Equitable Block Colourings for Systems of 4-Kites

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Abstract

An equitable colouring of a balanced $G$-design $(X, B)$ is a map $\phi : B \rightarrow C$ such that $|b_i(x) - b_j(x)| \leq 1$ for any $x \in X$ and $i, j$, with $i \neq j$, being $b_i(x)$ the number of blocks containing the vertex $x$ and coloured with the colour $i$. A $c$-colouring is a colouring in which exactly $c$ colours are used. A $c$-colouring of type $s$ is a colourings in which, for every vertex $x$, all the blocks containing $x$ are coloured exactly with $s$ colours. A bicolouring, tricolouring or quadricolouring is an equitable colouring with $s = 2$, $s = 3$ or $s = 4$. In this paper we consider systems of graphs consisting of a 4-cycle and a pendant edge. We call such a graph a 4-kite and we consider balanced 4-kite systems. In particular, we prove that $c$-bicolourings of balanced 4-kite systems exist if and only if $c = 2, 3$.

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1 Introduction

A $G$-design of order $v$ is a pair $\Sigma = (X, \mathcal{B})$, where $V$ is the vertex set of $\Sigma$ elements and $\mathcal{B}$ is a set of copies (called blocks) of $G$ decomposing $K_X$. For any $x \in V$ the degree $d(x)$ of $x$ is the number of blocks of $\mathcal{B}$ containing $x$. A $G$-design $\Sigma = (X, \mathcal{B})$ is called balanced if all the vertices $x \in X$ have the same degree.

A colouring of a $G$-design $\Sigma = (X, \mathcal{B})$ is a map $\phi : \mathcal{B} \rightarrow C$, where $C$ is a set of colours. A $c$-colouring is a colouring in which exactly $c$ colours are used.

For any $x \in X$ let $b_i(x)$ be the number of blocks incident with $x$ that are coloured with the colour $i$. An equitable colouring is a colouring such that $|b_i(x) - b_j(x)| \leq 1$ for any $x \in X$ and $i,j$, with $i \neq j$. The set of blocks coloured with a colour of $\mathcal{C}$ is a colour class. A $c$-colouring of type $s$ is a colourings in which, for every vertex $x$, all the blocks containing $x$ are coloured exactly with $s$ colours. A bicolouring, tricolouring or quadricolouring is an equitable colouring with $s = 2$, $s = 3$ or $s = 4$.

In [6] Colbourn and Rosa consider Steiner Triple Systems, which are, of course, balanced designs having all the vertices of degree $\frac{v-1}{2}$, being $v$ the order. In that paper the authors, considered a partition $\pi$ of $\frac{v-1}{2}$, introduce the idea of block colourings in which the colours of blocks containing a vertex in a $STS(v)$ are partitioned according to $\pi$.

Equitable colourings of $STS$ have been also considered in [8] and later the case of cycle systems ($C_4$, $C_6$ and $C_8$-systems) have been studied in [2, 3, 4, 5, 7, 12].

In this paper we consider equitable colourings of balanced $G$-designs, as this case appears to be the natural generalization of the case of equitable colourings of cycle systems. In general we consider the colour spectrum of a balanced $G$-design $\Sigma = (X, \mathcal{B})$ as the set:

$$\Omega^G_s(\Sigma) = \{c \mid \text{there exists an equitable } c\text{-block-colouring of type } s \text{ of } \Sigma\}.$$ 

It is also considered the set $\Omega^G_s(v) = \bigcup \Omega^G_s(\Sigma)$, where $\Sigma$ varies in the set of all the $G$-designs, with $G$ fixed.

The lower $s$-chromatic index is $\chi^G_s(\Sigma) = \min \Omega^G_s(\Sigma)$ and the upper $s$-chromatic index is $\overline{\chi}^G_s(\Sigma) = \max \Omega^G_s(\Sigma)$. If $\Omega^G_s(\Sigma) = \emptyset$, then we say that $\Sigma$ is uncolourable. In the same way we consider $\chi^G_s(v) = \min \Omega^G_s(v)$ and $\overline{\chi}^G_s(v) = \max \Omega^G_s(v)$.

In this paper, we study the case of 4-kite systems, where a 4-kite is a 4-cycle with a pendant edge (in literature it is also denoted by $C_4+e$). So, a 4-kite on the set of vertices $\{x_1, x_2, x_3, x_4, x_5\}$ is a graph having edges $\{x_1, x_2\}$, $\{x_2, x_3\}$, $\{x_3, x_4\}$, $\{x_4, x_1\}$ and $\{x_1, x_5\}$. A 4-kite system of order $v$ will be denoted by $4KS(v)$ (see [9] as a reference for $4KS$ and [10] as a reference for $G$-designs in general).
By [1] it is known that a 4KS(v) of order v exists if and only if \( v \equiv 0, 1 \mod 5, \ v > 6 \). Moreover, in [9, Theorem 3.8] it has been proved that a balanced 4KS of order v exists if and only if \( v \equiv 1, 5 \mod 10, \ v \geq 11 \), and in [9, Theorem 2.3] we see that in a balanced 4KS of order v all the vertices have degree \( \frac{v-1}{2} \).

This paper is divided in two parts. In the first one, given by the first two sections, we study decompositions of graphs in 4-kites, providing so the instruments for the second part, in which we study bicolourings of 4-kite systems. In particular, we prove in the main result of the paper that \( c \)-bicolourings of balanced 4-kite systems exist if and only if \( c = 2, 3 \).

2 Decomposition of bipartite graphs in 4-kites

In this section we study the decomposition of bipartite graphs in 4-kites, determining first a necessary and sufficient condition.

**Theorem 2.1.** The complete bipartite graph \( K_{m,n} \) can be decomposed in 4-kites if and only if \( 5 \mid mn \), with \( m, n \neq 3 \) and \( m, n > 1 \).

**Proof.** \( K_{m,n} \) has \( mn \) edges and clearly \( \frac{mn}{5} \) must be an integer. Moreover, since a 4-kite contains a 4-cycle, it must be also \( m, n > 1 \).

Now, supposed that \( m = 5m' \) for some \( m' \geq 1 \), we need to prove that \( K_{5m',3} \) cannot be decomposed in 4-kites. So let \( X = \{ x_1, \ldots, x_{5m'} \} \) and \( Y = \{ y_1, y_2, y_3 \} \) be two disjoint sets and consider a decomposition of \( K_{X,Y} \) in 4-kites \( B_1, \ldots, B_{3m'} \). In any \( B_j \) there are either 2 vertices of \( X \) and all the 3 of \( Y \) or 3 of \( X \) and 2 of \( Y \). Moreover, in any \( B_j \) there are exactly two elements of \( Y \) of degree at least 2 and in \( B_j \) they are both adjacent to 2 distinct elements of \( X \). Let \( B_1, \ldots, B_r \) be the 4-kites in which \( y_1 \) is a vertex of degree at least 2 and let \( x_1, \ldots, x_{2r} \in X \) be the vertices adjacent to \( y_1 \) and to either \( y_2 \) or \( y_3 \) and note that \( r < 3m' \). This means that in the remaining \( B_{r+1}, \ldots, B_{3m'} \), in which \( y_2 \) and \( y_3 \) are vertices of degree at least 2, \( y_2 \) and \( y_3 \) are both adjacent to \( 2(3m' - r) \) vertices in \( \{ x_{2r+1}, \ldots, x_{5m'} \} \). So it must be:

\[
2(3m' - r) \leq 5m' - 2r \Rightarrow m' \leq 0.
\]

This is a contradiction and so we have proved that \( K_{5m',3} \) cannot be decomposed in 4-kites.

