Hopf Bifurcation Analysis of a Dynamical Heart Model with Time Delay

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Abstract

This paper presents the stability and bifurcation analysis of a dynamical model of the heartbeat with time delay. The existence of Hopf bifurcations at the positive equilibrium is established by analyzing the distribution of the characteristic values. Moreover, the direction of the Hopf bifurcation and the stability of the bifurcating periodic solutions are determined by applying the center manifold theorem and the normal form theory.

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1 Introduction

Differential equations with time delay play an important role in economy, engineering, biology and social sciences, because a great deal of problems may be described with their help. The effects of time delays are of great interest as well, since their presence may include complex behavior. Recently, Páez-Hernández et al. [2] have investigated a model of the heartbeat using nonlinear dynamics and considering the time delays inherent in the system. The resulting model
is given by
\[
\begin{cases}
\dot{x} &= -\frac{1}{\varepsilon}(x^3 - Tx + y\xi), \\
\dot{y} &= x\xi + x_d,
\end{cases}
\]
where \(x\) denotes the length of the muscle fiber, \(y\) represents a variable related to electrochemical activity, \(\varepsilon\) is a small positive constant associated to the local stability of an endoreversible system with the fast eigenvalue of the system, \(x_d\) is a scalar quantity representing a typical length of muscle fiber in the diastolic state, \(T\) is tension in the muscle fiber, and \(\xi\) is a time delay. However, system (1) was only analyzed in the particular case \(\xi = \pi/2\). The aim of this paper is to give the explicitly mathematical analysis for stability and periodicity of system (1) for an arbitrary time delay \(\xi\).

2 Stability and existence of Hopf bifurcation

The corresponding characteristic equation of the linear system of (1) at its unique equilibrium \((x_*, y_*)\), where \(x_* = -x_d\) and \(y_* = x_d^3 - T x_d\), is
\[
\Delta_1(\lambda, \xi) = \lambda^2 + A\lambda + Be^{-2\lambda\xi} = 0,
\]
where
\[
A = \frac{1}{\varepsilon}(3x_d^2 - T), \quad B = \frac{1}{\varepsilon}.
\]
When \(\xi = 0\), Eq. (2) reduces to \(\lambda^2 + A\lambda + B = 0\). An analysis of the sign of the roots of this equation implies that the equilibrium \((x_*, y_*)\) is unstable if \(T \geq 3x_d^2\) and stable if \(T < 3x_d^2\).

Next we shall explore the distribution of characteristic roots of (2) when \(\xi > 0\). For computational purpose, we rewrite (2) as
\[
\Delta_1(\lambda, \xi) = e^{\lambda\xi} \Delta_1(\lambda, \xi) = (\lambda^2 + A\lambda) e^{\lambda\xi} + Be^{-\lambda\xi} = 0.
\]

Let \(\lambda = i\omega\) (\(\omega > 0\)) be a root of (2). Then, substituting it into (3) yields
\[
(-\omega^2 + B) \cos \omega\xi = A\omega \sin \omega\xi, \quad (-\omega^2 + B) \sin \omega\xi = -A\omega \cos \omega\xi.
\]

Squaring and adding Eqs. (4) we obtain
\[
\omega^4 - (A^2 + 2B) \omega^2 + B^2 = 0.
\]

**Lemma 2.1.**

1) If \(A = 0\) holds, then Eq. (5) has a unique positive root \(\omega_0\), where \(\omega_0 = B\).
2) If $A \neq 0$ holds, then Eq. (5) has two positive roots $\omega_- < \omega_+$, where

$$
\omega_{\pm} = \frac{\sqrt{A^2 + 2B \pm \sqrt{(A^2 + 2B)^2 - 4B^2}}}{2}.
$$

Proof. The statement follows noting that the discriminant of Eq. (5) is given by $(A^2 + 2B)^2 - 4B^2$, and recalling that $B > 0$.

Notice that solving Eqs. (4) directly, one can derive the values $\xi_0^j$ and $\xi_+^j$ $(j = 0, 1, 2, ...)$ of $\xi$ at which Eq. (3) has a pair of purely imaginary roots $\pm i\omega_0$ and $\pm i\omega_{\pm}$, respectively.

