Inference of Bivariate Generalized Exponential Distribution Based on Copula Functions

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Abstract

The generalized exponential (GE) distribution is an important lifetime distribution in survival analysis. It is considered as a suitable alternative to the most common lifetime distributions such as gamma and Weibull distributions. In this paper, a bivariate generalized exponential distribution (BVGE) is studied based on copula functions. Two illustrative examples are introduced to compare the studied estimation methods for the two proposed models using simulated and real data sets. Useful results are obtained.
Keywords: Bivariate generalized exponential distribution, Farlie-Gumbel-Morgenstern copula, Plackett copula

1 Introduction

The statistics literature has considerable lifetime distributions for modeling survival data. The generalized exponential distribution is one of the important lifetime distributions in survival analysis. It provides better fit than gamma and Weibull distributions, see Gupta and Kundu in [5] and [6]. This distribution has been studied extensively by Gupta and Kundu and others. Gupta and Kundu in [7] employed different estimation procedures to estimate the generalized exponential distribution parameters and compared the performance of these procedures. Gupta and Kundu in [8] used the ratio of the maximized likelihoods (RML) to distinguish between the Weibull and generalized exponential distributions. They reused the same techniques to distinguish between the gamma and generalized exponential distributions in [9]. Sarhan in [19] derived the estimates of the parameters of generalized exponential distribution in the presence of incomplete and censored data for a competing risks model. Danish and Aslam in [2] presented the Bayesian analysis of generalized exponential distribution in proportional hazards model of random censorship under asymmetric loss functions. Mohie El-Din et al. in [16] studied and estimated the parameters of a generalized exponential distribution based on an adaptive progressively type-II censored sample.

In the case of bivariate extension, recent researches have been made for the bivariate generalized exponential distribution. Kundu and Gupta in [13] introduced a bivariate generalized exponential so that the marginal distributions are generalized exponential distributions. Kundu and Gupta in [14] constructed a new absolute continuous bivariate generalized exponential distribution with generalized exponential distribution as marginal using Clayton copula. Achcar et al. in [1] introduced a Bayesian analysis for the bivariate generalized exponential distribution based on FGM copula in the presence of censored data.

The univariate generalized exponential distribution has the distribution function;

\[ F(t; \alpha, \lambda) = (1 - e^{-\lambda t})^\alpha ; \quad t > 0, \quad \alpha, \lambda > 0, \quad (1) \]

and the density function;

\[ f(t; \alpha, \lambda) = \alpha \lambda e^{-\lambda t} (1 - e^{-\lambda t})^{\alpha - 1} ; \quad t > 0, \quad \alpha, \lambda > 0, \quad (2) \]

where \( \alpha, \lambda \) are the shape and scale parameters respectively. The density function has great flexibility of fitting depending on the shape parameter \( \alpha \); if \( \alpha < 1 \), it becomes a decreasing function and if \( \alpha > 1 \), it becomes a unimodal function with mode given by \( \lambda^{-1} \log \alpha \). Observe that if \( \alpha = 1 \), it becomes an exponential distribution with parameter \( \lambda \). The reliability function is defined as
\[ R(t; \alpha, \lambda) = 1 - (1 - e^{-\lambda t})^\alpha \quad ; \quad t > 0 \quad , \quad \alpha, \lambda > 0, \]

and the hazard function is defined as

\[ h(t; \alpha, \lambda) = \frac{\alpha \lambda e^{-\lambda t} (1 - e^{-\lambda t})^{\alpha - 1}}{1 - (1 - e^{-\lambda t})^\alpha} \quad ; \quad t > 0 \quad , \quad \alpha, \lambda > 0. \]

The main aim of this paper is to study the bivariate extension of the generalized exponential distribution based on copula functions. Different estimation methods for estimating the parameters of the suggested models are used. A simulation study is conducted to compare the preferences between estimation methods. Also, a real data set is introduced and analyzed to investigate the new models.

This paper is organized as follows. In Section 2, the concept of copula and its estimation procedures are reviewed. The construction of the bivariate generalized exponential distribution based on copula are discussed in Section 3. The estimation of bivariate generalized exponential distribution based on copula are discussed in Section 4. The analysis of simulated and real data sets is provided in Section 5. Finally, the paper is concluded in Section 6.

2 Brief Review of Copula

The definition and the estimation methods of copula are reviewed in this section.

2.1 Copula Definition

Copula is a convenient approach to describe a multivariate (bivariate) distribution with dependence structure. It is a multivariate (bivariate) probability distribution for which the marginal probability distribution of each variable is uniform on the interval (0, 1).

Consider two random variables \( T_1 \) and \( T_2 \), each with distribution functions \( F_1(t_1) \) and \( F_2(t_2) \) respectively, each of \( F_1(t_1) \) and \( F_2(t_2) \) is uniform (0,1) distributed. According to Sklar in [20], there exists a copula \( C \) such that, for all \( t_1 \) and \( t_2 \), there is a joint distribution function

\[
F(t_1, t_2) = C(F_1(t_1), F_2(t_2)) = C(u, v),
\]

where \( u = F_1(t_1) \) and \( v = F_2(t_2) \). Therefore, the density function is given as

\[
f(t_1, t_2) = f_1(t_1)f_2(t_2) c(F_1(t_1), F_2(t_2)).
\]

Sklar theorem

*Let \( H \) be a joint distribution function with margins \( F \) and \( G \). Then there exists a copula \( C \) such that for all \( x, y \) in \( \mathbb{R} \),*
If $F$ and $G$ are continuous, then $C$ is unique; otherwise, $C$ is uniquely determined on $\text{Ran}F \times \text{Ran}G$. Conversely, if $C$ is a copula and $F$ and $G$ are distribution functions, then the function $H$ defined by (5) is a joint distribution function with margins $F$ and $G$.

2.2 Estimation of Copula-Based Model

There is a number of methods for fitting a copula model. In this paper, three different estimation methods are used to fit the proposed models. The first method is the maximum likelihood estimation (MLE). It is a one-step parametric method that employs the maximum likelihood estimator to estimate all model parameters jointly. Consider a bivariate distribution based on copula for a random vector $T = (T_1, T_2)$. The log-likelihood function is given as

$$
\ln L = \sum_{j=1}^{n} \left[ \ln f_1(t_{1j}) + \ln f_2(t_{2j}) + \ln c(F_1(t_{1j}), F_2(t_{2j})) \right].
$$

(6)

The parameters estimates are obtained by maximizing the log-likelihood function in (6) over each parameter separately. Therefore, the estimate of copula parameter is given as

$$
\hat{\theta} = \arg\max L(\theta).
$$

(7)

Another method of estimation is studied in this paper. It is known as inference functions for margins (IFM), see Joe in [10] and [11]. It is a parametric method with two-step of estimation. Firstly, estimating the marginal $F_1$ and $F_2$ respectively by maximizing the log-likelihood function of each marginal in (8)

$$
\ln L_{T_1} = \sum_{j=1}^{n} \ln f_1(t_{1j}), \quad \ln L_{T_2} = \sum_{j=1}^{n} \ln f_2(t_{2j}).
$$

(8)

The second step is estimating the copula parameter by maximizing the log-likelihood function of the copula density using the ML estimates of the marginal $\hat{F}_1(t_{1j})$ and $\hat{F}_2(t_{2j})$ from first step as given

$$
\hat{\theta} = \arg\max \sum_{j=1}^{n} \ln c(\hat{F}_1(t_{1j}), \hat{F}_2(t_{2j})).
$$

(9)

The feature of this method over the MLE is the flexibility in computations. The canonical maximum likelihood (CML) is another estimation method that used in this paper. It is a semi parametric method with two process. It is useful method, which is allowing the misspecification of marginal distributions. Kim et al. in [12]
confirm that the CML method is more efficient than the MLE and IFM methods when the marginal distributions are unknown. In this method, the observations are transformed into pseudo-observations using the empirical distribution function of each marginal distribution. The empirical distribution function is defined as

$$
\hat{F}_i(t) = \frac{\sum_{j=1}^{n} I\{T_{ij} \leq t\}}{n+1}.
$$

Then, the copula parameter is estimated by maximizing the log-likelihood function of the copula density using the transformed variables as

$$
\hat{\theta} = \arg \max \sum_{j=1}^{n} \ln c(\tilde{F}_1(t_{1j}), \tilde{F}_2(t_{2j})).
$$

### 3 Bivariate Generalized Exponential Distribution based on Copula

Basing on (3), the bivariate generalized exponential distribution based on copula is expressed as

$$
F(t_1, t_2) = C(F_1(t_1), F_2(t_2)) = C((1 - e^{-\lambda_1 t_1})^{\alpha_1}, (1 - e^{-\lambda_2 t_2})^{\alpha_2})
$$

And the density function is given as

$$
f(t_1, t_2) = f_1(t_1)f_2(t_2) c(F_1(t_1), F_2(t_2))
$$

$$
= \alpha_1 \alpha_2 \lambda_1 \lambda_2 e^{-\lambda_1 t_1} e^{-\lambda_2 t_2} (1 - e^{-\lambda_1 t_1})^{\alpha_1-1} (1 - e^{-\lambda_2 t_2})^{\alpha_2-1}
$$

$$
\times c((1 - e^{-\lambda_1 t_1})^{\alpha_1}, (1 - e^{-\lambda_2 t_2})^{\alpha_2})
$$

In statistics literature, there is a large collection of copulas with one or more parameter and different features. In this paper, the FGM and Plackett copulas are employed to form the bivariate extension of generalized exponential distribution.

