Homoclinic Solutions for Singular Impulsive Second Order Differential Equations

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Abstract

In this work we deal with the existence of homoclinic solutions for a class of second order nonautonomous differential equations under impulses effects when the nonlinear term presents a singularity. By simple sufficient conditions, we obtain homoclinic solutions, which are derived as the limit of periodic solutions of approximate problems. We use a variational approach.

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1 Introduction

Impulsive differential equations are recognized as adequate models to study the evolution of processes that are subject to sudden changes in their states. Processes with such a character arise naturally and often, especially in engineering, biotechnology and medicine ([12], [14]) optimal control ([15], [23]) and population dynamics [29]. For this reason, the theory of impulsive differential equations has become an important area of investigation in recent years. For an application of the basic theory of impulsive differential equations see ([11] and [3]).

Some classical tools have been used to study such problems in literature, such
as the coincidence degree theory [27], the method of upper and lower solutions with the monotone iterative technique, some fixed point theorems in cones see ([2], [10], [31]) and variational methods, see for instance ([30], [40], [41], [43]). In recent 40 years, variational methods are also widely applied to the existence of homoclinic orbits for Hamiltonian systems, for examples, ([1], [28], [33], [42]), however, the existence of homoclinic solutions for the impulsive equations is paid little attention. It is well known that the homoclinic orbit rupture phenomenon can lead to chaos, which has been studied in [26], [44]. In literature, Coti-Zelati [13] used dual variational methods, and Lions [24] Hofer and Wysocki [17] employed concentration compactness method with Ekeland variational principle, that they established the existence of homoclinic solutions of the first-order Hamiltonian systems. The case of singular Hamiltonian systems is rather important, due to their applications, in fact some potentials arising in physics (celestial mechanics [22]), present singularities.

It seems there are not many works on homoclinic solutions for singular differential equations. Let give a brief introduction about previous research. In 1996, in [38], P.H. Rabinowitz investigated a nonautonomous planar second order Hamiltonian system \( \ddot{q}(t) + V_q(t, q(t)) = 0 \). Assuming that \( V : \mathbb{R} \times (\mathbb{R}^2 \setminus \{\xi\}) \to \mathbb{R} \), is periodic with respect to real variable \( t \), singular at \( \xi \) and possesses a global maximum at the origin, he proved that there exist at least two homoclinic solutions: at least one of a positive rotation and at least one of a negative rotation. In [8], by the extra assumption about the existence of a minimal non contractible periodic orbit around \( \xi \) due to S. Bolotin [5], P. Caldiroli and L. Jeanjean established that if \( V \) does not depend on time variable, then for each \( k \in \mathbb{Z} \) there exists a homoclinic solution of rotation \( k \), ([4], [9]). In [6] M. Borges considered a planar second order Hamiltonian system with a potential having a global maximum at the origin and two singularities at points \( \xi_1 \) and \( \xi_2 \). By the additional assumptions that \( V : \mathbb{R}^2 \setminus \{\xi_1, \xi_2\} \to \mathbb{R} \) is of class \( C^2 \) and the second derivative of \( V \) at \( 0 \) is negative definite, she found homoclinic solutions winding around each singularity and around both singularities, periodic solutions and heteroclinic solutions joining \( 0 \) to periodic solutions. For \( n > 2 \) and \( V(q) \), the existence of homoclinic was shown by K. Tanaka in [39]. And, in [37] Rabinowitz proved the existence of so-called multi bump homoclinic solutions for a family of singular Hamiltonian systems in \( \mathbb{R}^2 \) which are subjected to almost periodic forcing in time [36].

Recently, in [20] J.Janczewska considers a class of second order Hamiltonian systems, where the potential has a singularity at a point \( \xi \in \mathbb{R}^2 \) and the unique global maximum \( 0 \in \mathbb{R} \) that is achieved at two distinct points \( a, b \in \mathbb{R}^2 \setminus \{\xi\} \). For a class of potentials that satisfy a strong force condition introduced by Gordon.W.B [16], via minimization of action integrals, she establish the existence of at least two solutions which wind around \( \xi \). One of
them, is a heteroclinic. and the second is either homoclinic or heteroclinic possessing a rotation number (a winding number) different from the first one. For more works see ([18], [21], [19]) and the references therein.

In this work, we study the existence of homoclinic solutions in presence of impulses, for a singular second order differential equations under assumptions different from those considered in the above papers: the nonlinearity does not possess a global maximum at the origin. In our setting unlike that of [34] and [35] we do not require any isolateness of the periodic orbit. Recent results in the literature are generalized and significantly improved.

The remain of paper is organized as follows, Section 2 contains the formulation of problem and the resolution process. Section 3 is devoted to the main result and an example is given at the end.

2 Preliminaries

In this work, we consider the following problem,

\[
\begin{aligned}
-u''(t) + u(t) &= g(t, u(t)) \quad t \neq t_j, \ t \in \mathbb{R} \\
\Delta u'(t_j) &= I_j(u(t_j)); \ j \in A \\
\lim_{t \to \pm \infty} u(t) &= \lim_{t \to \pm \infty} u'(t) = 0
\end{aligned}
\]

With, \( \Delta u'(t_j) = u'(t_j^+) - u'(t_j^-) \) and \( u'(t_j^+) = \lim_{t \to t_j^+} u'(t) \); \( t_j \) are the instants where the impulses occur, \( j \in A \), where \( A \) is a bounded subset of \( \mathbb{Z} \) with \( (\text{card}(A) = N) \). \(-T = t_0 < t_1 < t_2 < \ldots < t_{p-1} < t_p = T, \ t_{j+kp} = t_j + 2kT; \ k \in \mathbb{Z} \) and such that \( j + kp \in A \) (\( T \) is a fixed positive constant). The functions \( I_j \) represent the jump discontinuities of \( u' \) at the impulse moments.