Let \( X = \{ x_1, \ldots, x_5 \} \) and \( Y = \{ y_1, y_2 \} \) two disjoint sets. We can decompose \( K_{X,Y} \) in the following two 4-kites:

\[
B_1 = (x_3, y_2, x_2, y_1) - x_1 \quad \text{and} \quad B_2 = (x_5, y_1, x_4, y_2) - x_1.
\]
Let $X = \{x_1, \ldots, x_5\}$ and $Y = \{y_1, \ldots, y_5\}$ two disjoint sets. We can decompose $K_{X,Y}$ in the following 4-kites:

$$
B_1 = (x_1, y_2, x_2, y_1) - x_3, \quad B_2 = (x_2, y_4, x_3, y_3) - x_4,
$$

$$
B_3 = (x_3, y_5, x_4, y_2) - x_5, \quad B_4 = (x_4, y_1, x_5, y_4) - x_1,
$$

$$
B_5 = (x_5, y_3, x_1, y_2) - x_2.
$$

So we have proved that $K_{5,2}$ and $K_{5,5}$ can be decomposed in 4-kites. This implies easily that $K_{5m,n}$ can be decomposed in 4-kites for any $m' \geq 1$ and $n > 1, n \neq 3$.

Indeed, suppose, first, that $m = 1$ and $n = 2h$ for some $h \geq 1$. Let $Y_1, \ldots, Y_h$ pairwise disjoint sets such that $|Y_i| = 2$ for any $i$. Let us decompose $K_{X,Y_i}$ for any $i$ in a family $B_i$ of 4-kites. So the set $B_1 \cup \cdots \cup B_h$ is a decomposition of $K_{X,Y}$ in 4-kites.

Suppose that $m = 1$ and $n = 5 + 2h$ for some $h \geq 0$. Let $Y_1, \ldots, Y_h + 1$ pairwise disjoint sets such that $|Y_i| = 2$ for any $i = 1, \ldots, h$ and $|Y_{h+1}| = 5$. Let us decompose $K_{X,Y_i}$ for any $i = 1, \ldots, h + 1$ in a family $B_i$ of 4-kites. So the set $B_1 \cup \cdots \cup B_{h+1}$ is a decomposition of $K_{X,Y}$ in 4-kites.

If $m > 1$ we proceed in a similar way and so the statement is proved.

The following remark will be very useful in the next section.

**Remark 2.2.** Given two disjoint sets $X$ and $Y$, in the proof of Theorem 2.1 we have seen that, if $|X| = 5$ and $|Y| = 5$, $K_{X,Y}$ can be decomposed in 4-kites, in such a way that for any $x \in X$ we have $d(x) = 3$ and for any $y \in Y$ we have $d(y) = 2$.

The following result is useful because it is more precise about the number of blocks containing a vertex in a decomposition of a bipartite graph.

**Theorem 2.3.** Let $X$ and $Y$ be two disjoint sets such that $|X| = 10m$ and $|Y| = 2n$, for some $m \geq 1$ and $n \geq 2$. Then there exists a decomposition of $K_{X,Y}$ in 4-kites in such a way that for any $x \in X$ and $y \in Y$ $d(x) = n$ and $d(y) = 5m$.

**Proof.** Let $n = 2$, $Z = \{z_1, \ldots, z_5\}$ and $Y = \{y_1, \ldots, y_5\}$. Let us decompose $K_{Z,Y}$ in the following 4-kites:

$$
B_1 = (y_2, z_2, y_1, z_1) - y_3, \quad B_2 = (y_3, z_2, y_4, z_3) - y_1,
$$

$$
B_3 = (z_5, y_3, z_4, y_2) - z_3, \quad B_4 = (z_5, y_1, z_4, y_4) - z_1.
$$

Then it is easy to see that any element in $Z$ has degree 2, $y_1$ and $y_3$ have degree 3 and $y_2$ and $y_4$ have degree 2.

This implies easily the statement in the case $m = 1$ and $n = 2$. Indeed, let $X = X_1 \cup X_2$, with $X_1$ and $X_2$ disjoint sets such that $|X_1| = |X_2| = 5$. Then we
can decompose \( K_{X,Y} \) in a family \( B_1 \) of 4-kites in such a way that any \( x \in X_1 \) have degree 2, \( y_1 \) and \( y_3 \) have degree 3 and \( y_2 \) and \( y_4 \) have degree 2. Similarly, we can decompose \( K_{X,Y} \) in a family \( B_2 \) of 4-kites in such a way that any \( x \in X_2 \) have degree 2, \( y_1 \) and \( y_3 \) have degree 2 and \( y_2 \) and \( y_4 \) have degree 3. The set \( B_1 \cup B_2 \) is a decomposition of \( K_{X,Y} \) in 4-kites which gives us the statement in the case \( m = 1 \) and \( n = 2 \).

Let \( n = 3 \), \( Z = \{z_1, \ldots, z_5\} \) and \( Y = \{y_1, \ldots, y_6\} \). Let us decompose \( K_{Z,Y} \) in the following 4-kites:

\[
B_1 = (y_2, z_2, y_1, z_1) - y_3, \quad B_2 = (y_4, z_4, y_3, z_3) - y_5, \quad B_3 = (y_6, z_1, y_5, z_5) - y_1, \\
B_4 = (z_5, y_3, z_2, y_4) - z_1, \quad B_5 = (z_3, y_1, z_4, y_2) - z_5, \quad B_6 = (z_2, y_5, z_4, y_6) - z_3.
\]

Then it is easy to see that any element in \( Z \) has degree 3, \( y_1, y_3 \) and \( y_5 \) have degree 3 and \( y_2, y_4 \) and \( y_6 \) have degree 2.

In a similar way to the previous remark, this implies easily the statement in the case \( m = 1 \) and \( n = 3 \).

Together with the previous remark this proves the statement in the case \( m = 1 \) and \( n \geq 2 \). Indeed, suppose, first, that \( n = 2h \) for some \( h \geq 2 \). Let \( Y_1, \ldots, Y_h \) pairwise disjoint sets such that \( |Y_i| = 4 \) for any \( i \). Let us decompose \( K_{X,Y_i} \) for any \( i \) in a family \( B_i \) of 4-kites in such a way that \( d(x) = 2 \) for any \( x \in X \) and \( d(y) = 5 \) for any \( y \in Y_i \). Then set \( B_1 \cup \cdots \cup B_h \) is a decomposition of \( K_{X,Y} \) in 4-kites such that \( d(x) = 2h = n \) for any \( x \in X \) and \( d(y) = 5 \) for any \( y \in Y \).

Suppose that \( n = 2h + 1 \) for some \( h \geq 2 \). Let \( Y_1, \ldots, Y_h \) pairwise disjoint sets such that \( |Y_i| = 4 \) for any \( i = 1, \ldots, h-1 \) and \( |Y_h| = 3 \). Let us decompose \( K_{X,Y_i} \) for any \( i = 1, \ldots, h-1 \) and \( K_{X,Y_h} \) in a family \( B_i \) of 4-kites in such a way that \( d(x) = 2 \) for any \( x \in X \) and \( d(y) = 5 \) for any \( y \in Y_i \) and \( K_{X,Y_h} \) in a family \( B_h \) of 4-kites in such a way that \( d(x) = 3 \) for any \( x \in X \) and \( d(y) = 5 \) for any \( y \in Y_h \). Then set \( B_1 \cup \cdots \cup B_h \) is a decomposition of \( K_{X,Y} \) in 4-kites such that \( d(x) = 2h + 1 = n \) for any \( x \in X \) and \( d(y) = 5 \) for any \( y \in Y \).

Now, in a similar way we get the statement for any \( m \geq 1 \) and \( n \geq 2 \) \( \square \)

**Corollary 2.4.** For any \( m \geq 1 \) there exists a decomposition in 4-kites of the complete equipartite graph \( K_{10m,10m} \) such that any vertex has degree \( 5m \).