Proposition 2.2. Let $\xi_* \in \{\xi_0^j, \xi_-^j, \xi_+^j\}$ and $\omega_* \in \{\omega_0, \omega_-^j, \omega_+^j\}$. If $\lambda(\xi)$ is the root of (3) near $\xi = \xi_*$ such that $Re(\lambda(\xi_*)) = 0$ and $Im(\lambda(\xi_*)) = \omega_*$, then

$$
\left[ \frac{dRe(\lambda)}{d\xi} \right]_{\xi = \xi_*, \omega = \omega_*} > 0.
$$

Furthermore, $\lambda = i\omega_*$ is a simple root of the characteristic equation (3).

Proof. By substituting $\lambda(\xi)$ into equation (3) and differentiating both sides of this equation with respect to $\xi$, we obtain

$$
\{(2\lambda + A)e^{\lambda \xi} + (\lambda^2 + A\lambda) \xi e^{\lambda \xi} - Be^{-\lambda \xi} \frac{d\lambda}{d\xi} \}
= \lambda \left[ - (\lambda^2 + A\lambda) e^{\lambda \xi} + Be^{-\lambda \xi} \right] \tag{6}
$$

If $\lambda = i\omega_*$ is not a simple root of the characteristic equation (3), then, from (6) and using (3), i.e. $- (\lambda^2 + A\lambda) e^{\lambda \xi} - Be^{-\lambda \xi} = 0$, yields $i\omega_* (2Be^{-i\omega_* \xi}) = 0$, leading to an absurd. According to (6), we can get

$$
\left( \frac{d\lambda}{d\xi} \right)^{-1} = - \frac{2\lambda + A}{2B\lambda(\lambda^2 + A\lambda)} - \frac{\xi}{\lambda}.
$$

Recalling that

$$
\text{sign} \left\{ \frac{d(Re\lambda)}{d\xi} \right\}_{\xi = \xi_*, \omega = \omega_*} = \text{sign} \left\{ Re \left( \frac{d\lambda}{d\xi} \right)^{-1} \right\}_{\xi = \xi_*, \omega = \omega_*},
$$

we obtain

$$
\text{sign} \left\{ \frac{d(Re\lambda)}{d\xi} \right\}_{\xi = \xi_*, \omega = \omega_*} = \text{sign} \left\{ \frac{2\omega_*^2 + A^2}{2B\omega_*^2(\omega_*^2 + A^2)} \right\}.
$$

$\square$
From the previous discussions and the Hopf bifurcation theorem, we can obtain the following results.

**Theorem 2.3.**

1) Let $A = 0$.

   i) If the equilibrium $(x_*, y_*)$ of system (1) is locally asymptotically stable in absence of delay, then it is locally asymptotically stable for $0 \leq \xi < \xi_0^0$ and unstable for $\xi > \xi_0^0$. System (1) undergoes a Hopf bifurcation at $(x_*, y_*)$ when $\xi = \xi_0^0$.

   ii) If the equilibrium $(x_*, y_*)$ of system (1) is unstable in absence of delay, then it remains unstable for $\xi \geq 0$.

2) Let $A \neq 0$. The equilibrium $(x_*, y_*)$ of system (1) has many stability switches and a Hopf bifurcation occurs at each stability switch.

### 3 Stability and direction of bifurcating periodic solutions

In this section, formulae for determining the direction of Hopf bifurcation and the stability of bifurcating periodic solution of system (1) shall be presented by employing the normal form method and center manifold theorem introduced by Hassard et al. [1].