The problem (1) is considered under the following assumptions on functions \( g \) and \( I_j \):

\begin{enumerate}
\item[(H₁)] \( g : \mathbb{R} \times ]\xi, +\infty[ \to \mathbb{R} \) is a \( L^1_{\text{loc}} \) and \( 2T \)-periodic in \( t \); and continuous in its second argument with \( \lim g(t, u) = -\infty \), where \( \xi < 0 \).
\item[(ii)] \( K := \sup_{u \in [\xi, +\infty[} g(., u) \) is in \( L^1(\mathbb{R}) \).
\item[(iii)] \( \lim_{u \to 0^-} \frac{g(t, u)}{u} = +\infty \), for almost every \( t \in \mathbb{R} \).
\item[(iv)] \( \lim_{u \to +\infty} \frac{g(t, u)}{u} = 0 \) for almost every \( t \in \mathbb{R} \).
\end{enumerate}

\begin{enumerate}
\item[(H₂)] \( I_j : \mathbb{R} \to \mathbb{R} \), is a bounded continuous function such that, \( \max I_j(s) < 0 \) and \( I_{j+p} \equiv I_j \), for every \( j \in A \).
\end{enumerate}

Remark 1 There exist real constants \( m, M \) such that for every \( j \in A; \ m < I_j < M < 0 \)
Clearly, when the nonlinearity \( g(t, u) \) is \( 2T \)-periodic in \( t \), it is natural to look for homoclinic solutions as limits of \( 2nT \)-periodic solutions (subharmonic solutions) as \( n \to \infty \). So, homoclinic solutions of the problem (1), are considered as the limit, as \( n \to +\infty \), of periodic extensions of solutions \( u_n \) of approximating periodic problems (2), defined in the intervals \( I_n := [-nT, nT] \), by

\[
\begin{align*}
-u''(t) + u(t) &= g(t, u(t)) \quad t \neq t_j, t \in I_n \quad (1.1)_n \\
\Delta u'(t_j) &= I_j(u(t_j)) \quad t_j \in I_n \quad (1.2)_n, \\
u(nT) - u(-nT) &= u'(nT) - u'(-nT) = 0 \quad (1.3)_n
\end{align*}
\]

Throughout this paper we shall use the following notations, \( L^\infty_{2nT}(\mathbb{R}, \mathbb{R}) \), is the space of \( 2nT \) periodic essentially bounded measurable functions from \( \mathbb{R} \) into \( \mathbb{R} \), endowed with the norm, \( ||u||_{L^\infty_{2nT}} = \operatorname{ess \ sup}\{|u(t)| : t \in I_n\} \), \( L^p(I_n) \) is the classical Lebesgue space of functions \( u : I_n \to \mathbb{R} \) such that \( |u(\cdot)|^p \) is integrable, and for \( u \in L^p(I_n) \) its norm is given by,

\[
||u||_{L^p} = \left( \int_{-nT}^{nT} |u(t)|^p \, dt \right)^{\frac{1}{p}}
\]

For \( n \in \mathbb{N}^* \), we consider the Sobolev space \( H_n = \{ u \in W^{1,2}(I_n, \mathbb{R}) ; u(-nT) = u(nT) \} \), endowed with the inner product

\[
(u, v) = \int_{-nT}^{nT} [u'(t)v'(t) + u(t)v(t)] \, dt
\]

inducing the norm

\[
||u|| = \left( ||u'||_{L^2}^2 + ||u||_{L^2}^2 \right)^{\frac{1}{2}}
\]

\((H_n; ||\cdot||)\) is a reflexive Hilbert space. Also, \( H_n \) admits the orthogonal decomposition, \( H_n = E_n + F_n \), where \( F_n \) is the subspace of constant functions in \( H_n \) and \( E_n \) denotes the subspace of zero mean value functions in \( H_n \). \( E_n \) is a weakly closed subspace of \( H_n \).

The definition of solution for problem (2) is given by:

**Definition 2** \( u \in H_n \) is a solution of (2) that \( u \in C(I_n) \) such that for every \( j \), \( u_j = u|_{(t_j, t_{j+1})} \in H^2((t_j, t_{j+1})) \) satisfies the equation \((1.1)_n\) for a.e. \( t \in I_n \) \( (t \neq t_j) \), and the limits \( u'(t_j^-), u'(t_j^+) \) exist, and \((1.2)_n - (1.3)_n\) are satisfied.

The following lemma introduces the embedding of \( L^\infty_{2nT} \) in \( H_n \).
Lemma 3 see (lemma1.1 and corollary 1.1 of [45]): Let $u : \mathbb{R} \to \mathbb{R}$ be a continuous mapping such that $u' \in L^2_{loc}(\mathbb{R}, \mathbb{R})$. For every $t \in \mathbb{R}$ the following inequality holds:

$$|u(t)| \leq 2\left(\int_{t-\frac{1}{2}}^{t+\frac{1}{2}} (\dot{u}^2(t) + u^2(t))dt\right)^{\frac{1}{2}},$$

(3)

And, if $u \in H_n$, the following inequality holds:

$$||u||_{L^\infty_{2nT}} \leq 2||u||,$$

(4)

3 Main result

In this section we state and prove an existence result of $2nT$ periodic solutions, $n \in \mathbb{N}^*$ for problem (2), and an another one for problem (1).

3.1 Existence of solutions for (2)

We have the following theorem,

Theorem 4 If $(H_1)$ and $(H_2)$ hold, then for every $n \in \mathbb{N}^*$ the problem (2) possesses at least one non constant solution.

Proof. The proof is made in several steps,

step1 : modification of the problem (2).

Let denote by $d = d(\xi, 0) = |\xi|$ the distance between $\xi$ and 0 and $\gamma = \inf(d, 1)$, for $\beta \in (\xi, \xi + \gamma)$, we define the following truncation function $g_\beta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, by

$$f(t, u) = g_\beta(t, s) = \begin{cases} g(t, s) & \text{if } s > \beta \\ g(t, \beta) & \text{if } s \leq \beta \end{cases},$$

(5)

and the corresponding modified problem,

$$\begin{cases} -u''(t) + u(t) = f(t, u(t)) & t \neq t_j, t \in I_n \\ \Delta u(t_j) = I_j(u(t_j)) \text{, } t_j \in I_n \\ u(nT) - u(-nT) = u'(nT) - u'(-nT) = 0 \end{cases} \quad (1.1)_n, (1.2)_n, (1.3)_n$$

From $(H_1)$ the function $f$ satisfies the following assumption:

$(H_1)'$ (i) $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a $L^1_{loc}$ and $2T$–periodic function with respect to its first argument and continuous in the second one ,

$(ii) K' := \sup_{u \in \mathbb{R}} f(., u)$ is in $L^1(\mathbb{R})$ ,

$(iii) \lim_{u \to 0} \frac{f(t, u)}{u} = +\infty$, for almost every $t \in \mathbb{R}$

$(iv) \lim_{|u| \to +\infty} \frac{f(t, u)}{u} = 0$, for almost every $t \in \mathbb{R}$.
Step 2: Variational formulation

We study the existence of solutions for the problem (6) by converting the problem to the existence of critical points of some variational structure. And, to obtain non-constant solution we prove the existence of critical points in $E_n$. For this, we define the functional $\varphi_n : H_n \to \mathbb{R}$, by

$$\varphi_n(u) := \frac{1}{2}||u||^2 + \sum_{t_j \in I_n} \int_0^{u(t_j)} I_j(s)ds - \int_{-nT}^{nT} F(t,u(t))dt,$$

(7)

where $F(t,u) := \int_0^u f(t,s)ds$.