### 3 Decomposition of unions of graphs in 4-kites

In this section we are going to prove some lemmas that will be used later in some constructions to obtain equitable colourings of 4-kite systems. First, let us recall the following definitions.

**Definition 3.1.** Let \( G_1 = (V_1, E_1) \) and \( G_2 = (V_2, E_2) \) be two simple graphs, having \( V_1 \) and \( V_2 \) as sets of vertices and \( E_1 \) and \( E_2 \) as sets of edges. The union of \( G_1 \) and \( G_2 \) is the graph \( G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2) \) having as set of vertices \( V_1 \cup V_2 \) and as set of edges \( E_1 \cup E_2 \).
Definition 3.2. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two simple graphs, having $V_1$ and $V_2$ as sets of vertices and $E_1$ and $E_2$ as sets of edges, and suppose that $V_2 \subseteq V_1$. The difference of $G_1$ and $G_2$ is the graph $G_1 - G_2 = (V_1, E_1 \setminus E_2)$ having as set of vertices $V_1$ and as set of edges $E_1 \setminus E_2$.

The results in this section will be proved using some basic decompositions that we will show in examples. Here we have the first:

Example 3.3. Let $X = \{x_1, x_2, x_3, x_4, x_5\}$ and $Y = \{y_1, y_2, y_3, y_4, y_5, y_6\}$ two disjoint sets. Let us consider the following blocks:

$$B_1 = (y_6, x_2, y_4, x_1) - x_3, B_2 = (x_4, y_5, x_3, y_2) - x_5, B_3 = (x_4, y_6, x_5, y_3) - x_1,$$
$$B_4 = (y_2, x_1, y_1, x_2) - y_5, B_5 = (y_5, x_1, x_4, x_5) - x_2, B_6 = (y_3, x_2, x_4, x_3) - y_6,$$
$$B_7 = (x_4, y_4, x_5, y_1) - x_3, B_8 = (x_5, x_1, x_2, x_3) - y_4.$$

Then these blocks are a decomposition of the graph $K_{X,Y} \cup K_X$ and it is easy to see that for any $x \in X$ we have $d(x) = 5$, while $y_1, y_2, y_3$ have degree 2 and $y_4, y_5, y_6$ have degree 3.

The first of the technical lemmas proved in this section is the following.

Lemma 3.4. Let $X_1, \ldots, X_{2m}, Y$ pairwise disjoint sets such that $|X_i| = 5$ for any $i$ and $|Y| = 11$. Then there exists a decomposition in 4-kites of $K_{X_1 \cup \cdots \cup X_{2m} \cup Y} - \cup_{i=1}^m K_{X_i, X_{i+m} \cup K_Y}$ such that $d(x) = 5m + 3$ for any $x \in X_i$, with $i \in \{1, \ldots, m\}$, $d(x) = 5m + 2$ for any $x \in X_j$, with $j \in \{m+1, \ldots, 2m\}$, and $d(y) = 5m$ for any $y \in Y$.

Proof. Let $Y_1$ and $Y_2$ disjoint sets such that $Y = Y_1 \cup Y_2$, with $|Y_1| = 6$ and $|Y_2| = 5$. In particular let $Y_1 = \{y_1, \ldots, y_6\}$. Let us consider:

- the family $A_i$, for any $i \in \{1, \ldots, m\}$, decomposing $K_{X_i, Y_1} \cup K_{X_i}$ in 4-kites in such a way that $d(x) = 5$ for any $x \in X_i$, $y_1, y_2, y_3$ have degree 3 and $y_4, y_5, y_6$ have degree 2 (see Example 3.3);

- the family $B_i$, for any $i \in \{m+1, \ldots, 2m\}$, decomposing $K_{X_i, Y_1} \cup K_{X_i}$ in 4-kites in such a way that $d(x) = 5$ for any $x \in X_i$, $y_1, y_2, y_3$ have degree 2 and $y_4, y_5, y_6$ have degree 3 (see Example 3.3);

- the family $C_i$, for any $i \in \{1, \ldots, m\}$, decomposing $K_{X_i, Y_2}$ in 4-kites in such a way that $d(x) = 3$ for any $x \in X_i$ and $d(y) = 2$ for any $y \in Y_2$ (see Remark 2.2);

- the family $D_i$, for any $i \in \{m+1, \ldots, 2m\}$, decomposing $K_{X_i, Y_2}$ in 4-kites in such a way that $d(x) = 2$ for any $x \in X_i$ and $d(y) = 3$ for any $y \in Y_2$ (see Remark 2.2);
the family $E_{ij}$, for any $i, j \in \{1, \ldots, m\}$ with $i \neq j$, decomposing $K_{X_i \cup X_{i+1m}, X_j \cup X_{j+1m}}$ in 4-kites in such a way that all the vertices have degree 5 (see Corollary 2.4).

Then, called $\mathcal{F}$ the set of all these blocks, it easy to see that the blocks of $\mathcal{F}$ are a decomposition of $K_{X_1 \cup \cdots \cup X_{2m} \cup Y} = (\bigcup_{i=1}^{m} K_{X_i, X_{i+1m}} \cup K_Y)$ that satisfy the conditions of the statement.

The next lemmas will be proved using the following:

**Example 3.5.** Let $X = \{x_1, x_2, x_3, x_4, x_5\}$ and $Y = \{y_1, y_2, y_3, y_4, y_5\}$ be two disjoint sets. Let us consider the following blocks:

$$
B_1 = (x_3, x_4, y_1, x_2) - x_1, \quad B_2 = (x_3, y_2, x_4, x_1) - x_5, \\
B_3 = (x_4, x_2, y_3, x_5) - y_4, \quad B_4 = (x_5, x_2, y_5, x_3) - y_1, \quad B_5 = (y_3, x_3, y_4, x_4) - y_5, \\
B_6 = (y_1, x_1, y_5, x_5) - y_2, \quad B_7 = (y_2, x_2, y_4, x_1) - y_3.
$$

Then these blocks are a decomposition of the graph $K_{X, Y} \cup K_X$ and it is easy to see that for any $x \in X$ we have $d(x) = 4$ and for any $y \in Y$ we have $d(y) = 3$.

**Lemma 3.6.** Let $X_1, \ldots, X_{2m}$ pairwise disjoint sets such that $|X_i| = 5$ for any $i$. Then there exists a decomposition of $K_{X_1 \cup \cdots \cup X_{2m}} - (K_{X_i} \cup \cdots \cup K_{X_m})$ in such a way that $d(x) = 5m - 2$ for any $x \in X_1 \cup \cdots \cup X_m$ and $d(y) = 5m - 1$ for any $y \in X_{m+1} \cup \cdots \cup X_{2m}$.

**Proof.** By Example 3.5 we can decompose $K_{X_i, X_{i+1m}} \cup K_{X_{i+1m}}$ for any $i = 1, \ldots, m$ in a family $A_i$ of 4-kites in such a way that $d(x) = 3$ for any $x \in X_i$ and $d(y) = 4$ for any $y \in X_{i+1}$.

By Corollary 2.4 we can decompose $K_{X_i \cup X_{i+1m}, X_j \cup X_{j+1m}}$ for any $i, j = 1, \ldots, m$, with $i \neq j$, in a family $B_{ij}$ of 4-kites in such a way that $d(x) = 5$ for any vertex $x \in X_i \cup X_{i+1m} \cup X_j \cup X_{j+1m}$.

Let $\mathcal{C} = \bigcup_i A_i \cup \bigcup_{i \neq j} B_{ij}$. Then the blocks of the family $\mathcal{C}$ are a decomposition of $K_{X_1 \cup \cdots \cup X_{2m}} - (K_{X_1} \cup \cdots \cup K_{X_m})$ in 4-kites such that $d(x) = 5m - 2$ for any $x \in X_1 \cup \cdots \cup X_m$ and $d(y) = 5m - 1$ for any $y \in X_{m+1} \cup \cdots \cup X_{2m}$. 