Let $\xi = \xi_c + \mu$, $\mu \in \mathbb{R}$, where $\xi_c$ is a value for $\xi$ where a Hopf bifurcation occurs. Let us rewrite (1) in the form

$$u = L_\mu(u_t) + f_\mu(u_t),$$

where $u = (u_1, u_2)^T \in \mathbb{R}^2$, $u_1(\theta) = u(t + \theta) \in C = C([-\xi, 0], \mathbb{R}^2)$, and $L_\mu : C \to \mathbb{R}^2$, $f : \mathbb{R} \times C \to \mathbb{R}^2$ are defined, respectively, by

$$L_\mu(\varphi) = P_0 \varphi(0) + P_1 \varphi(-\xi)$$

with

$$P_0 = \begin{bmatrix} \frac{1}{\varepsilon}(3x_d - T) & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad P_1 = \begin{bmatrix} 0 & -\frac{1}{\varepsilon} \\ 0 & 0 \end{bmatrix},$$

and

$$f_\mu(\varphi) = \begin{bmatrix} -\frac{6x_d}{\varepsilon} \varphi(0)^2 - \frac{1}{\varepsilon} \varphi(0)^3 \\ 0 \end{bmatrix}.$$
By the Riesz representation theorem, there is a bounded variation function \( \eta(\theta, \mu) \) for \( \theta \in [-\xi, 0] \) such that

\[
L_{\mu}(\varphi) = \int_{-\xi}^{0} d\eta(\theta, \mu) \varphi(\theta), \quad \text{for } \varphi \in C.
\]

We can choose \( \eta(\theta, \mu) = P_{0}\delta(\theta) + P_{1}\delta(\theta + \xi) \), where \( \delta(\cdot) \) is the Dirac function. For \( \varphi \in C^{1}([-\xi, 0], \mathbb{R}^{2}) \), we define two operators

\[
A(\mu)(\varphi) = \begin{cases} 
  \frac{d\varphi(\theta)}{d\theta}, & \theta \in [-\xi, 0), \\
  \int_{-\xi}^{0} d\eta(s, \mu) \varphi(s), & \theta = 0,
\end{cases}
\]

and

\[
R(\mu)(\varphi) = \begin{cases} 
  0, & \theta \in [-\xi, 0), \\
  f(\mu, \varphi), & \theta = 0.
\end{cases}
\]

In this way, we can transform (7) into

\[
\dot{u} = A(\mu)u_{t} + R(\mu)u_{t}.
\]

For \( \psi \in C^{1}([0, \xi], \mathbb{R}^{2}) \), we define the operator \( A^{*} \) as

\[
A^{*}\psi(s) = \begin{cases} 
  -\frac{d\psi(s)}{ds}, & s \in (0, \xi], \\
  \int_{-\xi}^{0} d\eta^{T}(r, \mu)\psi(-r), & s = 0,
\end{cases}
\]

and consider the following bilinear form

\[
< \psi, \varphi > = \overline{\psi^{T}(0)}\varphi(0) - \int_{\theta=-\xi}^{0} \int_{r=0}^{\theta} \overline{\psi^{T}(r - \theta)}d\eta(\theta)\varphi(\theta)dr,
\]

where \( \eta(\theta) = \eta(\theta, 0) \) and the overline stands for complex conjugate. Since \( A^{*} \) and \( A \) are adjoint operators, if \( \pm i\omega_{c}\xi_{c} \) are eigenvalues of \( A \), then they are also eigenvalues of \( A^{*} \). One can compute the eigenvectors \( q(\theta) \) and \( q^{*}(\theta^{*}) \) of \( A \) and \( A^{*} \) corresponding to \( i\omega_{c}\xi_{c} \) and \( -i\omega_{c}\xi_{c} \), respectively, which satisfy the normalized conditions \( < q^{*}, q >= 1 \), and \( < q^{*}, \bar{q} >= 0 \). Hence, we set \( q(\theta) = q(0)e^{i\omega_{c}\theta}, q^{*}(\theta^{*}) = q^{*}(0)e^{-i\omega_{c}\theta^{*}} \), for \( \theta \in [-\xi, 0), \theta^{*} \in (0, \xi] \), and \( q(0) = (q_1, q_2)^{T}, q^{*}(0) = (1/\rho)(q_1^{*}, q_2^{*})^{T} \).

