quad (3.1.a)\]

\[-\nabla \cdot (|v|^{p-2} \nabla v) + g(x, v) + [c(x)u^2 + d(x)v^2]v = 0, \quad (3.1.b)\]

where \(u = u(x), v = v(x), a(x), b(x), c(x), \) and \(d(x)\) are real valued functions, \(x \in \mathbb{R}^n, n \geq 2.\) We assume \(b(x) = c(x).\)

Let \(F(x, u)\) and \(G(x, v)\) be the antiderivatives of \(f(x, u)\) with respect to \(u\) and \(g(x, v)\) with respect to \(v,\) respectively, such that \(F(x, 0) = 0\) and \(G(x, 0) = 0.\)

Multiplying Equation (3.1.a) by \(\zeta(u + ((n - 1)u)/(2r))\) and (3.1.b) by \(\zeta(v + ((n - 1)v)/(2r))\) and then adding them up, where \(\zeta \in C^1(\mathbb{R}^n)\) and \(\zeta(x) = \zeta(|x|) = \zeta(r),\) we get

\[0 = \left( -\nabla \cdot (|u|^p \nabla u) + f(x, u) + [a(x)u^2 + b(x)v^2]u \right) \zeta(u + ((n - 1)u)/(2r)) \]
\[\quad + \left( -\nabla \cdot (|v|^p \nabla v) + g(x, v) + [c(x)u^2 + d(x)v^2]v \right) \zeta(v + ((n - 1)v)/(2r)) \]
\[= \nabla \cdot Y + Z, \quad (3.2)\]

where

\[Y \text{ depends on } a(x), b(x), d(x), \zeta, 1/r, u, \nabla u, \text{ and } F(x, u), v, \nabla v, \text{ and } G(x, v), \text{ and} \]

\[Z = -((\zeta/r - \zeta))|u|^p [(u)^2 + ((n - 1)/(2r))u,u] - [(\zeta'/p) - (\zeta/(2pr))((n + 1)p - 2(n - 1))]|u|^p - ((\zeta - \zeta'))|v|^p [(v)^2 + ((n - 1)/(2r))v,v] - [(\zeta'/p) - (\zeta/(2pr))((n + 1)p - 2(n - 1))]|v|^p - [\zeta' + ((n - 1)/r)\zeta](F(x, u) + G(x, v)) \]
\[+ ((n - 1)/(2r))[F(x, u)u + g(x, v)v] - [\zeta(F(x, u) + G(x, v))] - (1/4)[\zeta' - ((n - 1)/r)\zeta](a(x)u^4 + d(x)v^4) - (1/4)(a(x)u^4 + d(x)v^4) - (1/2)[b(x)\zeta + b(x)\zeta'] \]
\[+ ((n - 1)/(2r))b(x)\zeta]|u|^2v^2. \]

**Theorem 2**

Let \(u\) and \(v\) be \(C^2\) solutions of (3.1.a) and (3.1.b).

(a) Assume \(1 < p < 2n/(n + 1), a \in PA_k(\mathbb{R}^n), b \in PB_m(\mathbb{R}^n),\) and \(d \in PA_h(\mathbb{R}^n).\) Let \(j = \max \{k, m, h, n\}.\)

If \(u \in D_{j, 1}(\mathbb{R}^n) \cap F_{j}(\mathbb{R}^n) \cap PF(\mathbb{R}^n)\) and \(v \in D_{j, 1}(\mathbb{R}^n) \cap G_{j}(\mathbb{R}^n) \cap PG(\mathbb{R}^n),\) then \(u \equiv 0.\)

(b) Assume \(p > 2n/(n + 1), a \in NA_k(\mathbb{R}^n), b \in NB_m(\mathbb{R}^n),\) and \(d \in NA_h(\mathbb{R}^n).\) Let \(j = \max \{k, m, h, n\}.\)
If \( u \in D_{j,1}(\mathbb{R}^n) \cap F_n(\mathbb{R}^n) \cap NF(\mathbb{R}^n) \) and \( v \in D_{j,1}(\mathbb{R}^n) \cap G_n(\mathbb{R}^n) \cap NG(\mathbb{R}^n) \), then \( u \equiv 0 \).

(c) Assume \( p = \frac{2n}{n+1} \), \( a \in E_k(\mathbb{R}^n) \), \( b \in E_m(\mathbb{R}^n) \), and \( d \in E_h(\mathbb{R}^n) \). Let \( j = \max \{ k, m, h, n \} \).

If \( u \in D_{j,1}(\mathbb{R}^n) \cap F_n(\mathbb{R}^n) \) and \( v \in D_{j,1}(\mathbb{R}^n) \cap G_n(\mathbb{R}^n) \), then we have a constraint

\[
\int_{\mathbb{R}^n} \left\{ \left[ \frac{(2n-(n+1)p)}{2p} \right] |\nabla u|^p - \left[ \frac{(2n-(n+1)p)}{2p} \right] |\nabla v|^p - \frac{1}{4} [ra(x) - (n-2)a(x)]u^4 + \frac{1}{4} [rd(x) - (n-2)d(x)]v^4 - \frac{1}{2} [rb(x) - (n-3)/2b(x)]u^2v^2 \right\} \, dx = 0.
\]

**Proof:**

Let \( \zeta(x) = \zeta(|x|) = \zeta(r) = r \). Following the same steps as in the proof of Theorem 1. We first integrate both sides of (3.2) in \(|x| \leq R\) and then let \( R \rightarrow \infty \) to get

\[
0 = \int_{\mathbb{R}^n} \left\{ \left[ \frac{(2n-(n+1)p)}{2p} \right] |\nabla u|^p - \left[ \frac{(2n-(n+1)p)}{2p} \right] |\nabla v|^p - \frac{1}{4} [ra(x) - (n-2)a(x)]u^4 + \frac{1}{4} [rd(x) - (n-2)d(x)]v^4 - \frac{1}{2} [rb(x) - (n-3)/2b(x)]u^2v^2 \right\} \, dx,
\]

since \( a \in E_k(\mathbb{R}^n) \), \( b \in E_m(\mathbb{R}^n) \), \( d \in E_h(\mathbb{R}^n) \), and \( v \in D_{j,1}(\mathbb{R}^n) \cap G_n(\mathbb{R}^n) \).

We now rewrite the above equation (3.3) into the form

\[
\int_{\mathbb{R}^n} \left\{ \left[ \frac{(2n-(n+1)p)}{2p} \right] |\nabla u|^p + \left[ \frac{(2n-(n+1)p)}{2p} \right] |\nabla v|^p \right\} \, dx
\]

\[
= \int_{\mathbb{R}^n} \left\{ \left[ \frac{(2n-(n+1)p)}{2p} \right] |\nabla F(x, u)| + \left[ \frac{(2n-(n+1)p)}{2p} \right] |\nabla G(x, v)| + \frac{1}{4} [ra(x) - (n-2)a(x)]u^4 + \frac{1}{4} [rd(x) - (n-2)d(x)]v^4 + \frac{1}{2} [rb(x) - (n-3)/2b(x)]u^2v^2 \right\} \, dx.
\]
The rest of the proof follows from (3.4) and is similar to the proof of Theorem 1.

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**References**


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