**Lemma 3.7.** Let $X_1, \ldots, X_{4m+2}$ pairwise disjoint sets such that $|X_i| = 5$ for any $i$. Then there exists a decomposition of:

$$
K_{X_1 \cup \cdots \cup X_{2m+1}, X_{2m+2} \cup \cdots \cup X_{4m+2}} \cup K_{X_1 \cup \cdots \cup X_{2m+2} \cup \cdots \cup X_{3m+2}}
$$

in such a way that $d(x) = 5m + 4$ for any $x \in X_i$, with $i \in \{1, \ldots, m\} \cup \{2m + 2, \ldots, 3m + 2\}$ and $d(y) = 5m + 3$ for any $y \in X_j$, with $j \in \{m + 1, \ldots, 2m + 1\} \cup \{3m + 3, \ldots, 4m + 2\}$.

**Proof.** Let us consider:
• for any $i = m + 1, \ldots, 2m$ the family $A_i$ of blocks obtained by decomposing $K_{x_1, x_{2m+2}}$ in 4-kites in such a way that $d(y) = 3$ for any $y \in X_i$ and $d(x) = 2$ for any $x \in X_{2m+2}$ (see Remark 2.2);
• for any $i = 1, \ldots, m$ the family $B_i$ of blocks obtained by decomposing $K_{x_1, x_{2m+2}} \cup K_{x_i}$ in 4-kites in such a way that $d(x) = 4$ for any $x \in X_i$ and $d(y) = 3$ for any $y \in X_{2m+2}$ (see Example 3.5);
• for any $i = 3m + 3, \ldots, 4m + 2$ the family $C_i$ of blocks obtained by decomposing $K_{x_i, x_{2m+1}}$ in 4-kites in such a way that $d(x) = 3$ for any $x \in X_i$ and $d(y) = 2$ for any $y \in X_{2m+1}$ (see Remark 2.2);
• for any $i = 2m + 3, \ldots, 3m + 2$ the family $D_i$ of blocks obtained by decomposing $K_{x_i, x_{2m+1}} \cup K_{x_i}$ in 4-kites in such a way that $d(x) = 4$ for any $x \in X_{2m+1}$ and $d(y) = 3$ for any $y \in X_{2m+1}$ (see Example 3.5);
• the family $E$ of blocks obtained by decomposing $K_{x_1 \cup \ldots \cup x_{2m}, x_{2m+2} \cup \ldots \cup x_{4m+2}}$ in 4-kites in such a way that all the vertices have degree $5m$ (see Corollary 2.4);
• the family $F$ of blocks obtained by decomposing $K_{x_{2m+1}, x_{2m+2}} \cup K_{x_{2m+2}}$ in 4-kites in such a way that $d(x) = 4$ for any $x \in X_{2m+2}$ and $d(y) = 3$ for any $y \in X_{2m+1}$ (see Example 3.5).

Then, called $G$ the set of all these blocks, it easy to see that the blocks of $G$ are a decomposition of $K_{x_1 \cup \ldots \cup x_{2m+1}, x_{2m+2} \cup \ldots \cup x_{4m+2}} \cup K_{x_1} \cup \ldots \cup K_{x_m} \cup K_{x_{2m+1}} \cup \ldots \cup K_{x_{4m+2}}$ that satisfy the conditions of the statement. \hfill \Box

For the next results, we need to fix some notation. Given $X = \{x_1, x_2, x_3, x_4, x_5\}$, we denote by $H_X$ the graph on the vertex set $X$ having edges:

$$\{\{x_1, x_2\}, \{x_1, x_3\}, \{x_2, x_3\}, \{x_3, x_4\}, \{x_4, x_5\}\}.$$

Recalling that a complete graph on 5 vertices cannot be decomposed in 4-kites, we have the following:

**Remark 3.8.** Given $X = \{x_1, x_2, x_3, x_4, x_5\}$, it is easy to see that the complete graph $K_X$ can be decomposed in the graph $H_X$ and in the 4-kite $G = (x_2, x_4, x_1, x_5) - x_3$.

**Example 3.9.** Let $X = \{x_1, x_2, x_3, x_4, x_5\}$ and $Y = \{y_1, y_2, y_3, y_4, y_5\}$ be two disjoint sets. Consider the following blocks:

$$B_1 = (y_1, x_3, x_1, x_2) - y_3, \ B_2 = (y_2, x_5, x_4, x_3) - y_4, \ B_3 = (y_3, x_1, y_5, x_3) - x_2, \ B_4 = (y_2, x_4, y_4, x_2) - y_5, \ B_5 = (y_1, x_5, y_4, x_1) - y_2, \ B_6 = (y_3, x_5, y_5, x_4) - y_1.$$
It is easy to see that these blocks are a decomposition of the graph $K_{X,Y} \cup H_X$ and, moreover, in such a decomposition all the vertices of $X$ and $Y$ have degree 3.

**Example 3.10.** Let $X = \{x_1, x_2, x_3, x_4, x_5\}$ and $Y = \{y_1, y_2, y_3, y_4, y_5\}$ be two disjoint sets. Consider the following blocks:

$$B_1 = (x_4, x_5, y_1, x_3) - x_1, \quad B_2 = (x_3, x_2, x_1, y_2) - x_4, \quad B_3 = (x_2, y_2, x_5, y_3) - x_3, $$

$$B_4 = (x_3, y_5, x_2, y_4) - x_5, \quad B_5 = (x_1, y_3, x_4, y_1) - x_2, \quad B_6 = (x_1, y_4, x_4, y_5) - x_5.$$

It is easy to see that these blocks are a decomposition of the graph $K_{X,Y} \cup H_X$ and, moreover, in such a decomposition $d(x) = 4$ for any $x \in X$ and $d(y) = 2$ for any $y \in Y$.

Keeping this notation we prove the following:

**Lemma 3.11.** Let $X_1, \ldots, X_{2m+1}$ pairwise disjoint sets such that $|X_i| = 5$ for any $i$. Let $G_i$ be a 4-kite having as set of vertices $X_i$, for any $i$, and let $H_{X_i} = K_{X_i} - G_i$ then there exists a decomposition in 4-kites of $K_{X_1 \cup \cdots \cup X_{2m+1}} - (G_1 \cup \cdots \cup G_{2m} \cup H_{X_{2m+1}})$ in such a way that all the vertices have degree $5m+1$.

**Proof.** Let us consider:

- the graph $G_{2m+1}$;
- for $i \in \{1, \ldots, m\}$ the family $A_i$ of blocks obtained by the decomposition of $K_{X_i, X_{2m+1}} \cup H_{X_i}$, in such a way that $d(x) = 4$ for any $x \in X_i$ and $d(y) = 2$ for any $y \in X_{2m+1}$ (see Example 3.10);
- for $i \in \{m+1, \ldots, 2m\}$ the family $B_i$ of blocks obtained by the decomposition of $K_{X_i, X_{2m+1}} \cup H_{X_i}$, in such a way that any vertex has degree 3 (see Example 3.9);
- if $m \geq 2$, for any $i, j \in \{1, \ldots, m\}$, with $i \neq j$, the family $C_{ij}$ of blocks obtained by the decomposition of $K_{X_i \cup X_{i+m}, X_j \cup X_{j+m}}$, in such a way that any vertex has degree 5 (see Corollary 2.4);
- for $i \in \{1, \ldots, m\}$ the family $D_i$ of blocks obtained by the decomposition of $K_{X_i, X_{i+m}}$ in such a way that $d(x) = 2$ for any $x \in X_i$ and $d(y) = 3$ for any $y \in X_{i+m}$ (see Remark 2.2).