$\varphi_n$ is well defined on $H_n$, by $(H_1)'$ and $(H_2)$, $\varphi_n$ is weakly lower semi-continuous and it is a differentiable functional whose derivative is the functional $\varphi_n'(u)$, defined for all $v \in H_n$, by

$$\varphi_n'(u)(v) = \int_{-nT}^{nT} [u'(t)v'(t) + u(t)v(t)]dt + \sum_{t_j \in I_n} I_j(u(t_j))v(t_j) - \int_{-nT}^{nT} f(t,u(t))v(t) dt,$$

(8)

the following result characterizes the critical points of $\varphi_n$.

**Lemma 5** The critical points of $\varphi_n$ are solutions of (6).

**Proof.** Let $u \in H_n$ be a critical point of the functional $\varphi_n$, we have for all $v \in H_n$,

$$\varphi_n'(u)(v) = \int_{-nT}^{nT} [u'(t)v'(t) + u(t)v(t)]dt + \sum_{t_j \in I_n} I_j(u(t_j))v(t_j)$$

$$- \int_{-nT}^{nT} f(t,u(t))v(t) dt = 0.$$  

(9)

For any $t_j \in I_n$, we consider $v \in W_0^{1,2}(]t_j, t_{j+1}[) \setminus \{0\}$ such that

$$v(t) = \begin{cases} v(t) & \text{if } t \in ]t_j, t_{j+1}[ \\ 0 & \text{if } t \in I_n \setminus ]t_j, t_{j+1}[ \end{cases}$$

(10)

Then (9) implies

$$\int_{t_j}^{t_{j+1}} [u'(t)v'(t) + u(t)v(t)]dt = \int_{t_j}^{t_{j+1}} f(t,u(t))v(t)dt$$

This means, for any $w \in W_0^{1,2}(t_j, t_{j+1})$, we have

$$\int_{t_j}^{t_{j+1}} [u'(t)w'(t) + u(t)w(t)]dt = \int_{t_j}^{t_{j+1}} f(t,u(t))w(t)dt$$
where \( u_j = u\big|_{(t_j, t_{j+1})} \), thus for \( t \in (t_j, t_{j+1}) \), \( u_j \) is a weak solution of the equation:

\[
-u''(t) + u(t) = f(t, u(t)),
\]

and \( u_j \in H^2((t_j, t_{j+1})) \subset C^1((t_j, t_{j+1})) \).

Indeed, \( h(t) = -u(t) + f(t, u(t)) \), the function \( h \) is \( L^1(I_n) \) then (11) becomes of the following form

\[
-u''(t) = h(t) \text{ on } (t_j, t_{j+1})
\]

Then the solution of (12) can be written as

\[
u(t) = c_1 + c_2 t + \int_{t_j}^{t} \int_{t_j}^{s} h(\tau) d\tau ds \quad \text{for } t \in (t_j, t_{j+1})
\]

where \( c_1 \) and \( c_2 \) are two constants. Then \( u_j \in H^2(t_j, t_{j+1}) \) and the limits \( u'(t^-_j), u'(t^+_j) \) exist.

On the other hand, by the choice of \( v \) (10) we have

\[
\int_{-nT}^{nT} [u'(t)v'(t) + u(t)v(t)] dt - \int_{-nT}^{nT} f(t, u(t))v(t) dt = 0
\]

where \( t_n \in I_n \), is the first instant where the impulses occur on \( I_n \), and

\[
\int_{t_n}^{nT} [u'(t)v'(t) + u(t)v(t)] dt = \int_{t_n}^{nT} f(t, u(t))v(t) dt
\]

where \( t_n^* \in I_n \), is the last instant where the impulses occur on \( I_n \). Therefore, \( u \) satisfies the equation \((1.1)_n \) a.e on \( I_n \).

Integrating per parts \( \int_{-nT}^{nT} u'(t)v(t) dt \), we obtain:

\[
\int_{-nT}^{nT} u'(t)v(t) dt = \sum_{j=0}^{p} \int_{t_j^-}^{t_j^+} u'(t)v(t) dt,
\]

\[
= \sum_{j=0}^{p} u'(t)v(t) \bigg|_{t_j^-}^{t_j^+} - \int_{-nT}^{nT} u''(t)v(t) dt
\]

\[
= \sum_{j=0}^{p} u'(t_{j+1})v(t_{j+1}) - u'(t_j)v(t_j) - \int_{-nT}^{nT} u''(t)v(t) dt
\]

\[
= - \sum_{j=0}^{p} \Delta u(t_j)v(t_j) - u'(-nT)v(-nT) + u'(nT)v(nT) - \int_{-nT}^{nT} u''(t)v(t) dt
\]
Since \( u \) satisfies the equation \((1.1)_n \) of the problem \((6)\), a.e. on \( I_n \), \((13)\), implies,

\[
\sum_{t_j \in I_n} (I_j(u(t_j)) - \Delta u'(t_j))v(t_j) + u'(nT)v(nT) - u'(-nT)v(-nT) = 0
\]

Next, we will show that \( u \) satisfies \((1.2)_n \) of the problem \((6)\). Suppose on the contrary that, (without loss of generality), there exists \( t_k \in I_n \), such that

\[
I_k(u(t_k)) - \Delta(u'(t_k)) \neq 0
\]

Let

\[
v(t) := \prod_{t_j \in I_n, j \neq k} (t - t_j)(t^2 - n^2T^2),
\]

obviously, \( v \in H_n \). By simple calculations, we obtain \( v(t_j) = 0 \) for \( t_j \in I_n \), \( j \neq k \). Then, by \((14)\), we get

\[
0 = \sum_{t_j \in I_n} (I_j(u(t_j)) - \Delta u'(t_j))v(t_j) + u'(nT)v(nT) - u'(-nT)v(-nT)
\]

\[
= (I_k(u(t_k)) - \Delta u'(t_k)) \prod_{t_j \in I_n, j \neq k} (t_k^2 - t_j^2)(t_k^2 - n^2T^2)
\]

Which contradicts \((15)\). So \( u \) satisfies \((1.2)_n \) of the problem \((6)\).