#### 3.1 Bivariate Generalized Exponential Distribution based on FGM Copula

The FGM family is one of the most popular parametric families of copulas that discussed by Morgenstern in [17], Gumbel in [4] and Farlie in [3]. The expression of distribution function for FGM copula is

$$
C(u, v) = uv[1 + \theta(1 - u)(1 - v)],
$$

and the density function is given by
\[ c(u, v) = [1 + \theta(1 - 2u)(1 - 2v)] , \quad (15) \]

where \( u \) and \( v \in I \), and \( \theta \in [-1,1] \) is a dependence parameter. If the dependence parameter \( \theta \) equals zero, then the FGM copula corresponds the independence.

Although the FGM copula family is tractable mathematically, it does not model high dependences. The range of the dependence measures Kendall’s tau \( \tau \) and Spearman’s rho \( \rho \) are \( \tau \in [-0.222, 0.222] \) and \( \rho \in [-0.333, 0.333] \) respectively.

Considering the Equations (12), (13), (14) and (15), the bivariate generalized exponential distribution based on FGM copula can be expressed as

\[
F(t_1, t_2) = \left( 1 - e^{-\lambda_1 t_1} \right)^{\alpha_1} \left( 1 - e^{-\lambda_2 t_2} \right)^{\alpha_2} \\
\times \left[ 1 + \theta \left( 1 - \left( 1 - e^{-\lambda_1 t_1} \right)^{\alpha_1} \right) \left( 1 - \left( 1 - e^{-\lambda_2 t_2} \right)^{\alpha_2} \right) \right], \quad (16)
\]

where \( t_i > 0, \alpha_i, \lambda_i > 0, \ i = 1,2 \) and the density function is

\[
f(t_1, t_2) = \alpha_1 \alpha_2 \lambda_1 \lambda_2 e^{-\lambda_1 t_1} e^{-\lambda_2 t_2} \left( 1 - e^{-\lambda_1 t_1} \right)^{\alpha_1-1} \left( 1 - e^{-\lambda_2 t_2} \right)^{\alpha_2-1} \\
\times \left[ 1 + \theta \left( 1 - 2 \left( 1 - e^{-\lambda_1 t_1} \right)^{\alpha_1} \right) \left( 1 - 2 \left( 1 - e^{-\lambda_2 t_2} \right)^{\alpha_2} \right) \right]. \quad (17)
\]

### 3.2 Bivariate Generalized Exponential Distribution based on Plackett Copula

Another proposed copula in this paper which is the Plackett copula. It is proposed by Plackett [18]. Its distribution function is defined as

\[
C(u, v) = \frac{1 + (\theta - 1)(u + v) - \sqrt{[1 + (\theta - 1)(u + v)]^2 - 4uv\theta(\theta - 1)}}{2(\theta - 1)} , \quad (18)
\]

and the density function is defined as

\[
c(u, v) = \frac{\theta[1 + (u - 2uv + v)(\theta - 1)]}{([1 + (\theta - 1)(u + v)]^2 - 4uv\theta(\theta - 1))^\frac{3}{2}}. \quad (19)
\]

Where \( \theta \in (0, \infty) \). The correlation measure Spearman’s rho is \( \rho = \frac{\theta + 1}{\theta - 1} - \frac{2\theta \log(\theta)}{(\theta - 1)^2} \).

There is no closed expression in \( \theta \) for the correlation measure Kendall’s tau.

The distribution function of a bivariate generalized exponential distribution based on Plackett copula is defined as
\[
F(t_1, t_2) = \frac{1}{2(\theta - 1)} [1 + (\theta - 1) \left( (1 - e^{-\lambda_1 t_1})^{\alpha_1} + (1 - e^{-\lambda_2 t_2})^{\alpha_2} \right) - \sqrt{[1 + (\theta - 1)((1 - e^{-\lambda_1 t_1})^{\alpha_1} + (1 - e^{-\lambda_2 t_2})^{\alpha_2})]^2 - 4\theta(\theta - 1)(1 - e^{-\lambda_1 t_1})^{\alpha_1}(1 - e^{-\lambda_2 t_2})^{\alpha_2}}]
\]

The density function is defined as
\[
f(t_1, t_2) = \alpha_1 \alpha_2 \lambda_1 \lambda_2 e^{-\lambda_1 t_1} e^{-\lambda_2 t_2} (1 - e^{-\lambda_1 t_1})^{\alpha_1-1} (1 - e^{-\lambda_2 t_2})^{\alpha_2-1} \times \frac{\theta [1 + ((1 - e^{-\lambda_1 t_1})^{\alpha_1} - 2(1 - e^{-\lambda_1 t_1})^{\alpha_1}(1 - e^{-\lambda_2 t_2})^{\alpha_2} + (1 - e^{-\lambda_2 t_2})^{\alpha_2})(\theta - 1)]}{([1 + (\theta - 1)((1 - e^{-\lambda_1 t_1})^{\alpha_1} + (1 - e^{-\lambda_2 t_2})^{\alpha_2})]^2 - 4\theta(\theta - 1)(1 - e^{-\lambda_1 t_1})^{\alpha_1}(1 - e^{-\lambda_2 t_2})^{\alpha_2})^{\frac{3}{2}}} \]

where \( t_i > 0, \alpha_i, \lambda_i \) and \( \theta > 0, i = 1,2 \).

4 Estimation of Bivariate Generalized Exponential Distribution based on Copula

Three different estimation methods are used to estimate the parameters of bivariate generalized exponential distribution based on copula which are maximum likelihood estimation, inference functions for margins and canonical maximum likelihood.

4.1 Estimation of Bivariate Generalized Exponential Distribution based on FGM Copula

In this subsection, the bivariate generalized exponential distribution based on FGM copula is estimated by the three proposed estimation methods.

4.1.1 Estimation by Maximum Likelihood Estimation (MLE)

Considering the Equation (6), the log likelihood function of a bivariate generalized exponential distribution based on FGM copula is defined as

\[
\ln L = \sum_{j=1}^{n} \left[ \ln f_1(t_{1j}) + \ln f_2(t_{2j}) + \ln c \left( F_1(t_{1j}), F_2(t_{2j}) \right) \right]
\]

\[
= \sum_{j=1}^{n} \left[ \ln \alpha_1 \lambda_1 e^{-\lambda_1 t_{1j}} (1 - e^{-\lambda_1 t_{1j}})^{\alpha_1-1} \right]
\]
\[ + \ln \alpha_2 \lambda_2 e^{-\lambda_2 t_{2j}} \left( 1 - e^{-\lambda_2 t_{2j}} \right)^{\alpha_2 - 1} \]

\[ + \ln \left[ 1 + \theta \left( 1 - 2 \left( 1 - e^{-\lambda_1 t_{1j}} \right)^{\alpha_1} \right) \left( 1 - 2 \left( 1 - e^{-\lambda_2 t_{2j}} \right)^{\alpha_2} \right) \right]. \quad (22) \]

The estimates of all parameters are obtained by differentiating the log-likelihood function in (22) with respect to each parameter separately. Basing on this, differentiating the log-likelihood function with respect to \( \lambda_1 \) is given as

\[
\frac{\partial L}{\partial \lambda_1} = \sum_{j=1}^{n} \left[ \frac{1}{\lambda_1} - t_{1j} + (\alpha_1 - 1) \frac{t_{1j} e^{-\lambda_1 t_{1j}}}{1 - e^{-\lambda_1 t_{1j}}} \right]
\]

\[
- \frac{2 \theta \alpha_1 t_{1j} e^{-\lambda_1 t_{1j}} \left( 1 - e^{-\lambda_1 t_{1j}} \right)^{\alpha_1 - 1} \left( 1 - 2 \left( 1 - e^{-\lambda_2 t_{2j}} \right)^{\alpha_2} \right)}{1 + \theta \left( 1 - 2 \left( 1 - e^{-\lambda_1 t_{1j}} \right)^{\alpha_1} \right) \left( 1 - 2 \left( 1 - e^{-\lambda_2 t_{2j}} \right)^{\alpha_2} \right)} = 0. \quad (23) \]