Let the $\mathcal{E}$ be the collection of all these blocks. Then it is easy to see that the blocks of $\mathcal{F}$ are a decomposition of $K_{X_1 \cup \cdots \cup X_{2m+1}} - (G_1 \cup \cdots \cup G_{2m} \cup H_{X_{2m+1}})$ and that in this decomposition all the vertices have degree $5m+1$.

At last, we need the following example.
Example 3.12. Let $X = \{x_1, x_2, x_3, x_4, x_5\}$ and $Y = \{y_1, y_2, y_3, y_4, y_5\}$ be two disjoint sets. Consider the following blocks:

$$B_1 = (x_3, y_1, y_2, x_4) - x_5, \quad B_2 = (y_3, x_1, x_2, y_4) - y_5,$$
$$B_3 = (x_3, x_1, y_3) - x_5, \quad B_4 = (y_2, x_3, x_2, y_3) - x_4, \quad B_5 = (x_4, y_4, x_5, y_1) - x_2,$$
$$B_6 = (x_2, y_5, x_5, y_2) - x_1, \quad B_7 = (x_3, y_4, x_1, y_5) - x_4.$$

It is easy to see that these blocks are a decomposition of the graph $K_{X,Y} \cup H_X \cup H_Y$ and, moreover, in such a decomposition $d(x) = 4$ for any $x \in X$ and $d(y) = 3$ for any $y \in Y$.

Lemma 3.13. Let $X_1, \ldots, X_{2m+1}$ pairwise disjoint sets such that $|X_i| = 5$ for any $i$. Let $G_i$ be a 4-kite having as set of vertices $X_i$, for any $i$. Then there exists a decomposition in 4-kites of $K_{X_1 \cup \cdots \cup X_{2m+1}} - (G_1 \cup \cdots \cup G_{2m+1})$ in such a way that all the vertices have degree $5m + 1$.

Proof. In Remark 3.8 we have seen that for any $i$ the complete graph $K_{X_i}$ can be decomposed in the 4-kite $G_i$ and in a graph that we denoted by $H_{X_i}$. Keeping this notation, we can consider:

- a family $\mathcal{A}$ of blocks obtained by the decomposition of $K_{X_1, X_{2m+1}} \cup H_{X_1} \cup H_{X_{2m+1}}$, in such a way that $d(x) = 4$ for any $x \in X_1$ and $d(y) = 3$ for any $y \in X_{2m+1}$ (see Example 3.12);

- if $m \geq 2$, for $i \in \{2, \ldots, m\}$ the family $\mathcal{B}_i$ of blocks obtained by the decomposition of $K_{X_i, X_{2m+1}} \cup H_{X_i}$, in such a way that $d(x) = 4$ for any $x \in X_i$ and $d(y) = 2$ for any $y \in X_{2m+1}$ (see Example 3.10);

- for $i \in \{m+1, \ldots, 2m\}$ the family $\mathcal{C}_i$ of blocks obtained by the decomposition of $K_{X_i, X_{2m+1}} \cup H_{X_i}$, in such a way that any vertex has degree $3$ (see Example 3.9);

- if $m \geq 2$, for any $i, j \in \{1, \ldots, m\}$, with $i \neq j$, the family $\mathcal{D}_{ij}$ of blocks obtained by the decomposition of $K_{X_i \cup X_{i+m}, X_j \cup X_{j+m}}$, in such a way that any vertex has degree $5$ (see Corollary 2.4);

- for $i \in \{1, \ldots, m\}$ the family $\mathcal{E}_i$ of blocks obtained by the decomposition of $K_{X_i, X_{i+m}}$ in such a way that $d(x) = 2$ for any $x \in X_i$ and $d(y) = 3$ for any $y \in X_{i+m}$ (see Remark 2.2).

Let $\mathcal{F}$ be set of all these blocks. Then it is easy to see that the blocks of $\mathcal{F}$ are a decomposition of $K_{X_1 \cup \cdots \cup X_{2m+1}} - (G_1 \cup \cdots \cup G_{2m+1})$ and that in this decomposition all the vertices have degree $5m + 1$. 

\[\square\]
4 On the upper 2-chromatic index

In this section we are going to determine for any order an upper bound for the number of colours in a bicolouring of a 4-kite system.

**Lemma 4.1.** Let $\mathcal{B}$ a family of 4-kites having set of vertices $X$. If any vertex of $X$ belongs to at least $n$ blocks of $\mathcal{B}$, then $|X| \geq 2n + 1$.

*Proof.* Let us suppose that every vertex in $X$ is adjacent to at most $2n - 1$ vertices of $X$. Given $x \in X$, let $a_x$ be the number of blocks in which $x$ has degree 1, $b_x$ the number of blocks in which $x$ has degree 2 and $c_x$ the number of blocks in which $x$ has degree 3. Then, we have:

$$a_x + 2b_x + 3c_x \leq 2n - 1. \quad (1)$$

Moreover, $\sum_{x \in X} a_x = |\mathcal{B}|$, $\sum_{x \in X} b_x = 3|\mathcal{B}|$ and $\sum_{x \in X} c_x = |\mathcal{B}|$. So by (1) we get:

$$10|\mathcal{B}| \leq |X|(2n - 1). \quad (2)$$

However, by hypothesis it must be $|\mathcal{B}| \geq \frac{n|X|}{5}$. This contradicts (2) and so the statement is proved. \hfill \square

Now we can prove the following:

**Theorem 4.2.** Given a 4-kite $G$, $\overline{\chi}_2^G(v) \leq 3$ for any $v \equiv 1, 5 \ (\text{mod } 10)$, $v \geq 11$.

*Proof.* Let $\Sigma = (V, \mathcal{B})$ be a 4KS($v$) and let $\phi: \mathcal{B} \to C$ be a $c$-bicolouring of $\Sigma$. So $|C| = c$ and let $\gamma \in C$. We know that any element $v \in V$ is incident with exactly $\frac{v - 1}{2}$ blocks of $\mathcal{B}$. In particular, any element $v \in V$ incident with blocks colored with $\gamma$ must be incident with at least $\left\lceil \frac{v - 1}{4} \right\rceil$ blocks colored with $\gamma$. So by Lemma 4.1 there are at least $2\left\lceil \frac{v - 1}{4} \right\rceil + 1$ vertices incident with blocks colored with $\gamma$. This means that:

$$c \left( 2 \left\lceil \frac{v - 1}{4} \right\rceil + 1 \right) \leq 2v,$$

so that $c \leq 3$ if $v \equiv 1, 5 \ (\text{mod } 10)$. If $v \equiv 11, 15 \ (\text{mod } 20)$, then $\left\lceil \frac{v - 1}{4} \right\rceil = \frac{v - 3}{4}$ and we easily get that $c \leq 4$.

Let $c = 4$ and $v \equiv 11, 15 \ (\text{mod } 20)$. Let $\mathcal{B}_i$ be the set of blocks coloured with $i$, for $i = 1, 2, 3, 4$, and let $X_i$ be the set of vertices incident with the blocks of $\mathcal{B}_i$. By Lemma 4.1 we know that for any $i$ $|X_i| = 2\left\lceil \frac{v - 1}{4} \right\rceil + 1 + k_i = \frac{v - 1}{2} + k_i$ for some $k_i \geq 0$. Since $\sum_{i=1}^4 |X_i| = 2v$, we get that $\sum_{i=1}^4 k_i = 2$. So there are two possibilities: either $k_1 = k_2 = k_3 = 0$ and $k_4 = 2$ or $k_1 = k_2 = k_3 = 0$ and $k_3 = k_4 = 1$.