Now, for \( w \in H_n \) such that \( w(t) := \prod_{t_j \in I_n} (t^2 - t_j^2) \), we have, \( w(t_j) = 0 \) for \( t_j \in I_n \), and \( w(\pm nT) := \prod_{t_j \in I_n} (n^2T^2 - t_j^2) > 0 \). Then \((14)\) becomes of the following form:

\[
\left[ u'(nT) - u'(-nT) \right] w(nT) = 0
\]

\((17)\) shows that \( u'(nT) - u'(-nT) = 0 \). Therefore, \( u \) is a solution of the problem \((6)\).

The following auxiliary result gives a property of \( \varphi_n \).
Lemma 6 \( \varphi_n \) is coercive on \( E_n \).

**Proof.** First, from (\( H_1' \))', we establish a property for \( f \). The assumption (\( H_1' \))(iv), implies that for every \( \varepsilon, 0 < \varepsilon < 1 \), there exists a \( \delta = \delta (\varepsilon) > 0 \) such that, for almost every \( t \in I_n \), we have

\[
|f(t, s)| < \varepsilon |s|,
\]
whenever \( |s| > \delta \).

By (\( H_1' \))(i), \( f \) is continuous in \( u \), this implies that, for almost every \( t \) in \([-nT, nT]\),

\[
|f(t, s)| < \varepsilon |s| + \max_{|s| \leq \delta} |f(t, s)|,
\]
whenever \( |s| > \delta \).

An integration of (18) yields to,

\[
|F(t, u)| < \frac{\varepsilon}{2} |u|^2 + \max_{|s| \leq \delta} |f(t, s)| |u|
\]

By (\( H_1' \))(i) we have \( \max_{|s| \leq \delta} |f(t, s)| = |f(t, s_0)| \), for some \( s_0 \in (-\delta, \delta) \), let put \( C(t) := |f(t, s_0)| \), then

\[
|F(t, u)| < \frac{\varepsilon}{2} |u|^2 + C(t) |u|
\]

where \( C(t) \in L^1(I_n, \mathbb{R}) \), is a positive function.

Now, in view of (20) and (\( H_2 \))(\( m < 0 \)), we have for \( u \) in \( E_n \),

\[
\varphi_n(u) = \int_{-nT}^{nT} \left( \frac{1}{2} |u'(t)|^2 + |u(t)|^2 - F(t, u(t)) \right) dt + \sum_{t_j \in I_n} \int_{0}^{u(t_j)} I_j(s) ds
\]

\[
\geq \int_{-nT}^{nT} \frac{1}{2} |u'(t)|^2 + |u(t)|^2 - \frac{\varepsilon}{2} |u(t)|^2 dt - \int_{-nT}^{nT} C(t) u(t) dt + \sum_{t_j \in I_n} \int_{0}^{u(t_j)} I_j(s) ds
\]

\[
\geq \frac{1}{2} |u|^2 - \frac{\varepsilon}{2} |u|_{L^2}^2 - |u|_{\infty} \|C\|_{L^1} + n (p - 1) m u(t_j),
\]

\[
\geq \frac{1}{2} |u|^2 - \frac{\varepsilon}{2} |u|_{L^2}^2 - |u|_{\infty} \|C\|_{L^1} + n (p - 1) m |u|_{\infty},
\]

and using (4), we obtain

\[
\varphi_n(u) \geq \frac{1}{2} |u|^2 (1 - \varepsilon) - 2 |u| (\|C\|_{L^1} + N |m|).
\]

the choice of \( \varepsilon < \frac{1}{2} \), (\( H_2 \)) and \( \|C\|_{L^1} \) bounded imply \( \varphi_n(u) \geq -\infty \), then for every \( n \in \mathbb{N}^* \), \( \varphi_n(u) \) is bounded from below and hence there exists a positive constant \( B = B(n) \), such that

\[
\varphi_n(u) \geq -B,
\]
Considering \( \varepsilon < \frac{1}{2} \), we have from (21) that, \( \varphi_n(u) \to +\infty \) when \( \|u\| \to +\infty \).

So, \( \varphi_n \) is coercive on \( E_n \).

\( E_n \) is a weakly closed subspace of \( H_n \), then using the direct method of the calculus of variations, we immediately obtain that for all \( n \in \mathbb{N}^* \) there exists \( u_n \in E_n \), such that,

\[
\varphi_n(u_n) \to +\infty
\]

We have the following result giving an order on \((c_n)_n\).

**Lemma 7** For all \( n \in \mathbb{N}^* \), we have

\[-\infty < c_n < c_1 < 0,
\]

**Proof.** From (21), we plainly obtain that \( c_n > -\infty \). Moreover, since for all \( n \in \mathbb{N}^* \), we have \( E_1 \subset E_n \), and hence \( c_n \leq c_1 \),

\[
c_n = \inf_{E_n} \varphi_n \leq c_1 = \inf_{E_1} \varphi_1 \quad \text{for all } n \in \mathbb{N}^*
\]

Finally, to prove that \( c_1 < 0 \), we use \((H_1)(iii)\), for \( \alpha > 0 \), there exists, \( 1 > \delta_\alpha > 0 \), such that, for almost every \( t \) and for all \( \|u\| \leq \delta_\alpha \),

\[
F(t,u) > \frac{\alpha}{2} |u|^2
\]

Now, let \( \lambda_1 \) be the first eigenvalue of \( Lu = -u'' + u \) on \( H_1 \) and \( v_1 \) the corresponding eigenfunction in \( E_1 \),

\[
\|v_1\|^2 = \lambda_1 \int_{-T}^{T} |v_1(t)|^2 dt,
\]