Differentiating the log-likelihood function in (22) with respect to \( \lambda_2 \) is given as

\[
\frac{\partial L}{\partial \lambda_2} = \sum_{j=1}^{n} \left[ \frac{1}{\lambda_2} - t_{2j} + (\alpha_2 - 1) \frac{t_{2j} e^{-\lambda_2 t_{2j}}}{1 - e^{-\lambda_2 t_{2j}}} \right]
\]

\[
- \frac{2 \theta \alpha_2 t_{2j} e^{-\lambda_2 t_{2j}} \left( 1 - e^{-\lambda_2 t_{2j}} \right)^{\alpha_2 - 1} \left( 1 - 2 \left( 1 - e^{-\lambda_1 t_{1j}} \right)^{\alpha_1} \right)}{1 + \theta \left( 1 - 2 \left( 1 - e^{-\lambda_1 t_{1j}} \right)^{\alpha_1} \right) \left( 1 - 2 \left( 1 - e^{-\lambda_2 t_{2j}} \right)^{\alpha_2} \right)} = 0. \quad (24) \]

Maximizing the log-likelihood function in (22) with respect to \( \alpha_1 \) is given as
\[
\frac{\partial L}{\partial \alpha_1} = \sum_{j=1}^{n} \left[ \frac{1}{\alpha_1} - \ln(1 - e^{-\lambda_1 t_{1j}}) - \frac{2\theta (1 - e^{-\lambda_1 t_{1j}})^{\alpha_1} \ln(1 - e^{-\lambda_1 t_{1j}})(1 - 2(1 - e^{-\lambda_2 t_{2j}})^{\alpha_2})}{1 + \theta(1 - 2(1 - e^{-\lambda_1 t_{1j}})^{\alpha_1})(1 - 2(1 - e^{-\lambda_2 t_{2j}})^{\alpha_2})} \right].
\]

\[
\frac{\partial L}{\partial \alpha_1} = \frac{n}{\alpha_1} - \sum_{j=1}^{n} \ln(1 - e^{-\lambda_1 t_{1j}})
\times \left[ 1 + \frac{2\theta (1 - e^{-\lambda_1 t_{1j}})^{\alpha_1} (1 - 2(1 - e^{-\lambda_2 t_{2j}})^{\alpha_2})}{1 + \theta(1 - 2(1 - e^{-\lambda_1 t_{1j}})^{\alpha_1})(1 - 2(1 - e^{-\lambda_2 t_{2j}})^{\alpha_2})} \right] = 0. \tag{25}
\]

Maximizing the log-likelihood function in (22) with respect to \( \alpha_2 \) is given as

\[
\frac{\partial L}{\partial \alpha_2} = \sum_{j=1}^{n} \left[ \frac{1}{\alpha_2} - \ln(1 - e^{-\lambda_2 t_{2j}}) - \frac{2\theta (1 - e^{-\lambda_2 t_{2j}})^{\alpha_2} \ln(1 - e^{-\lambda_2 t_{2j}})(1 - 2(1 - e^{-\lambda_1 t_{1j}})^{\alpha_1})}{1 + \theta(1 - 2(1 - e^{-\lambda_1 t_{1j}})^{\alpha_1})(1 - 2(1 - e^{-\lambda_2 t_{2j}})^{\alpha_2})} \right].
\]

\[
\frac{\partial L}{\partial \alpha_2} = \frac{n}{\alpha_2} - \sum_{j=1}^{n} \ln(1 - e^{-\lambda_2 t_{2j}})
\times \left[ 1 + \frac{2\theta (1 - e^{-\lambda_2 t_{2j}})^{\alpha_2} (1 - 2(1 - e^{-\lambda_1 t_{1j}})^{\alpha_1})}{1 + \theta(1 - 2(1 - e^{-\lambda_1 t_{1j}})^{\alpha_1})(1 - 2(1 - e^{-\lambda_2 t_{2j}})^{\alpha_2})} \right] = 0. \tag{26}
\]

Maximizing the log-likelihood function in (22) with respect to \( \theta \) results

\[
\frac{\partial L}{\partial \theta} = \sum_{j=1}^{n} \left[ \frac{1 - 2(1 - e^{-\lambda_1 t_{1j}})^{\alpha_1}}{1 + \theta(1 - 2(1 - e^{-\lambda_1 t_{1j}})^{\alpha_1})(1 - 2(1 - e^{-\lambda_2 t_{2j}})^{\alpha_2})} \right] = 0.
\]

The estimates of parameters are obtained numerically through a statistical software.
4.1.2 Estimation by Inference Functions for Margins (IFM)

By considering the first step with Equation (8) for a bivariate generalized exponential distribution based on FGM copula, the parameters of each marginal distribution will be estimated by MLE separately. The equation becomes

\[
\ln L_{T_i}(\alpha_i, \lambda_i) = n\ln(\alpha_i) + n\ln(\lambda_i) + (\alpha_i - 1) \sum_{j=1}^{n} \ln(1 - e^{-\lambda_i t_{ij}}) - \lambda_i \sum_{j=1}^{n} t_{ij}.
\]  

(27)

Maximizing the log-likelihood function in (27) over \( \alpha_i \) is given as

\[
\frac{\partial L_{T_i}}{\partial \alpha_i} = \frac{n}{\alpha_i} + \sum_{j=1}^{n} \ln(1 - e^{-\lambda_i t_{ij}}) = 0.
\]

\[
\frac{n}{\alpha_i} = -\sum_{j=1}^{n} \ln(1 - e^{-\lambda_i t_{ij}}).
\]

Therefore, the estimate of \( \alpha_i \), say \( \hat{\alpha}_i \), is given as

\[
\hat{\alpha}_i = -\frac{n}{\sum_{j=1}^{n} \ln(1 - e^{-\lambda_i t_{ij}})}.
\]  

(28)

Maximizing the log-likelihood function in (27) over \( \lambda_i \) is given as

\[
\frac{\partial L_{T_i}}{\partial \lambda_i} = \frac{n}{\lambda_i} + (\alpha_i - 1) \sum_{j=1}^{n} t_{ij} e^{-\lambda_i t_{ij}} - \sum_{i=1}^{n} t_{ij} = 0.
\]

It is noted that the MLE of \( \alpha_i \) is a function of \( \lambda_i \), say \( \hat{\alpha}_i(\lambda_i) \), where

\[
\hat{\alpha}_i(\lambda_i) = -\frac{n}{\sum_{j=1}^{n} \ln(1 - e^{-\lambda_i t_{ij}})}.
\]  

(29)

Substituting \( \hat{\alpha}_i(\lambda_i) \) in (27), the equation is re-expressed as

\[
\ln L_{T_i}(\alpha_i, \lambda_i) = n\ln(\hat{\alpha}_i(\lambda_i)) + n\ln(\lambda_i) + (\hat{\alpha}_i(\lambda_i))
\]
\[ + (\hat{\alpha}_i (\hat{\lambda}_i) - 1) \sum_{j=1}^{n} \ln(1 - e^{-\lambda_i t_{ij}}) - \lambda_i \sum_{j=1}^{n} t_{ij}. \]

\[ \ln L_{T_i}(\alpha_i, \lambda_i) = n \ln \left( -\frac{n}{\sum_{j=1}^{n} \ln(1 - e^{-\lambda_i t_{ij}})} \right) + n \ln(\lambda_i) \]
\[ + \left( -\frac{n}{\sum_{j=1}^{n} \ln(1 - e^{-\lambda_i t_{ij}})} - 1 \right) \]
\[ \times \sum_{j=1}^{n} \ln(1 - e^{-\lambda_i t_{ij}}) - \lambda_i \sum_{j=1}^{n} t_{ij}. \]

\[ = n \left[ \ln n - \ln \sum_{j=1}^{n} (-\ln(1 - e^{-\lambda_i t_{ij}})) \right] + n \ln(\lambda_i) \]
\[ - n - \sum_{j=1}^{n} \ln(1 - e^{-\lambda_i t_{ij}}) - \lambda_i \sum_{j=1}^{n} t_{ij}. \]

\[ = n \ln n - n \ln \sum_{j=1}^{n} \left( -\ln(1 - e^{-\lambda_i t_{ij}}) \right) + n \ln(\lambda_i) \]
\[ - \sum_{j=1}^{n} \ln(1 - e^{-\lambda_i t_{ij}}) - \lambda_i \sum_{j=1}^{n} t_{ij}. \]

(30)

By differentiating the log likelihood function in (30) with respect to \( \lambda_i \), the equation is given as

\[ \frac{\partial L_{T_i}}{\partial \lambda_i} = - \frac{n}{\sum_{j=1}^{n} \ln(1 - e^{-\lambda_i t_{ij}})} \sum_{j=1}^{n} \frac{t_{ij} e^{-\lambda_i t_{ij}}}{1 - e^{-\lambda_i t_{ij}}} \]
\[ + \frac{n}{\lambda_i} - \sum_{j=1}^{n} \frac{t_{ij} e^{-\lambda_i t_{ij}}}{1 - e^{-\lambda_i t_{ij}}} - \sum_{j=1}^{n} t_{ij}. \]

\[ \frac{\partial L_{T_i}}{\partial \lambda_i} = n \left[ - \frac{\sum_{j=1}^{n} t_{ij} e^{-\lambda_i t_{ij}}}{\sum_{j=1}^{n} \ln(1 - e^{-\lambda_i t_{ij}})} + \frac{1}{\lambda_i} \right] - \sum_{j=1}^{n} \frac{t_{ij} e^{-\lambda_i t_{ij}}}{1 - e^{-\lambda_i t_{ij}}} - \sum_{j=1}^{n} t_{ij} = 0. \]
The fixed point solution of Equation (31) will provide the MLE of $\lambda_i$, say $\hat{\lambda}_i$. Substituting $\hat{\lambda}_i$ in (28) will provide the MLE of $\alpha_i$, say $\hat{\alpha}_i$.