Suppose that $k_1 = k_2 = k_3 = 0$ and $k_4 = 2$, so that $|X_1| = |X_2| = |X_3| = \frac{v - 1}{2}$ and $|X_4| = \frac{v + 3}{2}$. So:

$$\frac{v(v - 1)}{10} = |\mathcal{B}| = \sum_{i=1}^4 |\mathcal{B}_i| \leq \frac{3}{10} \frac{v - 1}{2} - \frac{3}{2} \frac{v - 3}{2} + \frac{1}{10} \frac{v + 3}{2} + \frac{1}{2} \frac{v + 1}{2}.$$
This leads to a contradiction. So suppose that $k_1 = k_2 = 0$ and $k_3 = k_4 = 1$. In this case, $|X_1| = |X_2| = \frac{v-1}{2}$ and $|X_3| = |X_4| = \frac{v+1}{2}$ and so:

$$\frac{v(v-1)}{10} = |B| = \sum_{i=1}^{4} |B_i| \leq \frac{2}{10} \cdot \frac{v-1}{2} - \frac{2}{2} + \frac{2}{10} \cdot \frac{v+1}{2} - \frac{2}{2}.$$ 

We get again a contradiction and this shows that it must be $c \leq 3$. 

The main result of the paper is the following:

**Theorem 4.3.** For a 4-kite $G$, $\Omega_5^G(v) = \{2, 3\}$ for any $v \equiv 1, 5 \mod 10$, with $v \geq 11$.

In the next sections we will see the proof of this theorem, since we will need to distinguish the cases $v \equiv 1 \mod 20$, $v \equiv 11 \mod 20$, $v \equiv 5 \mod 20$ and $v \equiv 15 \mod 20$.

## 5 Bicolourings for $v = 10h + 1$

In this section we deal with the case that the order $v$ is $\equiv 1 \mod 10$. We need to distinguish two cases: $v \equiv 1 \mod 20$ and $v \equiv 11 \mod 20$. In this cases we will use the fact there exists for such orders a cyclic $4KS$.

**Proof of Theorem 4.3: case $v \equiv 1 \mod 20$.** By [9, Theorem 4.1] there exists a cyclic decomposition of the complete graph $K_v$ on $\mathbb{Z}_{20h+1}$ in 4-kites, with $v = 20h + 1$. Let $B_1, \ldots, B_{2h}$ the base block of such a decomposition and let $\Sigma = (\mathbb{Z}_{20h+1}, B)$ the $4KS$ generated, so that the blocks of $B$ are all the translates of the blocks $B_i$ for any $i = 1, \ldots, 2h$. Assign the colour 1 to the blocks $B_i$ and all their translates, for $i \in \{1, \ldots, h\}$ and the colour 2 to the blocks $B_i$ and all their translates, for $i \in \{h+1, \ldots, 2h\}$: this assignment determines a 2-bicolouring of $\Sigma$.

Let $A$ and $B$ two disjoint sets such that $|A| = |B| = 10h$ and let $\infty \notin A \cup B$. By [9, Theorem 3.8] there exists a balanced 4-kite design $\Sigma_1 = (A \cup \{\infty\}, B_1)$, so that any element in $A \cup \{\infty\}$ has degree 5$h$. Similarly, there exists a balanced 4-kite design $\Sigma_2 = (B \cup \{\infty\}, B_2)$, so that any element in $B \cup \{\infty\}$ has degree 5$h$. By Corollary 2.4 there exists a decomposition in 4-kites $C_j$, for $j = 1, \ldots, 20h^2$, of the complete equipartite graph $K_{A,B}$, in such a way that any element in $A \cup B$ has degree 5$h$. So the system $\Sigma = (A \cup B \cup \{\infty\}, B_1 \cup B_2 \cup C_j)$ is a balanced $4KS(v)$. Assign the colour 1 to the blocks of $B_1$, the colour 2 to the blocks $B_2$ and the colour 3 to the blocks $C_j$: this assignment determines a 3-bicolouring of $\Sigma$. Now by Theorem 4.2 we get the statement in the case $v \equiv 1 \mod 20$. 

In the case $v \equiv 11 \mod 20$ we will use both the difference method technique and the decomposition techniques introduced previously.
Proof of Theorem 4.3: case $v \equiv 11 \mod 20$. (1) Consider on $\mathbb{Z}_{20h+11}$ the blocks $B_i = (2i, 10h + 6, 2i - 1, 0) - (4h + 2 + i)$ for $i = 1, \ldots, 2h$ and the block $C = (4h + 1, 10h + 6, 14h + 8, 0) - (6h + 4)$. Then the system $\Sigma = (\mathbb{Z}_{20h+11}, \mathcal{B})$ having as blocks all the translates of the blocks $B_i$ and of $C$ is a balanced $4KS$ of order $20h + 11$.

Let us assign the colour 1 to the blocks $B_i$ and all their translates, for $i \in \{1, \ldots, h\}$, and to the blocks $C + i = (4h + 1 + i, 10h + 6 + i, 14h + 8 + i, i) - (6h + 4 + i)$ for $i \in \{0, \ldots, 10h + 5\}$, and the colour 2 to the blocks $B_i$ and all their translates, for $i \in \{h + 1, \ldots, 2h\}$ and to the blocks $C + i = (4h + 1 + i, 10h + 6 + i, 14h + 8 + i, i) - (6h + 4 + i)$ for $i \in \{10h + 6, \ldots, 20h + 10\}$: this assignment determines a 2-bicolouring of $\Sigma$. Indeed, considered:

$$Y = \{0, 4h + 1, 4h + 2\} \cup \{6h + 4, \ldots, 14h + 6\} \cup \{14h + 8, \ldots, 16h + 9\},$$

the vertices in $Y$ are incident with $5h + 3$ blocks coloured with 1, while the remaining ones are incident with $5h + 2$ blocks coloured with 1 and, conversely, the vertices in $Y$ are incident with $5h + 2$ blocks coloured with 2, while the remaining ones are incident with $5h + 3$ blocks coloured with 2. This shows that $2 \in \Omega_2^C(20h + 11)$ for any $h \geq 0$.

(2) Let $h = 0$. Let us consider on $X = \{0, 1, \ldots, 10\}$ the following blocks:

$$B_1 = (4, 0, 5, 1) - 2, B_2 = (6, 1, 7, 0) - 3, B_3 = (4, 2, 5, 3) - 1,$$
$$B_4 = (6, 3, 7, 2) - 0, B_5 = (5, 8, 6, 4) - 7, B_6 = (6, 5, 9, 7) - 10,$$
$$B_7 = (7, 5, 10, 8) - 4, B_8 = (4, 10, 6, 9) - 8, B_9 = (8, 2, 9, 0) - 1,$$
$$B_{10} = (8, 3, 10, 1) - 9, B_{11} = (2, 3, 9, 10) - 0.$$

Then it is easy to see that the system $\Sigma = (X, \cup_{i=1}^{11} B_i)$ is a balanced $4KS(11)$. Let us assign a colouring in the following way:

1. assign the colour 1 to the blocks $B_1, B_2, B_3$ and $B_4$,
2. assign the colour 2 to the blocks $B_5, B_6, B_7$ and $B_8$,
3. assign the colour 3 to the blocks $B_9, B_{10}$ and $B_{11}$.

It is easy to see that this is a 3-bicolouring of $\Sigma$ and so $3 \in \Omega_2^C(11)$.

(3) Let $h \geq 1$. Let $X_1, \ldots, X_{4h}, Y$ pairwise disjoint sets such that $|X_1| = \cdots = |X_{4h}| = 5$ and $|Y| = 11$. We will construct a $4KS$ of order $20h + 11$ on $X = \cup_{i=1}^{4h} X_i \cup Y$. Let us consider:

- the families $\mathcal{A}_1$ and $\mathcal{A}_2$ of blocks such that $\Sigma = (Y, \mathcal{A}_1 \cup \mathcal{A}_2)$ is a $4KS(11)$ and there exists a 2-bicolouring of $\Sigma$ having colour classes $\mathcal{A}_1$ and $\mathcal{A}_2$ (possible by what we just proved);
Proof of Theorem 4.3: case $v$ dealt with in the next section, and even in the case $v \not\equiv 6$ the statement in the case $v \in \mathbb{N}$ proves that 3.