So, for \( \tau > 0 \), such that \( \|\tau v_1\|_\infty < \delta_\alpha \), (24) and \((H_2)\) imply

\[
\varphi_1(\tau v_1) = \frac{1}{2} \tau^2 \|v_1\|^2 - \int_{-T}^{T} F(t,\tau v_1(t))dt + \sum_{j=1}^{p-1} \int_{0}^{\tau v_1(t_j)} I_j(s)ds
\]

\[
\leq \frac{1}{2} \tau^2 \|v_1\|^2 - \frac{\alpha}{2} \int_{-T}^{T} |\tau v_1|^2 dt + \sum_{j=1}^{p-1} \int_{0}^{\tau v_1(t_j)} I_j(s)ds
\]

\[
\leq \tau^2 \left( \lambda_1 \|v_1\|^2_{L^2} - \alpha \|v_1\|^2_{L^2} \right) + (p-1) \|m\tau v_1(t_j)\|
\]

\[
\leq T(\lambda_1 - \alpha) \|\tau v_1\|^2 + (p-1) |m| \|\tau v_1\|_\infty
\]

\[
\leq T(\lambda_1 - \alpha) \delta_\alpha^2 + (p-1) |m| \delta_\alpha.
\]
Choosing $\alpha$ sufficiently large $\alpha > \lambda_1$, (26) and $(H_2)$ imply that
\[ c_1 = \inf_{E_1} \varphi_1 < 0 \]

\[ \boxed{} \]

\textbf{Remark 8} $u_n$ is a non constant solution of (6), indeed, from (7) and lemma (7), $\varphi_n(0) = 0 > c_n = \inf_{E_n} \varphi_n = \varphi_n(u_n)$, which implies that $u_n \neq 0$, now from $u_n \in E_n$, we deduce that $u_n$ is a non constant solution of the problem (6).

It remains to show that $u_n$ is a solution of the problem (2).

\textbf{Lemma 9} There exist $\beta_0 = \beta_0(n) \in ]\xi, \xi + \gamma[$ and a constant $D_0 > 0$ such that each solution $u \in E_n$ of (6), satisfies $\beta_0 < u(t) < D_0$, for all $t$.

In particular, any solution of (6) is a solution of (2).

\textbf{Proof.} Here, we shall use some ideas from [7].

We proceed by contradiction. Suppose, on the contrary, that for each $\beta \in ]\xi, \xi + \gamma[$ and for each constant $D > 0$, there exists a solution $u_n \in E_n$ of the problem (6) which satisfies

\[ u_n(t) > D, \text{ or } u_n(t) < \beta \text{ for some } t \in [−nT, nT]. \]

In particular, if for each integer $k > 1$ we consider $\beta_k = \xi + \frac{1}{k}$, and $D = k$, the above supposition implies that there exists a solution $u_n \in E_n$ of the problem (6) for $\beta = \beta_k$ such that

\[ \{ u_n^k(t); t \in \mathbb{R} \} \not\subseteq [\beta_k, k]. \]

We will show that this assumption leads to a contradiction.

First, we claim that for every $k > 1$, there must exist $\tau_k \in [−nT, nT]$ such that

\[ u_n^k(\tau_k) \in [\beta_k, k]. \]

Indeed, suppose on the contrary, that there exists a subsequence of $(u_n^k)_k$, that we label the same, for which $\min u_n^k(t) > k$. It follows from $(H_1)(iv)$ and Fatou lemma that

\[ n(p - 1)M > \liminf_{k \to +\infty} \sum_{t_j \in I_n} I_j(u_n^k(t_j)) = \liminf_{k \to +\infty} \int_{-nT}^{nT} g_{\beta_k}(t, u_n^k(t))dt \]

\[ \geq \int_{-nT}^{nT} \liminf_{k \to +\infty} g_{\beta_k}(t, u_n^k(t))dt \geq \int_{-nT}^{nT} \lim_{x \to +\infty} g(t, x)dt \geq 0 \]
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So, \( M > 0 \), this is a contradiction to \((H_2)\).
Similarly, we will arrive at a contradiction with \((H_2)\), if we assume that \( \max u_n^k < \beta_k \). In fact, by Fatou lemma we have,

\[
\limsup_{k \to +\infty} \sum_{j=1}^{p-1} I_j(u_n^k(t_j)) = \limsup_{k \to +\infty} \int_{-nT}^{nT} g_{\beta_k}(t, u_n(t))dt \\
\leq \int_{-nT}^{nT} \limsup_{k \to +\infty} g_{\beta_k}(t, u_n(t))dt \\
\leq \int_{-nT}^{nT} \limsup_{x \to \xi^+} g(t, x)dt
\]

Hence,

\[
\limsup_{k \to +\infty} \sum_{j=1}^{p-1} I_j(u_n(t_j)) \leq \int_{-nT}^{nT} \lim_{x \to \xi^+} g(t, x)dt = -\infty.
\]

Secondly, we show that \( (u_n^k) \) is bounded from above. Since for all \( k > 1 \), \( u_n^k \) is a weak solution of \((6)\), we have that, for all \( v \in H_n \) and for all \( k > 1 \),

\[
\int_{-nT}^{nT} \left[ (u_n^k)'(t)v'(t) + u_n^k(t)v(t) - g_{\beta_k}(t, u_n^k(t))v(t) \right] dt + \sum_{j=1}^{p-1} I_j(u_n^k(t_j))v(t_j) = 0
\]

Taking \( v(t) \equiv 1 \) in the above inequality, we obtain,

\[
- \int_{-nT}^{nT} g_{\beta_k}(t, u_n^k(t))dt + \sum_{t_j \in I_n} I_j(u_n(t_j)) = 0
\]

\((30)\) and \((H_2)\) imply that,

\[
\int_{-nT}^{nT} g_{\beta_k}(t, u_n^k(t))dt = \sum_{t_j \in I_n} I_j(u_n(t_j)) \leq \sum_{t_j \in I_n} |I_j(u_n(t_j))| \]

Let

\[
I_{n,1}^k := \{ t \in I_n; g_{\beta_k}(t, u_n^k(t)) \geq 0 \},
\]
and 
\[ I^k_{n,2} := \{ t \in I_n; \ g_{\beta_k}(t, u^k_n(t)) < 0 \} . \]

It follows from (31), (H1) and (5) that for all \( k \geq 1 \),
\[
\left| \int_{I^k_{n,2}} g_{\beta_k}(t, u^k_n(t)) dt \right| \leq |m| N + \int_{I^k_{n,1}} g_{\beta_k}(t, u^k_n(t)) dt \leq |m| N + \| K \|_{L^1} \tag{32}
\]