The second step is estimating the copula density using marginal estimates $\hat{F}_1(t_1)$ and $\hat{F}_2(t_2)$ from first step as follows

$$\ln L_C = \sum_{j=1}^{n} \ln c \left( \hat{F}_1(t_{1j}), \hat{F}_2(t_{2j}) \right)$$

$$= \sum_{j=1}^{n} \ln \left[ 1 + \theta \left( 1 - 2 \left( 1 - e^{-\hat{\lambda}_{1tj}} \hat{a}_1 \right) \right) \left( 1 - 2 \left( 1 - e^{-\hat{\lambda}_{2tj}} \hat{a}_2 \right) \right) \right].$$

$$\frac{\partial L_C}{\partial \theta} = \sum_{j=1}^{n} \frac{\left( 1 - 2 \left( 1 - e^{-\hat{\lambda}_{1tj}} \hat{a}_1 \right) \right) \left( 1 - 2 \left( 1 - e^{-\hat{\lambda}_{2tj}} \hat{a}_2 \right) \right)}{1 + \theta \left( 1 - 2 \left( 1 - e^{-\hat{\lambda}_{1tj}} \hat{a}_1 \right) \right) \left( 1 - 2 \left( 1 - e^{-\hat{\lambda}_{2tj}} \hat{a}_2 \right) \right)} = 0. \quad (32)$$

The estimates of parameters are handled numerically through a statistical software.

**4.1.3 Estimation by Canonical Maximum Likelihood (CML)**

Basing on this method, the bivariate generalized exponential distribution based on FGM copula is estimated as follows

$$\ln L_C = \sum_{j=1}^{n} \ln \left[ 1 + \theta \left( 1 - 2 \hat{F}_1(t_{1j}) \right) \left( 1 - 2 \hat{F}_2(t_{2j}) \right) \right].$$
Inference of bivariate generalized exponential distribution...

\[
\frac{\partial L_C}{\partial \theta} = \sum_{j=1}^{n} \left[\frac{(1 - 2F_1(t_{1j}))(1 - 2F_2(t_{2j}))}{1 + \theta (1 - 2F_1(t_{1j}))(1 - 2F_2(t_{2j}))}\right] = 0. \tag{33}
\]

In this paper, the estimation of parameters of bivariate generalized exponential distribution based on FGM copula is handled numerically through a statistical software.

### 4.2 Estimation of Bivariate Generalized Exponential Distribution based on Plackett Copula

In this subsection, the bivariate generalized exponential distribution based on Plackett copula is estimated by the three proposed estimation methods.

#### 4.2.1 Estimation by Maximum Likelihood Estimation (MLE)

The log likelihood function of the bivariate generalized exponential distribution based on Plackett copula is defined as

\[
\ln L = \sum_{j=1}^{n} \left[\ln f_1(t_{1j}) + \ln f_2(t_{2j}) + \ln c(F_1(t_{1j}), F_2(t_{2j}))\right]
\]

\[
\ln L = \sum_{j=1}^{n} \left[\ln \alpha_1 \lambda_1 e^{-\lambda_1 t_{1j}}(1 - e^{-\lambda_1 t_{1j}})^{\alpha_1 - 1} + \ln \alpha_2 \lambda_2 e^{-\lambda_2 t_{2j}}(1 - e^{-\lambda_2 t_{2j}})^{\alpha_2 - 1}\right]
\]

\[
+ \ln \frac{\theta \left[1 + (1 - e^{-\lambda_1 t_{1j}})^{\alpha_1} - 2(1 - e^{-\lambda_1 t_{1j}})^{\alpha_1}(1 - e^{-\lambda_2 t_{2j}})^{\alpha_2} + (1 - e^{-\lambda_2 t_{2j}})^{\alpha_2}\right](\theta - 1)}{\left[1 + (\theta - 1)(1 - e^{-\lambda_1 t_{1j}})^{\alpha_1}(1 - e^{-\lambda_2 t_{2j}})^{\alpha_2}\right]^2 - 4\theta(\theta - 1)(1 - e^{-\lambda_1 t_{1j}})^{\alpha_1}(1 - e^{-\lambda_2 t_{2j}})^{\alpha_2}}\]

Let \( g_{1j}(\lambda_1) = (1 - e^{-\lambda_1 t_{1j}}) \) and \( g_{2j}(\lambda_2) = (1 - e^{-\lambda_2 t_{2j}}) \). The previous equation can be rewritten as

\[
\ln L = \sum_{j=1}^{n} \left[\ln \alpha_1 \lambda_1 e^{-\lambda_1 t_{1j}} g_{1j}(\lambda_1)^{\alpha_1 - 1} + \ln \alpha_2 \lambda_2 e^{-\lambda_2 t_{2j}} g_{2j}(\lambda_2)^{\alpha_2 - 1}\right]
\]

\[
+ \ln \frac{\theta \left[1 + (g_{1j}(\lambda_1)^{\alpha_1} - 2g_{1j}(\lambda_1)^{\alpha_1} g_{2j}(\lambda_2)^{\alpha_2} + g_{2j}(\lambda_2)^{\alpha_2})(\theta - 1)\right]}{\left[1 + (\theta - 1)(g_{1j}(\lambda_1)^{\alpha_1} + g_{2j}(\lambda_2)^{\alpha_2})\right]^2 - 4\theta(\theta - 1)g_{1j}(\lambda_1)^{\alpha_1} g_{2j}(\lambda_2)^{\alpha_2}}\]
\[
\ln L = \sum_{j=1}^{n} \left[ \ln \alpha_1 + \ln \lambda_1 - \lambda_1 t_{1j} + (\alpha_1 - 1) \ln g_{1j}(\lambda_1) \right] \\
+ \ln \alpha_2 + \ln \lambda_2 - \lambda_2 t_{2j} + (\alpha_2 - 1) \ln g_{2j}(\lambda_2) \\
+ \ln \theta + \ln \left[ 1 + \left( g_{1j}(\lambda_1)^{\alpha_1} - 2g_{1j}(\lambda_1)^{\alpha_1} g_{2j}(\lambda_2)^{\alpha_2} + g_{2j}(\lambda_2)^{\alpha_2} \right)(\theta - 1) \right] \\
- \frac{3}{2} \ln \left[ \left( 1 + (\theta - 1)(g_{1j}(\lambda_1)^{\alpha_1} + g_{2j}(\lambda_2)^{\alpha_2}) \right)^2 - 4\theta(\theta - 1)g_{1j}(\lambda_1)^{\alpha_1}g_{2j}(\lambda_2)^{\alpha_2} \right].
\] (34)

Maximizing the log likelihood function in (34) over \( \alpha_1 \) is given as

\[
\frac{\partial L}{\partial \alpha_1} = \sum_{j=1}^{n} \left[ \frac{1}{\alpha_1} + \ln g_{1j}(\lambda_1) \right] \\
\times \frac{(\theta - 1) g_{1j}(\lambda_1)^{\alpha_1} \ln g_{1j}(\lambda_1) \left[ 1 - 2g_{2j}(\lambda_2)^{\alpha_2} \right]}{1 + (\theta - 1)(g_{1j}(\lambda_1)^{\alpha_1} - 2g_{1j}(\lambda_1)^{\alpha_1} g_{2j}(\lambda_2)^{\alpha_2} + g_{2j}(\lambda_2)^{\alpha_2})} \\
- 3(\theta - 1)g_{1j}(\lambda_1)^{\alpha_1} \ln g_{1j}(\lambda_1) \\
\times \frac{\left[ \left( 1 + (\theta - 1)(g_{1j}(\lambda_1)^{\alpha_1} + g_{2j}(\lambda_2)^{\alpha_2}) \right) - 2\theta g_{2j}(\lambda_2)^{\alpha_2} \right]}{\left[ 1 + (\theta - 1)(g_{1j}(\lambda_1)^{\alpha_1} + g_{2j}(\lambda_2)^{\alpha_2}) \right]^2 - 4\theta(\theta - 1)g_{1j}(\lambda_1)^{\alpha_1}g_{2j}(\lambda_2)^{\alpha_2}}.
\]