It is easy to see that with this assignment $\phi$ for $i \in \{1, \ldots, h\}$ and $j \in \{h+1, \ldots, 4h\}$, and $d(y) = 5h$ for any $y \in Y$ (see Lemma 3.4);.

for $i \in \{1, \ldots, h\}$ the families $D_i$ of blocks obtained by decomposing $K_{X_{i+1}}$ in such a way that $d(x) = 2$ for any $x \in X_i$ and $d(y) = 3$ for any $y \in X_{i+h}$ (see Remark 2.2);

the family of blocks $E$ obtained by decomposing $K_{X_1 \cup \cdots \cup X_{2h+1}}$ in such a way that all the vertices have degree $5h$ (see Corollary 2.4).

It is not difficult to see that, called $F$ the set of all these blocks, $\Sigma' = (X, F)$ is a balanced 4KS of order $20h + 11$. Let us assign a colouring $\phi : F \to \{1, 2, 3\}$ in the following way:

1. let us assign the colour 1 to the families of blocks $A_1$ and $B$;
2. let us assign the colour 2 to the families of blocks $A_2$ and $C$;
3. let us assign the colour 3 to the families of blocks $E$ and $D_i$ for $i \in \{1, \ldots, h\}$.

It is easy to see that with this assignment $\phi$ is a 3-bicolouring of $\Sigma$, so that this proves that $3 \in \Omega_2^G(20h + 11)$ for any $h \geq 1$. Now by Theorem 4.2 we get the statement in the case $v \equiv 1 \mod 20$.

6 Bicolourings for $v = 20h + 5$

Even in the case $v \equiv 5 \mod 10$ we will distinguish two cases, $v \equiv 5 \mod 20$, which will be dealt with in this section, and $v \equiv 15 \mod 20$, which will be dealt with in the next section.

Proof of Theorem 4.3: case $v \equiv 5 \mod 20$. (1) Let us consider on $X = \mathbb{Z}_{4h+1} \times \{1, 2, 3, 4, 5\}$ the following 4-kites:

- $A_{ij} = ((j+1)_{i+3}, (2h+1)_{i}, (j+1)_{i+1}, 0_i) - j_i$ for any $i \in \{1, \ldots, 5\}$, $j \in \{1, \ldots, 2h\}$, with $(i, j) \neq (1, 1)$, where the indices are taken mod 5,
- $B = ((2h+1), 2, 0, 2) - 3$,
\begin{itemize}
  \item $C = (2, 1, 1, 0, 5, 1) - 2_4$,
  \item $D = (0, 1, 4, 2, 5, 1) - 2_3$.
\end{itemize}

Denoted by $\mathcal{B}$ the set of all the translates of these blocks, it is easy to see that the system $\Sigma = (X, \mathcal{B})$ is a balanced $4K \Sigma(20h + 5)$. Moreover, let us assign the colour 1 to the blocks $B, C, A_{ij}$ for any $i$ and $j \in \{1, \ldots, h\}$, with $(i, j) \neq (1, 1)$, and all their translates and the colour 2 to the blocks $D, A_{ij}$ for any $i$ and $j \in \{h + 1, \ldots, 2h\}$ and all their translates. Then, this assignment determines a 2-bicolouring of $\Sigma$ and this shows that $2 \in \Omega_2^G(20h + 5)$ for any $h \geq 1$.

\(2\) Let $h \geq 2$. We will show another construction in order to prove that $2 \in \Omega_2^G(20h + 5)$ for $h \geq 2$. In this case we will use 2-bicolourings of $4K \Sigma(20k + 1)$ for some $k$.

Let us consider $X_1, \ldots, X_6$ and $Y_1, \ldots, Y_4$ pairwise disjoint sets such that $|X_i| = 4$ for any $i = 1, \ldots, 6$ and $|Y_j| = 5h - 5$ for $j = 1, \ldots, 4$. Let us consider also an element $\infty \notin X \cup Y$. Let us consider:

\begin{itemize}
  \item a $4K \Sigma \Sigma_1 = \{(\bigcup_{i=1}^6 X_i \cup \{\infty\}, \mathcal{B}_1 \cup \mathcal{B}_2)\}$ with a 2-bicolouring, where $\mathcal{B}_1$ and $\mathcal{B}_2$ are the two colour classes (possible by the case $h = 1$);
  \item a $4K \Sigma \Sigma_2 = \{(\bigcup_{j=1}^4 Y_j \cup \{\infty\}, \mathcal{C}_1 \cup \mathcal{C}_2)\}$ with a 2-bicolouring, where $\mathcal{C}_1$ and $\mathcal{C}_2$ are the two colour classes (see the case $v = 1 \mod 20$ already proved);
  \item the family $\mathcal{D}_{ij}$, for $(i, j) \in \{(1, 1), (1, 3), (4, 1), (4, 3)\}$ obtained by the decomposition of $K_{X_i \cup X_i+1 \cup X_i+2 \cup Y_j \cup Y_j+1}$ in 4-kites in such a way that $d(x) = 5h - 5$ for any $x \in X_i \cup X_i+1 \cup X_i+2$ and $d(y) = 6$ for any $y \in Y_j \cup Y_j+1$ (see Theorem 2.3).
\end{itemize}

Let now $\mathcal{E}$ be the family of the blocks of $\mathcal{B}_1, \mathcal{B}_2, \mathcal{C}_1, \mathcal{C}_2$ and $\mathcal{D}_{ij}$, for $(i, j) \in \{(1, 1), (1, 3), (4, 1), (4, 3)\}$. Then it is easy to see that the system $\Sigma = \{(\bigcup_{j=1}^4 Y_j \cup \bigcup_{i=1}^6 X_i \cup \{\infty\}, \mathcal{E})\}$ is a balanced $4K \Sigma$ of order $20h + 5$.

Let us assign the colour 1 to the blocks of $\mathcal{B}_1, \mathcal{C}_1, \mathcal{D}_{11}$ and $\mathcal{D}_{43}$ and the colour 2 to the blocks of $\mathcal{B}_2, \mathcal{C}_2, \mathcal{D}_{13}$ and $\mathcal{D}_{41}$. It is easy to see that this is a 2-bicolouring of $\Sigma$, because any vertex is incident with $5h + 1$ blocks coloured with 1 and with $5h + 1$ blocks coloured with 2.

\(3\) Now we prove that $3 \in \Omega_2^G(20h + 5)$ for any $h \geq 1$. Let $X_1, \ldots, X_{4h+1}$ be pairwise disjoint sets such that $|X_i| = 5$ for any $i$. By Remark 3.8 for any $i$ we can decompose the complete graph $K_{X_i}$ in a 4-kite $G_i$ and in a graph that we denote by $H_{X_i}$. Let us consider:

\begin{itemize}
  \item the family $\mathcal{A}$ of blocks obtained by a decomposition of $K_{X_1 \cup \ldots \cup X_{2h}, X_{2h+1} \cup \ldots \cup X_{4h}}$ in such a way that any vertex has degree $5h$ (see Corollary 2.4);
\end{itemize}
• the family \( B \) of blocks obtained by a decomposition in 4-kites of \( K_{X_1 \cup \cdots \cup X_{2h} \cup X_{4h+1}} \) in such a way that all the vertices have degree \( 5h + 1 \) (see Lemma 3.13);

• the family \( C \) of blocks obtained by a decomposition in 4-kites of \( K_{X_{2h+1} \cup \cdots \cup X_{4h} \cup X_{4h+1}} \) in such a way that all the vertices have degree \( 5h + 1 \) (see Lemma 3.11);

Denoted by \( D \) the set of all the previous blocks and of the blocks \( G_1, \ldots, G_{4h} \), it is not difficult to see that \( \Sigma = (\cup_{i=1}^{4h+1} X_i, \mathcal{G}) \) is a balanced \( 4KS \) of order \( 20h + 5 \).