Hence,
\[
\int_{-nT}^{nT} \left| g_{\beta_k}(t, u^k_n(t)) \right| dt \leq |m| N + 2 \| K \|_{L^1}
\]

Also, taking \( v = u^k_n \) in (29), we obtain
\[
\| u^k_n \|^2 - \int_{-nT}^{nT} g_{\beta_k}(t, u^k_n(t)) u^k_n(t) dt + \sum_{t_j \in I_n} I_j(u^k_n(t_j)) u^k_n(t_j) = 0 \tag{33}
\]

which means that,
\[
\int_{-nT}^{nT} g_{\beta_k}(t, u^k_n(t)) u^k_n(t) dt - \sum_{t_j \in I_n} I_j(u^k_n(t_j)) u^k_n(t_j) = \| u^k_n \|^2 \geq 0
\]

So, (33) can be write
\[
0 = \| u^k_n \|^2 - \left| \int_{-nT}^{nT} g_{\beta_k}(t, u^k_n(t)) u^k_n(t) dt - \sum_{j \in I_n} I_j(u^k_n(t_j)) u^k_n(t_j) \right| \tag{34}
\]
\[
\geq \| u^k_n \|^2 - \left( \left| \int_{-nT}^{nT} g_{\beta_k}(t, u^k_n(t)) u^k_n(t) dt \right| + \left| \sum_{j \in I_n} I_j(u^k_n(t_j)) u^k_n(t_j) \right| \right)
\]
\[
\geq \| u^k_n \|^2 - \| u^k_n \|_{\infty} \left[ \int_{-nT}^{nT} \left| g_{\beta_k}(t, u^k_n(t)) \right| dt + \left| \sum_{j \in I_n} I_j(u^k_n(t_j)) \right| \right]
\]

Thus, (4) combined with (34) imply that,
\[
\| u^k_n \|^2 \leq 2 \| u^k_n \| \left[ \int_{-nT}^{nT} \left| g_{\beta_k}(t, u^k_n(t)) \right| dt + |m| N \right], \tag{35}
\]
We deduce from (32) that, for \( k > 1 \),
\[
\| u_n^k \| \leq D
\]
where,
\[
D = 4 \left( \| K \|_{L^1} + |m| N \right), \tag{36}
\]
Notice that \( D \) is independent of \( k \). Hence \((u_n^k)_k\) is bounded in \( H_n \). For all \( k \), \( u_n^k \in E_n \), then (36), implies that, there exists \( D_0 > 0 \)
\[
u_n^k(t) \leq D_0
\]
Consequently, for \( k \) sufficiently large \((k > D_0)\), for all \( t \in [-nT, nT] \), we have \( u_n^k(t) < k \). Furthermore, we cannot have \( u_n^k(t) \geq \beta_k \) for all \( t \in [-nT, nT] \); otherwise we would get, \( \beta_k \leq u_n^k(t) \leq k \) for all \( t \in [-nT, nT] \) and this contradicts the assumption (27). Therefore, for \( k \) sufficiently large \((k > D_0)\), there must exist a \( t_k^* \in [-nT, nT] \) such that \( u_n^k(t_k^*) < \beta_k \). This means that \( t_k^* \in I_{\beta_k} \), where \( I_{\beta_k} \) is the set defined by
\[
I_{\beta_k} = \{ t \in [nT, nT]; u_n^k(t) < \beta_k \}. \tag{37}
\]
Then, the set \( I_{\beta_k} \) is not empty, the continuity of the solution \( u_n^k(t) \) at \( t = t_k^* \), implies that \( \lim \text{meas}(I_{\beta_k}) > 0 \), which implies
\[
\int_{I_{\beta_k}} [g_{\beta_k}(t, u_n^k(t))]dt \neq 0.
\]
Now, consider the sets
\[
I_{0,D_0} = \{ t \in [0,T]; 0 \leq u_n^k(t) \leq D_0 \}, \tag{38}
\]
\[
I_{\beta_k,0} = \{ t \in [0,T]; \beta_k \leq u_n^k(t) < 0 \}, \tag{39}
\]
so that, we can write
\[
[-nT, nT] = I_{\beta_k} \cup I_{\beta_k,0} \cup I_{0,D_0}.
\]
Then, integrating the equation \(- (u_n^k)^"(t) + u_n^k(t) = g_{\beta_k}(t, u_n^k(t))\) from \(-nT\) to \(nT\) we obtain,
\[
\Upsilon_k : = \int_{-nT}^{nT} \left( (u_n^k)^" + u_n^k(t) \right) dt = \int_{-nT}^{nT} g_{\beta_k}(t, u_n^k(t))dt \tag{40}
\]
\[
= \int_{I_{\beta_k}} g_{\beta_k}(t, u_n^k(t))dt + \int_{I_{\beta_k,0}} g_{\beta_k}(t, u_n^k(t))dt + \int_{I_{0,D_0}} g_{\beta_k}(t, u_n^k(t))dt
\]
1) Assume we are integrating positively on all subintervals of \([-nT,nT]\).
If \(t \in I_{\beta_k}\) then \(u_n^k(t) < \beta_k\). Hence, (37) and \((H_1)\) imply
\[
\int_{I_{\beta_k}} g_{\beta_k}(t,u_n^k(t))dt = \int_{I_{\beta_k}} g(t,\beta_k)dt < 0,
\]
which yields,
\[
\Upsilon_k < \int_{I_{\beta_k,0}} g_{\beta_k}(t,u_n^k(t))dt + \int_{I_{0,D_0}} g_{\beta_k}(t,u_n^k(t))dt \tag{41}
\]
If \(t \in I_{0,D_0}\) then \(u_n^k(t) \in [0,D_0]\). This means that \(u_n^k(t)\) is bounded on \(I_{0,D_0}\), since \(g_{\beta_k}\) is continuous in \(u\), then \(g_{\beta_k}\) is bounded almost everywhere in \(I_{0,D_0}\). Let
\[
\alpha = \alpha(D_0) = \max \left\{ \left| g_{\beta_k}(t,x) \right| ; t \in [-nT,nT], 0 \leq x \leq D_0 \right\} \tag{42}
\]
Then,
\[
\left| \int_{I_{0,D_0}} g_{\beta_k}(t,u_n^k(t))dt \right| \leq \int_{I_{0,D_0}} \left| g_{\beta_k}(t,u_n^k(t)) \right| dt \leq 2\alpha nT \tag{43}
\]
(41) and (43) lead to
\[
\Upsilon_n \leq \int_{I_{\beta_k,0}} g_{\beta_k}(t,u_n^k(t))dt + 2\alpha nT. \tag{44}
\]
By \((H_1)\), \(\lim_{u \to \xi^-} g(t,u) = -\infty\) so, for every \(\sigma > 0\), there exists \(\gamma_\sigma > 0\) such that \(g(t,u) < -\sigma\), for all \(u \in [\xi,\xi + \gamma_\sigma]\) and for almost every \(t \in [-nT,nT]\). Then, for \(k\) large enough, we have by, \((H_1)\) \((i)\),
\[
\int_{I_{\beta_k,0}} g_{\beta_k}(t,u_n^k(t))dt < -\sigma \lim \text{meas}(I_{\beta_k,0}) \tag{45}
\]
taking \(\sigma = \frac{1}{\lim \text{meas}(I_{\beta_k,0})} k^2 2\alpha nT\), we obtain,
\[
\Upsilon_k \to k \to +\infty -\infty. \tag{46}
\]
Then, \(\Upsilon_k\) is not bounded.