Maximizing the log likelihood function in (34) over \( \alpha_2 \) is given as

\[
\frac{\partial L}{\partial \alpha_2} = \frac{n}{\alpha_1} + \ln \sum_{j=1}^{n} g_{1j}(\lambda_1) \\
\times \left[ 1 + \frac{(\theta - 1) g_{1j}(\lambda_1)^{\alpha_1} \left[ 1 - 2g_{2j}(\lambda_2)^{\alpha_2} \right]}{1 + (\theta - 1)(g_{1j}(\lambda_1)^{\alpha_1} - 2g_{1j}(\lambda_1)^{\alpha_1} g_{2j}(\lambda_2)^{\alpha_2} + g_{2j}(\lambda_2)^{\alpha_2})} \right] \\
- \frac{3(\theta - 1)g_{1j}(\lambda_1)^{\alpha_1} \left[ \left( 1 + (\theta - 1)(g_{1j}(\lambda_1)^{\alpha_1} + g_{2j}(\lambda_2)^{\alpha_2}) \right) - 2\theta g_{2j}(\lambda_2)^{\alpha_2} \right]}{\left[ 1 + (\theta - 1)(g_{1j}(\lambda_1)^{\alpha_1} + g_{2j}(\lambda_2)^{\alpha_2}) \right]^2 - 4\theta(\theta - 1)g_{1j}(\lambda_1)^{\alpha_1}g_{2j}(\lambda_2)^{\alpha_2}} = 0.
\] (35)
\[
\frac{\partial L}{\partial \alpha_2} = \sum_{j=1}^{n} \left[ \frac{1}{\alpha_2} + \ln g_{2j}(\lambda_2) \right] 
+ \frac{(\theta - 1)g_{2j}(\lambda_2)^{\alpha_2} \ln g_{2j}(\lambda_2) [1 - 2g_{1j}(\lambda_1)^{\alpha_1}]}{1 + (\theta - 1)(g_{1j}(\lambda_1)^{\alpha_1} - 2g_{1j}(\lambda_1)^{\alpha_1}g_{2j}(\lambda_2)^{\alpha_2} + g_{2j}(\lambda_2)^{\alpha_2})} 

- 3(\theta - 1)g_{2j}(\lambda_2)^{\alpha_2} \ln g_{2j}(\lambda_2) 
\times \left[ \frac{1}{1 + (\theta - 1)(g_{1j}(\lambda_1)^{\alpha_1} + g_{2j}(\lambda_2)^{\alpha_2})} \right] 
\]

\[
\frac{\partial L}{\partial \alpha_2} = \frac{n}{\alpha_2} + \ln \sum_{j=1}^{n} g_{2j}(\lambda_2) 
\times \left[ 1 + \frac{(\theta - 1)g_{2j}(\lambda_2)^{\alpha_2} [1 - 2g_{1j}(\lambda_1)^{\alpha_1}]}{1 + (\theta - 1)(g_{1j}(\lambda_1)^{\alpha_1} - 2g_{1j}(\lambda_1)^{\alpha_1}g_{2j}(\lambda_2)^{\alpha_2} + g_{2j}(\lambda_2)^{\alpha_2})} \right] 
- 3(\theta - 1)g_{2j}(\lambda_2)^{\alpha_2} \left[ \frac{1}{1 + (\theta - 1)(g_{1j}(\lambda_1)^{\alpha_1} + g_{2j}(\lambda_2)^{\alpha_2})} \right] = 0. \tag{36} \]

Maximizing the log likelihood function in (34) over \( \lambda_1 \) is given as

\[
\frac{\partial L}{\partial \lambda_1} = \sum_{j=1}^{n} \left[ \frac{1}{\lambda_1} - t_{1j} + \frac{(\alpha_1 - 1)g_{1j}(\lambda_1)}{g_{1j}(\lambda_1)} \right] 
\times \frac{\alpha_1(\theta - 1)g_{1j}(\lambda_1)g_{1j}(\lambda_1)^{\alpha_1 - 2} [1 - 2g_{2j}(\lambda_2)^{\alpha_2}]}{[1 + (\theta - 1)(g_{1j}(\lambda_1)^{\alpha_1} - 2g_{1j}(\lambda_1)^{\alpha_1}g_{2j}(\lambda_2)^{\alpha_2} + g_{2j}(\lambda_2)^{\alpha_2})]} 

- 3\alpha_1(\theta - 1)g_{1j}(\lambda_1)g_{1j}(\lambda_1)^{\alpha_1 - 1} \times \left[ \frac{1}{1 + (\theta - 1)(g_{1j}(\lambda_1)^{\alpha_1} + g_{2j}(\lambda_2)^{\alpha_2})} \right] \right].
\]
Where \( g'_{1j}(\lambda_1) = t_{1j}e^{-\lambda_1 t_{1j}}. \)

\[
\frac{\partial L}{\partial \lambda_1} = \frac{n}{\lambda_1} - \sum_{j=1}^{n} t_{1j} + \sum_{j=1}^{n} \frac{g'_{1j}(\lambda_1)}{g_{1j}(\lambda_1)} ([\alpha_1 - 1) + \frac{\alpha_1(\theta - 1)g_{1j}(\lambda_1)^{\alpha_1} [1 - 2g_{2j}(\lambda_2)^{\alpha_2}]}{1 + (\theta - 1)(g_{1j}(\lambda_1)^{\alpha_1} - 2g_{1j}(\lambda_1)^{\alpha_1}g_{2j}(\lambda_2)^{\alpha_2} + g_{2j}(\lambda_2)^{\alpha_2})}\]

\[
-3\alpha_1(\theta - 1)g(\lambda_1)^{\alpha_1}
\]

\[
\times \frac{[\left(1 + (\theta - 1)(g_{1j}(\lambda_1)^{\alpha_1} + g_{2j}(\lambda_2)^{\alpha_2})\right) - 2\theta g_{2j}(\lambda_2)^{\alpha_2}]}{[1 + (\theta - 1)(g_{1j}(\lambda_1)^{\alpha_1} + g_{2j}(\lambda_2)^{\alpha_2})]^2 - 4\theta(\theta - 1)g_{1j}(\lambda_1)^{\alpha_1}g_{2j}(\lambda_2)^{\alpha_2}]} = 0. (37)
\]

Maximizing the log likelihood function in (34) over \( \lambda_2 \) is given as

\[
\frac{\partial L}{\partial \lambda_2} = \sum_{j=1}^{n} \frac{1}{\lambda_2} - t_{2j} + \frac{(\alpha_2 - 1)g_{2j}(\lambda_2)}{g_{2j}(\lambda_2)}
\]

\[
+ \frac{\alpha_2(\theta - 1)g_{2j}(\lambda_2)g_{2j}(\lambda_2)^{\alpha_2 - 1} [1 - 2g_{1j}(\lambda_1)^{\alpha_1}]}{1 + (\theta - 1)(g_{1j}(\lambda_1)^{\alpha_1} - 2g_{1j}(\lambda_1)^{\alpha_1}g_{2j}(\lambda_2)^{\alpha_2} + g_{2j}(\lambda_2)^{\alpha_2})}\]

\[
-3\alpha_2(\theta - 1)g_{2j}(\lambda_2)g_{2j}(\lambda_2)^{\alpha_2 - 1}
\]

\[
\times \frac{[\left(1 + (\theta - 1)(g_{1j}(\lambda_1)^{\alpha_1} + g_{2j}(\lambda_2)^{\alpha_2})\right) - 2\theta g_{1j}(\lambda_1)^{\alpha_1}]}{[1 + (\theta - 1)(g_{1j}(\lambda_1)^{\alpha_1} + g_{2j}(\lambda_2)^{\alpha_2})]^2 - 4\theta(\theta - 1)g_{1j}(\lambda_1)^{\alpha_1}g_{2j}(\lambda_2)^{\alpha_2}} = 0.
\]