Let us consider the colouring \( \phi : \mathcal{G} \to \{1, 2, 3\} \) defined in the following way:

1. assign the colour 1 to the blocks of the family \( A \) and to the blocks \( G_1, \ldots, G_{4h} \);

2. assign the colour 2 to the blocks of the family \( B \);

3. assign the colour 2 to the blocks of the family \( \mathcal{C} \).

Then it is easy to see that this assignment is a 3-bicolouring of \( \Sigma \), so that \( 3 \in \Omega_2^3(20h + 5) \) for any \( h \geq 1 \). So by Theorem 4.2 we get the statement in the case \( v \equiv 1 \mod 20 \).

\[ \square \]

7 Bicolourings for \( v = 20h + 15 \)

At last we suppose that \( v \equiv 15 \mod 20 \).

**Proof of Theorem 4.3: case \( v \equiv 15 \mod 20 \).** (1) Let \( X = \mathbb{Z}_{10h+7} \times \{1, 2\} \cup \{\infty\} \). Consider the following blocks:

• \( A_i = (\infty, (i + 5h + 3)_1, (i - 1)_1, i_2) - (i + 1)_1 \) for \( i \in \{0, \ldots, 10h + 6\} \),

• \( B_1 = (0_2, 3_2, 2_2, 0_1) - 2_1 \),

• \( B_2 = ((10h + 5)_2, (10h + 3)_2, 0_1, 1_1) - (10h + 6)_2 \),

• \( C_i = ((i + 2)_2, (10h + 6)_2, (i + h + 2)_2, 0_1) - (i + 4h + 2)_1 \) for \( i \in \{1, \ldots, h\} \), if \( h > 0 \),

• \( D_i = (h_1, (i + 3h + 2)_1, 0_1, (i + 3h + 2)_2) - (9h + 6)_2 \) for \( i \in \{1, \ldots, h\} \), if \( h > 0 \),

• \( E_i = ((i + 2h + 3)_2, (9h + 7)_2, (i + 4)_1, 0_2) - (i + 2h + 4)_1 \) for \( i \in \{1, \ldots, h\} \), if \( h > 0 \),

• \( F_i = ((i + 2)_1, (9h + 7)_1, (i + 4h + 2)_2, 0_1) - (i + 6h + 2)_2 \) for \( i \in \{1, \ldots, h\} \), if \( h > 0 \).
Let $\mathcal{B}$ be the set of the blocks $A_i$ and of all the translates of $B_1$, $B_2$, $C_i$, $D_i$, $E_i$ and $F_i$. It is easy to see that $\Sigma = (X, \mathcal{B})$ is a balanced $4KS$ of order $20h + 15$ for any $h \geq 0$.

Now let us assign a 2-bicolouring of $\Sigma$ in the following way:

1. let us assign the colour 1 to the blocks $A_i$ for $i \in \{0, \ldots, 5h + 2\}$ and to the blocks $B_1$, $C_i$ and $D_i$ (these last two only if $h > 0$) and all their translates,

2. let us assign the colour 2 to the blocks $A_i$ for $i \in \{5h + 3, \ldots, 10h + 6\}$ and to the blocks $B_2$, $E_i$ and $F_i$ (these last two only if $h > 0$) and all their translates.

Then this is a 2-bicolouring of $\Sigma$ and it shows that $2 \in \Omega_2^G(20h + 15)$.

(2) Let $X$, $Y$ and $Z$ pairwise disjoint sets such that $|X| = |Y| = |Z| = 5$. Let us consider:

- the family $\mathcal{B}_1$ of blocks blocks obtained by decomposing $K_{X,Y} \cup K_X$ (see Example 3.5), in such a way that $d(x) = 4$ for any $x \in X$ and $d(y) = 3$ for any $y \in Y$;

- the family $\mathcal{B}_2$ of blocks blocks obtained by decomposing $K_{X,Z} \cup K_Z$ (see Example 3.5), in such a way that $d(x) = 3$ for any $x \in X$ and $d(z) = 4$ for any $z \in Z$;

- the family $\mathcal{B}_3$ of blocks blocks obtained by decomposing $K_{Y,Z} \cup K_Y$ (see Example 3.5), in such a way that $d(y) = 4$ for any $y \in Y$ and $d(z) = 3$ for any $z \in Z$.

It is easy to see that $\Sigma = (X \cup Y \cup Z, \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3)$ is a balanced $4KS$ of order 15. We can assign a 3-bicolouring of $\Sigma$ by assigning the colour 1 to the blocks of $\mathcal{B}_1$, the colour 2 to the blocks of $\mathcal{B}_2$ and the colour 3 to the blocks of $\mathcal{B}_3$. This shows that $3 \in \Omega_2^G(15)$.

(3) Let $h > 0$. Let $X_i$ for $i \in \{1, \ldots, 4h + 3\}$ pairwise disjoint sets such that $|X_i| = 5$ for any $i$ and let $X = \bigcup_{i=1}^{4h+3} X_i$. Consider:

- the family $\mathcal{A}$ of blocks decomposing $K_{X_{1+1}} \cup \cdots \cup X_{2h+2} - (K_{X_1} \cup \cdots \cup K_{X_{h+1}})$ in such a way that $d(x) = 5h + 4$ for any $x \in X_i$, with $i \in \{h+2, \ldots, 2h+2\}$, and $d(y) = 5h + 3$ for any $y \in X_j$, with $j \in \{1, \ldots, h+1\}$ (see Lemma 3.6);

- the family $\mathcal{B}$ of blocks decomposing $K_{X_{2h+3}} \cup \cdots \cup X_{4h+3} - (K_{X_{2h+2}} \cup \cdots \cup K_{X_{3h+2}})$ in such a way that $d(x) = 5h + 4$ for any $x \in X_i$, with $i \in \{3h+3, \ldots, 4h+3\}$, and $d(y) = 5h + 3$ for any $y \in X_j$, with $j \in \{2h+2, \ldots, 3h+2\}$ (see Lemma 3.6);
the family $\mathcal{C}$ of blocks decomposing $K_{X_1 \cup \cdots \cup X_{2h+1} \cup X_{2h+3} \cup \cdots \cup K_{X_{h+1} \cup K_{X_{2h+3}} \cup \cdots \cup K_{X_{2h+1}}}}$ in such a way that $d(x) = 5h + 4$ for any $x \in X_i$, with $i \in \{1, \ldots, h+1\} \cup \{2h+3, \ldots, 3h+2\}$, and $d(y) = 5h + 3$ for any $y \in \{h+2, \ldots, 2h+1\} \cup \{3h+3, \ldots, 4h+3\}$ (see Lemma 3.7).

Let $\mathcal{D}$ be the set of all these blocks. It is not difficult to see that $\Sigma = (X, \mathcal{D})$ is a $4KS(20h+15)$. Now we can assign a 3-bicolouring of $\Sigma$ in the following way:

1. assign the colour 1 to the blocks of the families $\mathcal{A}$;
2. assign the colour 2 to the blocks of the families $\mathcal{B}$;
3. assign the colour 3 to the blocks of the families $\mathcal{C}$.

This shows that $3 \in \Omega_2^G(20h+15)$ and so by Theorem 4.2 we get the statement in the case $v \equiv 1 \mod 20$. \hfill \qed

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**References**


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