2) If we suppose that we integrate negatively on all subintervals of \([-nT,nT]\), then, instead of (45) we get,
\[
\int_{I_{\beta_k,0}} g_{\beta_k}(t,u_n^k(t))dt > \sigma \lim \text{meas}(I_{J})
\]
These together with (42) lead to
\[ \Upsilon_k \to +\infty, \text{ as } k \to +\infty. \quad (47) \]

On the other hand, integrating the equation \((-u^{(k)}_n)^{''} + u^{(k)}_n(t)\) from \(-nT\) to \(nT\) and using \(2nT\)-periodicity of \((u^{(k)}_n)\), and \(u^{(k)}_n \in E_n\), we obtain,
\[ \Upsilon_k = \int_{-nT}^{nT} \left((-u^{(k)}_n)^{''} + u^{(k)}_n(t)\right) dt = -\sum_{t_j \in I_n} \int_{t_j}^{t_{j+1}} u^{''}_n(t) dt \]
\[ = \sum_{t_j \in I_n} \Delta u'_n(t_j) = \sum_{t_j \in I_n} I_j(u_n(t_j)) \]

thus by \((H_2)\),
\[ n(p-1)m \leq \Upsilon_n \leq n(p-1)M \]

(48) contradicts (46) and (47) and the lemma (9) is proved.

Lemma (9), shows that, there exists \(\beta \in ]\xi, \xi + \gamma[\), such that every solution \(u\) of (6) is a solution of (2), since it satisfies \(u(t) \geq \beta\) for all \(t \in \mathbb{R}\) and \(g_\beta(t,u(t)) = g(t,u(t))\), if \(u(t) \geq \beta\).

3.2 Existence of homoclinic solution for (1)

We present an existence result of homoclinic solution.

**Theorem 10** If \((H_1)\) and \((H_2)\) hold, then the problem (1) possesses at least one solution.

**Proof.** Observe that where as the function \(f\) is a \(2T\)-periodic functions in \(t\), and \(I_{j+p} = I_j\), if \(u\) is a \(2nT\)-periodic solution of (1.1), then \(u(., + 2kT)\) is also a \(2nT\)-periodic solution of (1.1), for all \(k \in \mathbb{Z}\). Therefore, replacing \(u(t)\) by some \(u(t + 2kT)\) if necessary, we still obtain \(2nT\)-periodic solutions of (1.1).

Let us denote by \(\tilde{H}_n\) the space that consists of \(\tilde{u}\) the \(2T\)-periodic extensions of the functions \(u\), of \(H_n\) and denote by \(\tilde{\varphi}_n\) the functional defined on \(\tilde{H}_n\) by \(\tilde{\varphi}_n(\tilde{u}) = \varphi_n(u)\). Moreover, by a change of variable, it is easy to see that the
functional $\tilde{\varphi}_n$ is invariant by a translation $\tau_k$ of $t$ by $2kT$, that is, $\tilde{\varphi}_n(\tau_k u) = \tilde{\varphi}_n(u)$, where $\tau_k u(t) = u(t + 2kT)$.

Let now consider $\tilde{u}_n$ the periodic extensions on $\tilde{H}_n$, of the solutions $u_n$ of the problem (6). By lemma (9) $u_n$ is bounded in $H_n$, then $\tilde{u}_n$ is bounded in $\tilde{H}_n$. If $(\tilde{u}_n^k)_k$ is a sequence of such functions, from the reflexivity of $\tilde{H}_n$, we may extract a weakly convergent subsequence of $(\tilde{u}_n^k)_k$ that, for simplicity, we call $(\tilde{u}_n^k)_k$, such that $\tilde{u}_n^k \rightharpoonup u_0$ in $\tilde{H}_n$. The Sobolev embedding theorem implies $\tilde{u}_n^k \to u_0$ in $C_{2nT}$, the space of periodic extensions of continuous functions on $I_n$ and

$$\tilde{u}_n^k \to u_0 \text{ in } L^2_{2nT}, \text{space of periodic extensions of functions in } L^2(I_n) \quad (49)$$

Now, using (8), we have,

$$(\tilde{\varphi}_n'(\tilde{u}_n^k) - \tilde{\varphi}_n'(u_0))(\tilde{u}_n^k - u_0) = ||\tilde{u}_n^k - u_0||^2 - \int_{-nT}^{nT} (f(t, \tilde{u}_n^k) - f(t, u_0))(\tilde{u}_n^k(t) - u_0(t))dt + \sum_{j \in I_n} [I_j(\tilde{u}_n^k(t_j)) - I_j(u_0(t_j))](\tilde{u}_n^k(t_j) - u_0(t_j)),$$

$(H_1)$ and (49) imply that,

$$\int_{-nT}^{nT} (f(t, \tilde{u}_n^k) - f(t, u_0))|\tilde{u}_n^k(t) - u_0(t)|dt \to 0 \text{ as } k \to +\infty \quad (50)$$

$$\sum_{t_j \in I_n} |I_j(\tilde{u}_n^k(t_j)) - I_j(u_0(t_j))| |\tilde{u}_n^k(t_j) - u_0(t_j)| \to 0 \text{ as } k \to +\infty$$

Now, from $\tilde{\varphi}_n'(\tilde{u}_n^k) = 0$ for all $k$, we have $\lim_{k \to \infty} \tilde{\varphi}_n'(\tilde{u}_n^k) = 0$, consequently by $\tilde{u}_n^k \rightharpoonup u_0$, we obtain

$$(\tilde{\varphi}_n'(\tilde{u}_n^k) - \tilde{\varphi}_n'(u_0))(\tilde{u}_n^k - u_0) \to 0, \text{ as } k \to \infty. \quad (51)$$

By (50), (51) and $\tilde{u}_n^k \to u_0$ in $L^2([-nT, nT])$ we obtain,

$$||\tilde{u}_n^k - u_0|| \to 0, \text{ as } k \to \infty. \quad (52)$$

That is, $(\tilde{u}_n^k)_k$ strongly converges to $u_0$ in $\tilde{H}_n$.