Where \( g'_{2j}(\lambda_2) = t_{2j}e^{-\lambda_2 t_{2j}}. \)

\[
\frac{\partial L}{\partial \lambda_2} = \frac{n}{\lambda_2} - \sum_{j=1}^{n} t_{2j} + \sum_{j=1}^{n} \frac{g'_{2j}(\lambda_2)}{g_{2j}(\lambda_2)} ([\alpha_2 - 1) + \frac{\alpha_2(\theta - 1)g_{2j}(\lambda_2)^{\alpha_2} [1 - 2g_{1j}(\lambda_1)^{\alpha_1}]}{1 + (\theta - 1)(g_{1j}(\lambda_1)^{\alpha_1} - 2g_{1j}(\lambda_1)^{\alpha_1}g_{2j}(\lambda_2)^{\alpha_2} + g_{2j}(\lambda_2)^{\alpha_2})}\]

\[
-3\alpha_2(\theta - 1)g_{2j}(\lambda_2)^{\alpha_2}
\]
\[
\left[1 + (\theta - 1)(g_{1j}(\lambda_1)^{a_1} + g_{2j}(\lambda_2)^{a_2})\right] - 2\theta g_{1j}(\lambda_1)^{a_1} \right]
\left[1 + (\theta - 1)(g_{1j}(\lambda_1)^{a_1} + g_{2j}(\lambda_2)^{a_2})\right]^2 - 4\theta(\theta - 1)g_{1j}(\lambda_1)^{a_1}g_{2j}(\lambda_2)^{a_2}\right] = 0. \tag{38}
\]

Maximizing the log likelihood function in (34) over copula parameter \( \theta \) is given as

\[
\frac{\partial L}{\partial \theta} = \sum_{j=1}^{n} \left[ \frac{1}{\theta} + \frac{g_{1j}(\lambda_1)^{a_1} - 2g_{1j}(\lambda_1)^{a_1}g_{2j}(\lambda_2)^{a_2} + g_{2j}(\lambda_2)^{a_2}}{1 + (\theta - 1)(g_{1j}(\lambda_1)^{a_1} - 2g_{1j}(\lambda_1)^{a_1}g_{2j}(\lambda_2)^{a_2} + g_{2j}(\lambda_2)^{a_2})} \right] - \frac{3\left[(g_{1j}(\lambda_1)^{a_1} + g_{2j}(\lambda_2)^{a_2})[1 + (\theta - 1)(g_{1j}(\lambda_1)^{a_1} + g_{2j}(\lambda_2)^{a_2})] - 2(\theta - 1)g_{1j}(\lambda_1)^{a_1}g_{2j}(\lambda_2)^{a_2}\right]}{[1 + (\theta - 1)(g_{1j}(\lambda_1)^{a_1} + g_{2j}(\lambda_2)^{a_2})]^2 - 4\theta(\theta - 1)g_{1j}(\lambda_1)^{a_1}g_{2j}(\lambda_2)^{a_2}} = 0.
\]

The parameters estimates are handled numerically through a statistical software.

### 4.2.2 Estimation by Inference Functions for Margins (IFM)

The ML estimates of parameters of the marginal \( F_1(t_1) \) and \( F_2(t_2) \) are obtained in (28) and (31). Basing on this, the copula parameter is estimated as follows

\[
\ln L = \sum_{j=1}^{n} \ln c \left( \hat{F}_1(t_{1j}), \hat{F}_2(t_{2j}) \right),
\]

where \( \hat{F}_1(t_{1j}) = \left(1 - e^{-\lambda_1 t_{1j}}\right)^{\hat{\alpha}_1} \) and \( \hat{F}_2(t_{2j}) = \left(1 - e^{-\lambda_2 t_{2j}}\right)^{\hat{\alpha}_2} \).
\[
\ln L = \sum_{j=1}^{n} \left[ \ln \theta + \ln \left( 1 + (\theta - 1) \left( \hat{F}_1(t_{1j}) - 2\hat{F}_1(t_{1j})\hat{F}_2(t_{1j}) + \hat{F}_2(t_{1j}) \right) \right) \right] \\
- \frac{3}{2} \ln \left[ \left( 1 + (\theta - 1) \left( \hat{F}_1(t_{1j}) + \hat{F}_2(t_{2j}) \right) \right)^2 - 4\theta (\theta - 1) \hat{F}_1(t_{1j})\hat{F}_2(t_{2j}) \right] \\

\]

Differentiating the log likelihood function with respect to copula parameter \( \theta \) is given as

\[
\frac{\partial L}{\partial \theta} = \sum_{j=1}^{n} \left[ \frac{1}{\theta} + \frac{\left( \hat{F}_1(t_{1j}) - 2\hat{F}_1(t_{1j})\hat{F}_2(t_{2j}) + \hat{F}_2(t_{2j}) \right)}{1 + (\theta - 1) \left[ \left( \hat{F}_1(t_{1j}) - 2\hat{F}_1(t_{1j})\hat{F}_2(t_{2j}) + \hat{F}_2(t_{2j}) \right) \right]} \right] \\
- 3 \left[ \left( \hat{F}_1(t_{1j}) + \hat{F}_2(t_{2j}) \right) \left( 1 + (\theta - 1) \left( \hat{F}_1(t_{1j}) + \hat{F}_2(t_{2j}) \right) \right) \right] - 2(2\theta - 1) \hat{F}_1(t_{1j})\hat{F}_2(t_{2j}) \\
\left[ 1 + (\theta - 1) \left( \hat{F}_1(t_{1j}) + \hat{F}_2(t_{2j}) \right) \right]^2 - 4\theta (\theta - 1) \hat{F}_1(t_{1j})\hat{F}_2(t_{2j}) \\
= 0.
\]

The estimate of copula parameter is obtained numerically through a statistical software.

4.2.3 Estimation by Canonical Maximum Likelihood (CML)

Using the empirical distribution function of each marginal distribution

\[
\hat{F}_i(t) = \frac{\sum_{j=1}^{n} I(T_{ij} \leq t)}{n + 1},
\]

the copula parameter is estimated by maximizing the log-likelihood function of the copula density as

\[
\ln L_c = \sum_{j=1}^{n} \ln c \left( \hat{F}_1(t_{1j}), \hat{F}_2(t_{2j}) \right)
\]
\[ \ln L_c = \sum_{j=1}^{n} [\ln \theta + \ln \left(1 + (\theta - 1) \left(\tilde{F}_1(t_{1j}) - 2\tilde{F}_1(t_{1j})\tilde{F}_2(t_{2j}) + \tilde{F}_2(t_{2j})\right)\right) ] \\
- \frac{3}{2} \ln \left(1 + (\theta - 1) \left(\tilde{F}_1(t_{1j}) + \tilde{F}_2(t_{2j})\right)^2 - 4(\theta - 1)\tilde{F}_1(t_{1j})\tilde{F}_2(t_{2j})\right). \]

Differentiating the log likelihood function with respect to copula parameter \(\theta\) is given as

\[
\frac{\partial L}{\partial \theta} = \sum_{j=1}^{n} \left[ \frac{1}{\theta} + \frac{\left(F_1(t_{1j}) - 2F_1(t_{1j})F_2(t_{2j}) + F_2(t_{2j})\right)}{1 + (\theta - 1)\left(F_1(t_{1j}) - 2F_1(t_{1j})F_2(t_{2j}) + F_2(t_{2j})\right)} \right] \\
- \frac{3}{2} \left[ \left(\tilde{F}_1(t_{1j}) + \tilde{F}_2(t_{2j})\right)^2 - 4(\theta - 1)\tilde{F}_1(t_{1j})\tilde{F}_2(t_{2j})\right].
\]

\[
\frac{\partial L}{\partial \theta} = n \left[ \frac{F_1(t_{1j}) - 2F_1(t_{1j})F_2(t_{2j}) + F_2(t_{2j})}{1 + (\theta - 1)\left(F_1(t_{1j}) - 2F_1(t_{1j})F_2(t_{2j}) + F_2(t_{2j})\right)} \right] \\
- \frac{3}{2} \left[ \left(\tilde{F}_1(t_{1j}) + \tilde{F}_2(t_{2j})\right)^2 - 4(\theta - 1)\tilde{F}_1(t_{1j})\tilde{F}_2(t_{2j})\right] = 0.
\]

In this paper, the parameters estimation of bivariate generalized exponential distribution based on Plackett copula is handled through a statistical software numerically.

5 Application

An analysis of simulated and real data sets is provided in this section.

5.1 Simulation Data

Considering the following values of marginal and copula parameters of bivariate generalized exponential distribution based on FGM and Plackett copula with different sizes of sample (n = 25, 50, 75, 100, 125, and 150):

1. \(\alpha_1 = 3.5, \lambda_1 = 2.5, \alpha_2 = 2, \lambda_2 = 1.5\), FGM copula parameter \(\theta_F = 0.88\) and Plackett copula parameter \(\theta_P = 2.48\).
2. \(\alpha_1 = 2, \lambda_1 = 1.5, \alpha_2 = 4.5, \lambda_2 = 1.5\), FGM copula parameter \(\theta_F = 0.92\) and Plackett copula parameter \(\theta_P = 2.60\).
3. $\alpha_1 = 3, \lambda_1 = 1.8, \alpha_2 = 3, \lambda_2 = 2.5$, FGM copula parameter $\theta_F = 0.80$ and Plackett copula parameter $\theta_P = 2.27$.