Next, using \( q^{*} \) and \( q \), we can compute the coordinates describing the center manifold at \( \mu = 0 \). Let \( u_{t} \) be the solution of (8) with \( \mu = 0 \). Define

\[
z = < q^{*}, u_{t} >, \quad W(t, \theta) = u_{t}(\theta) - 2Re[zq(\theta)].
\]
On the center manifold, one has \( W(t, \theta) = W(z, \bar{z}, \theta) \), where
\[
W(z, \bar{z}, \theta) = W_{20}(\theta) \frac{z^2}{2} + W_{11}(\theta) z \bar{z} + W_{02}(\theta) \frac{\bar{z}^2}{2} + \cdots ,
\]
z and \( \bar{z} \) are the local coordinates for the center manifold in the direction of \( q^* \) and \( \bar{q}^* \), respectively. One has
\[
\dot{z} = i \omega_c z + g(z, \bar{z}),
\]
with
\[
g(z, \bar{z}) = q^* T(0) f_0(z, \bar{z}) = g_{20} \frac{z^2}{2} + g_{11} z \bar{z} + g_{02} \frac{\bar{z}^2}{2} + g_{21} z^2 \bar{z} + \cdots .
\]
Notice that, from the definition of \( f(\mu, u_t) \), we have
\[
f_0(z, \bar{z}) = f_{z} \frac{z^2}{2} + f_{\bar{z}} \frac{\bar{z}^2}{2} + f_{z \bar{z}} z \bar{z} + f_{z \bar{z}} \frac{z^2 \bar{z}}{2} + \cdots .
\]
Hence, we get
\[
g_{20} = \bar{q}^* (0) f_z z, \quad g_{02} = \bar{q}^* (0) f_{\bar{z}} \bar{z}, \quad g_{11} = \bar{q}^* (0) f_{z \bar{z}}, \quad g_{21} = \bar{q}^* (0) f_{z^2 \bar{z}}.
\]
We remark that \( g_{21} \) will depend on the \( W_{20}(\theta) \) and \( W_{11}(\theta) \). Note that
\[
W_{20}(\theta) = \frac{ig_{20}}{\omega_c} q(0)e^{i \omega_c \theta} - \frac{g_{02}}{3i \omega_c} \bar{q}(0)e^{-i \omega_c \theta} + E_1 e^{2i \omega_c \theta},
\]
\[
W_{11}(\theta) = \frac{g_{11}}{i \omega_c} q(0)e^{i \omega_c \theta} - \frac{\bar{g}_{11}}{i \omega_c} \bar{q}(0)e^{-i \omega_c \theta} + E_2,
\]
where \( E_1 = (E_1^{(1)}, E_1^{(2)}) \in \mathbb{R}^2 \) is a constant vector that can be determined with a further analysis.

Based on the above analysis, we have all formulas to find the values of \( g_{20}, g_{02}, g_{11} \) and \( g_{21} \). Thus, we can explicitly compute the following quantities:
\[
c_1(0) = \frac{i}{2 \omega_c} \left[ g_{11} g_{20} - 2 |g_{11}|^2 - \frac{|g_{02}|^2}{3} + \frac{g_{21}}{2} \right], \quad \mu_2 = -\frac{\text{Re} \left[ c_1(0) \right]}{\text{Re} \left[ \lambda(\xi_c) \right]},
\]
\[
\beta_2 = 2 \text{Re} \left[ c_1(0) \right] , \quad \tau_2 = -\frac{\text{Im} \left[ c_1(0) \right] + \mu_2 \text{Im} \left[ \lambda(\xi_c) \right]}{\omega_c},
\]
where \( \lambda \) is the root of characteristic equation (3).

**Theorem 3.1.** Let \( \xi_c \) be a value for \( \xi \) where system (1) has a Hopf bifurcation.
1) If $\mu_2 > 0$ ($\mu_2 < 0$), then the Hopf bifurcation is supercritical (subcritical) and the bifurcating periodic solutions exist for $\xi > \xi_c$ ($\xi < \xi_c$).

2) The Floquet exponent $\beta_2$ determines the stability of bifurcating periodic solutions. The bifurcating periodic solutions are stable (unstable) if $\beta_2 < 0$ ($\beta_2 > 0$).

3) The quantity $\tau_2$ determines the period of the bifurcating periodic solutions. The period increases (decreases) if $\tau_2 > 0$ ($\tau_2 < 0$).

References


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