Now, we show that $u_0$ is a solution of (1.1), (1.2). For any given interval $(a, b) \subset [-nT, nT]$ and any $h_1 \in W^{1,2}_0((a, b), \mathbb{R})$, define by

$$\tilde{h}_1 = \begin{cases} h_1 & t \in (a, b) \\ 0 & t \in [-nT, nT] \setminus (a, b) \end{cases} \quad (53)$$
Therefore, one has
\[
\tilde{\varphi}'_{\gamma}(\tilde{u}_j)\tilde{h}_1 = \int_a^b [\tilde{u}_n^k(t)h_1'(t) + \tilde{u}_n^k(t)h_1(t)]dt + \sum_{t_j \in (a,b)} I_j(\tilde{u}_n^k(t_j))h_1(t_j) \quad (54)
\]
\[
+ \int_a^b f(t, \tilde{u}_n^k(t))h_1(t) dt = 0
\]

So \(u_0\) is solution of the problem (1.1), (1.2).

Secondly, we show that \(u_0(t) \to 0\), as \(t \to \pm \infty\).

By (52), \(\tilde{u}_n^k\) is bounded in \(H_n\), which implies that there exist a positive constant \(D\) (36) such that

\[
\int_{-\infty}^{+\infty} (|u_0|^2 + |u'_0|^2) dt = \int_{-nT}^{+nT} (|u_0|^2 + |u'_0|^2) dt + \int_{\mathbb{R} \setminus [-nT,nT]} (|u_0|^2 + |u'_0|^2) dt
\]

\[
\int_{-\infty}^{+\infty} (|u_0|^2 + |u'_0|^2) dt \leq \lim_{n \to +\infty} \liminf_{n \to +\infty} \int_{-nT}^{+nT} (|\tilde{u}_n|^2 + |\tilde{u}'_n|^2) dt \leq D,
\]

hence

\[
\int_{|t| \geq r} (|u_0|^2 + |u'_0|^2) dt \to 0, \text{ as } r \to +\infty \quad (55)
\]

By the first inequality of lemma (3) and (55), we obtain

\[
u_0(t) \to 0, \text{ as } t \to \pm \infty.
\]

Thirdly, we prove that, \(u_0'(t^\pm) \to 0\) as \(t \to \pm \infty\). We have proved \(u_0(t)\) is a solution of (1.1), so we have

\[
\int_{t_{j-1}}^{t_j} |u_0''(s)|^2 ds = \int_{t_{j-1}}^{t_j} \left[-u_0(s) + f(s, u_0(s))\right]^2 ds
\]

\[
= \int_{t_{j-1}}^{t_j} \left[-u_0(s) + f(s, u_0(s))\right]^2 ds
\]

\[
\leq 2 \int_{t_{j-1}}^{t_j} \left(u_0(s)\right)^2 + (f(s, u_0(s)))^2 ds
\]
By (H2) and (56), we have \( \int_{t_{j-1}}^{t_j} |u_0''(s)|^2 ds \to 0 \) as \( t_j \to \pm \infty \).

Using (2), lemma (3), and \( u_0 \) solution of (1.1), we have for all \( t \in [t_j, t_{j+1}[ \)

\[
|u'_0(t)| \leq 2 \int_{t_{j-1}}^{t_j} (|u_0''(s)|^2 + |u'_0(s)|^2) ds
\]

\[
\leq 2 \int_{t_{j-1}}^{t_j} (|u_0(s)|^2 + |u'_0(s)|^2) ds + 2 \int_{t_{j-1}}^{t_j} |u''_0(s)|^2 ds.
\]

Therefore \( u'_0(t) \to 0 \) as \( t \to \pm \infty \).

We conclude that the problem (1) has at least a solution.

4 Example

Consider the following problem

\[
\begin{cases}
-u''(t) + u(t) = g(t, u) & t \neq t_j , \ t \in \mathbb{R} \\
\Delta u'(t_j) = I_j(u(t_j)) & j \in \{-5; \ldots; 5\} \subset \mathbb{Z} \\
\lim_{t \to \pm \infty} u(t) = \lim_{t \to \pm \infty} u'(t) = 0
\end{cases}
\]  

(1.1)  

(1.2)  

(1.3)

(57)

where, \( g : \mathbb{R} \times ]-1, +\infty[ \to \mathbb{R} \), is such that \( g(t, u) = (\cos t) \left( \frac{u}{1+u} \right)^{\frac{1}{2}} \), so, it is \( L^1_{loc} \) and \( 2\pi \)-periodic in \( t \); and continuous in its second argument ; \( \lim_{u \to -1} g(t, u) = -\infty \) and \( I_j : \mathbb{R} \to \mathbb{R} \), is defined by,

\[
I_j(s) = \sin s - 2
\]

We see that \( I_j \) is a bounded continuous function such that, \( \max_{s} I_j(s) = -1 < 0 \), for all \( j \),

\[
K := \sup_{u \in ]-1, +\infty[} g(., u) = \cos t \text{ is in } L^1(\mathbb{R}) .
\]

\[
\lim_{u \to 0} \frac{g(t,u)}{u} = +\infty , \text{ for almost every } t \in \mathbb{R}
\]

\[
\lim_{u \to +\infty} \frac{g(t,u)}{u} = 0^+ \text{ for almost every } t \in \mathbb{R} .
\]

\( g \) and \( I_j \) verify the assumptions (H1) and (H2) so from the theorem (10), we obtain the existence of at least one homoclinic solution for (1).

References


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