The average estimates for these parameters of two models by different ways of estimation and the corresponding mean squared errors (in brackets) based on 1000 replications are reported in Table 1, 2, 3, 4, 5 and 6.

Table 1. The average estimates of parameters and the mean squared errors (in brackets) by simulation study for bivariate generalized exponential distribution based on FGM copula at $\alpha_1=3.5, \lambda_1=2.5, \alpha_2=2, \lambda_2=1.5$ and $\theta_F=0.88$.

<table>
<thead>
<tr>
<th>Sample Size</th>
<th>Estimation Method</th>
<th>$\alpha_1 = 3.5$</th>
<th>$\lambda_1 = 2.5$</th>
<th>$\alpha_2 = 2$</th>
<th>$\lambda_2 = 1.5$</th>
<th>$\theta_F = 0.88$</th>
</tr>
</thead>
<tbody>
<tr>
<td>n=25</td>
<td>MLE</td>
<td>3.8626 (0.1315)</td>
<td>2.6037 (0.0107)</td>
<td>2.3974 (0.1579)</td>
<td>1.6695 (0.0287)</td>
<td>0.7218 (0.0250)</td>
</tr>
<tr>
<td></td>
<td>IFM</td>
<td>4.3001 (0.6402)</td>
<td>2.6908 (0.0364)</td>
<td>2.3498 (0.1224)</td>
<td>1.6449 (0.0210)</td>
<td>0.6936 (0.0347)</td>
</tr>
<tr>
<td></td>
<td>CML</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>0.6916 (0.0355)</td>
</tr>
<tr>
<td>n=50</td>
<td>MLE</td>
<td>3.5907 (0.0082)</td>
<td>2.5210 (0.0004)</td>
<td>2.2217 (0.0492)</td>
<td>1.5913 (0.0083)</td>
<td>0.8069 (0.0053)</td>
</tr>
<tr>
<td></td>
<td>IFM</td>
<td>3.7993 (0.0896)</td>
<td>2.5853 (0.0073)</td>
<td>2.1779 (0.0316)</td>
<td>1.5730 (0.0053)</td>
<td>0.7780 (0.0104)</td>
</tr>
<tr>
<td></td>
<td>CML</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>0.7785 (0.0103)</td>
</tr>
<tr>
<td>n=75</td>
<td>MLE</td>
<td>3.5315 (0.0010)</td>
<td>2.4932 (4.6\times10^{-5})</td>
<td>2.1472 (0.0217)</td>
<td>1.5602 (0.0036)</td>
<td>0.8361 (0.0019)</td>
</tr>
<tr>
<td></td>
<td>IFM</td>
<td>3.7193 (0.0481)</td>
<td>2.5613 (0.0038)</td>
<td>2.1061 (0.0113)</td>
<td>1.5452 (0.0020)</td>
<td>0.8054 (0.0056)</td>
</tr>
<tr>
<td></td>
<td>CML</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>0.8040 (0.0058)</td>
</tr>
<tr>
<td>n=100</td>
<td>MLE</td>
<td>3.4675 (0.0011)</td>
<td>2.4711 (0.0008)</td>
<td>2.1158 (0.0134)</td>
<td>1.5481 (0.0023)</td>
<td>0.8552 (0.0006)</td>
</tr>
<tr>
<td></td>
<td>IFM</td>
<td>3.6375 (0.0189)</td>
<td>2.5375 (0.0014)</td>
<td>2.0771 (0.0059)</td>
<td>1.5338 (0.0011)</td>
<td>0.8256 (0.0030)</td>
</tr>
<tr>
<td></td>
<td>CML</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>0.8249 (0.0030)</td>
</tr>
<tr>
<td>n=125</td>
<td>MLE</td>
<td>3.4420 (0.0034)</td>
<td>2.4618 (0.0014)</td>
<td>2.0975 (0.0095)</td>
<td>1.5411 (0.0017)</td>
<td>0.8678 (0.0001)</td>
</tr>
</tbody>
</table>
Table 1. (Continued): The average estimates of parameters and the mean squared errors (in brackets) by simulation study for bivariate generalized exponential distribution based on FGM copula at $\alpha_1=3.5$, $\lambda_1=2.5$, $\alpha_2=2$, $\lambda_2=1.5$ and $\theta=0.88$.

<table>
<thead>
<tr>
<th>Sample Size</th>
<th>Estimation Method</th>
<th>$\alpha_1 = 3.5$</th>
<th>$\lambda_1 = 2.5$</th>
<th>$\alpha_2 = 2$</th>
<th>$\lambda_2 = 1.5$</th>
<th>$\theta_p = 2.48$</th>
</tr>
</thead>
<tbody>
<tr>
<td>n=150</td>
<td>IFM</td>
<td>3.6088 (0.0118)</td>
<td>2.5298 (0.0009)</td>
<td>2.0604 (0.0036)</td>
<td>1.5280 (0.0008)</td>
<td>0.8379 (0.0018)</td>
</tr>
<tr>
<td></td>
<td>CML</td>
<td>-</td>
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<td>-</td>
<td>0.8368 (0.0019)</td>
</tr>
<tr>
<td></td>
<td>MLE</td>
<td>3.4259 (0.0055)</td>
<td>2.4551 (0.0020)</td>
<td>2.0935 (0.0087)</td>
<td>1.5370 (0.0014)</td>
<td>0.8710 (8.1×10^{-5})</td>
</tr>
<tr>
<td></td>
<td>IFM</td>
<td>3.5946 (0.0089)</td>
<td>2.5251 (0.0006)</td>
<td>2.0574 (0.0033)</td>
<td>1.5245 (0.0006)</td>
<td>0.8402 (0.0016)</td>
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<tr>
<td></td>
<td>CML</td>
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<td>-</td>
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<td>0.8390 (0.0017)</td>
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</tbody>
</table>

Table 2. The average estimates of parameters and the mean squared errors (in brackets) by simulation study for bivariate generalized exponential distribution based on Plackett copula at $\alpha_1=3.5$, $\lambda_1=2.5$, $\alpha_2=2$, $\lambda_2=1.5$ and $\theta_p=2.48$.

<table>
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<th>$\alpha_2 = 2$</th>
<th>$\lambda_2 = 1.5$</th>
<th>$\theta_p = 2.48$</th>
</tr>
</thead>
<tbody>
<tr>
<td>n=25</td>
<td>MLE</td>
<td>3.9470 (0.1998)</td>
<td>2.6580 (0.0250)</td>
<td>2.3891 (0.1514)</td>
<td>1.6702 (0.0290)</td>
<td>2.6207 (0.0198)</td>
</tr>
<tr>
<td></td>
<td>IFM</td>
<td>4.2848 (0.6159)</td>
<td>2.7067 (0.0427)</td>
<td>2.3317 (0.1100)</td>
<td>1.6401 (0.0196)</td>
<td>2.511 (0.0010)</td>
</tr>
<tr>
<td></td>
<td>CML</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>2.4855 (3.0×10^{-5})</td>
</tr>
<tr>
<td>n=50</td>
<td>MLE</td>
<td>3.7272 (0.0516)</td>
<td>2.5883 (0.0078)</td>
<td>2.2347 (0.0551)</td>
<td>1.5952 (0.0091)</td>
<td>2.7231 (0.0591)</td>
</tr>
<tr>
<td></td>
<td>IFM</td>
<td>3.8439 (0.1183)</td>
<td>2.6027 (0.0105)</td>
<td>2.1645 (0.0271)</td>
<td>1.5674 (0.0045)</td>
<td>2.6009 (0.0146)</td>
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<tr>
<td></td>
<td>CML</td>
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<td>-</td>
<td>-</td>
<td>-</td>
<td>2.6092 (0.0167)</td>
</tr>
<tr>
<td>n=75</td>
<td>MLE</td>
<td>3.6425 (0.0203)</td>
<td>2.5556 (0.0031)</td>
<td>2.1665 (0.0277)</td>
<td>1.5686 (0.0047)</td>
<td>2.7031 (0.0498)</td>
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<tr>
<td></td>
<td>IFM</td>
<td>3.7207 (0.0487)</td>
<td>2.5676 (0.0046)</td>
<td>2.0959 (0.0092)</td>
<td>1.5400 (0.0016)</td>
<td>2.5714 (0.0084)</td>
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<tr>
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<td>CML</td>
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<td>-</td>
<td>-</td>
<td>-</td>
<td>2.5652 (0.0073)</td>
</tr>
<tr>
<td>n=100</td>
<td>MLE</td>
<td>3.6101 (0.0121)</td>
<td>2.5475 (0.0023)</td>
<td>2.1315 (0.0173)</td>
<td>1.5550 (0.0030)</td>
<td>2.7005 (0.0486)</td>
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</table>
Table 2. (Continued): The average estimates of parameters and the mean squared errors (in brackets) by simulation study for bivariate generalized exponential distribution based on Plackett copula at $\alpha_1=3.5$, $\lambda_1=2.5$, $\alpha_2=2$, $\lambda_2=1.5$ and $\theta_P=2.48$.

<table>
<thead>
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<th>Estimates of Parameter</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>IFM</td>
<td>$\alpha_1$ = 3.6664 (0.0277) $\lambda_1$ = 2.5516 (0.0027) $\lambda_2$ = 2.0600 (0.0036) $\theta_P$ = 2.5641 (0.0071)</td>
</tr>
<tr>
<td></td>
<td>CML</td>
<td>- - - - 2.5649 (0.0072)</td>
</tr>
<tr>
<td>n=125</td>
<td>MLE</td>
<td>3.5897 (0.0080) $\lambda_1$ = 2.5386 (0.0015) $\lambda_2$ = 2.1176 (0.0138) $\theta_P$ = 2.7023 (0.0494)</td>
</tr>
<tr>
<td></td>
<td>IFM</td>
<td>3.6451 (0.0211) $\lambda_1$ = 2.5446 (0.0020) $\lambda_2$ = 2.0469 (0.0022) $\theta_P$ = 2.5306 (0.0026)</td>
</tr>
<tr>
<td></td>
<td>CML</td>
<td>- - - - 2.5248 (0.0020)</td>
</tr>
<tr>
<td>n=150</td>
<td>MLE</td>
<td>3.5544 (0.0030) $\lambda_1$ = 2.5226 (0.0005) $\lambda_2$ = 2.1126 (0.0127) $\theta_P$ = 2.7346 (0.0648)</td>
</tr>
<tr>
<td></td>
<td>IFM</td>
<td>3.6124 (0.0126) $\lambda_1$ = 2.5341 (0.0012) $\lambda_2$ = 2.0467 (0.0022) $\theta_P$ = 2.5335 (0.0029)</td>
</tr>
<tr>
<td></td>
<td>CML</td>
<td>- - - - 2.5349 (0.0030)</td>
</tr>
</tbody>
</table>

From Table 1 and 2, the ML estimates of parameters of marginal $F_1$ for two models were better than IFM estimates for the same marginal, while IFM estimates of parameters of marginal $F_2$ for two models were better than ML estimates, knowing that $\alpha_1 > \alpha_2$. For copula parameter, it is observed that the ML estimates of FGM copula parameter $\theta_F$ were more efficient than the corresponding IFM and CML estimates. It is also observed that the IFM and CML estimates are close together. The Plackett copula parameter $\theta_P$ had efficient estimates by IFM and CML estimation methods compared with MLE.

Table 3. The average estimates of parameters and the mean squared errors (in brackets) by simulation study for bivariate generalized exponential distribution based on FGM copula at $\alpha_1=2.5$, $\lambda_1=1.5$, $\alpha_2=4$, $\lambda_2=2$ and $\theta_F=0.92$.

<table>
<thead>
<tr>
<th>Sample Size</th>
<th>Estimation Method</th>
<th>Estimates of Parameter</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>MLE</td>
<td>$\alpha_1$ = 2.4008 (0.0098) $\lambda_1$ = 1.6518 (0.0230) $\alpha_2$ = 4.3418 (0.1168) $\lambda_2$ = 2.0732 (0.0054) $\theta_F$ = 0.7549 (0.0272)</td>
</tr>
<tr>
<td>n=25</td>
<td>IFM</td>
<td>2.3547 (0.0211) $\lambda_1$ = 1.6278 (0.0163) $\alpha_2$ = 4.9521 (0.9065) $\lambda_2$ = 2.1685 (0.0284) $\theta_F$ = 0.7207 (0.0397)</td>
</tr>
<tr>
<td></td>
<td>CML</td>
<td>- - - - 0.7177 (0.0409)</td>
</tr>
</tbody>
</table>
Table 3. (Continued): The average estimates of parameters and the mean squared errors (in brackets) by simulation study for bivariate generalized exponential distribution based on FGM copula at $\alpha_1=2.5$, $\lambda_1=1.5$, $\alpha_2=4$, $\lambda_2=2$ and $\theta_F=0.92$.

<table>
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<tr>
<th></th>
<th>MLE</th>
<th>IFM</th>
<th>CML</th>
</tr>
</thead>
<tbody>
<tr>
<td>n=50</td>
<td>2.1904 (0.0959)</td>
<td>2.1367 (0.1320)</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>1.5765 (0.0058)</td>
<td>1.5577 (0.0033)</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>4.0651 (0.0042)</td>
<td>4.4648 (0.2160)</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>2.0031 (9.6\times10^{-6})</td>
<td>2.0851 (0.0072)</td>
<td>0.8374 (0.0068)</td>
</tr>
<tr>
<td>n=75</td>
<td>2.1493 (0.1230)</td>
<td>2.1015 (0.1588)</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>1.5567 (0.0032)</td>
<td>1.5410 (0.0017)</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>3.9293 (0.0050)</td>
<td>4.2757 (0.076)</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>1.9718 (0.0008)</td>
<td>2.0526 (0.0028)</td>
<td>0.8343 (0.0073)</td>
</tr>
<tr>
<td>n=100</td>
<td>2.1109 (0.1514)</td>
<td>2.0636 (0.1904)</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>1.5405 (0.0016)</td>
<td>1.5253 (0.0006)</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>3.8765 (0.0153)</td>
<td>4.2006 (0.0402)</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>1.9590 (0.0017)</td>
<td>2.0393 (0.0015)</td>
<td>0.8536 (0.0044)</td>
</tr>
<tr>
<td>n=125</td>
<td>2.0950 (0.1640)</td>
<td>2.0504 (0.2021)</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>1.5346 (0.0012)</td>
<td>1.5201 (0.0004)</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>3.8299 (0.0289)</td>
<td>4.1573 (0.0247)</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>1.9488 (0.0026)</td>
<td>2.0323 (0.0010)</td>
<td>0.8657 (0.0028)</td>
</tr>
<tr>
<td>n=150</td>
<td>2.0876 (0.1701)</td>
<td>2.0442 (0.2078)</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>1.5310 (0.0009)</td>
<td>1.5169 (0.0003)</td>
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</tr>
<tr>
<td></td>
<td>3.8039 (0.0385)</td>
<td>4.1458 (0.0213)</td>
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</tr>
<tr>
<td></td>
<td>1.9401 (0.0036)</td>
<td>2.0283 (0.0008)</td>
<td>0.8700 (0.0025)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>0.8684 (0.0027)</td>
</tr>
</tbody>
</table>
Table 4. The average estimates of parameters and the mean squared errors (in brackets) by simulation study for bivariate generalized exponential distribution based on Plackett copula at $\alpha_1=2.5$, $\lambda_1=1.5$, $\alpha_2=4$, $\lambda_2=2$ and $\theta_P=2.60$.

<table>
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<tr>
<th>Sample Size</th>
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<th>Estimates of Parameter</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$\alpha_1=2.5$</td>
</tr>
<tr>
<td>n=25</td>
<td>MLE</td>
<td>2.4390 (0.0037)</td>
</tr>
<tr>
<td></td>
<td>IFM</td>
<td>2.3554 (0.0209)</td>
</tr>
<tr>
<td></td>
<td>CML</td>
<td>src:</td>
</tr>
<tr>
<td>n=50</td>
<td>MLE</td>
<td>2.2517 (0.0617)</td>
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<tr>
<td></td>
<td>IFM</td>
<td>2.1564 (0.1181)</td>
</tr>
<tr>
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<td>CML</td>
<td>src:</td>
</tr>
<tr>
<td>n=75</td>
<td>MLE</td>
<td>2.1937 (0.0938)</td>
</tr>
<tr>
<td></td>
<td>IFM</td>
<td>2.1025 (0.1580)</td>
</tr>
<tr>
<td></td>
<td>CML</td>
<td>src:</td>
</tr>
<tr>
<td>n=100</td>
<td>MLE</td>
<td>2.1717 (0.1078)</td>
</tr>
<tr>
<td></td>
<td>IFM</td>
<td>2.0770 (0.1789)</td>
</tr>
<tr>
<td></td>
<td>CML</td>
<td>src:</td>
</tr>
<tr>
<td>n=125</td>
<td>MLE</td>
<td>2.1609 (0.1078)</td>
</tr>
<tr>
<td></td>
<td>IFM</td>
<td>2.0675 (0.1871)</td>
</tr>
<tr>
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<td>CML</td>
<td>src:</td>
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<tr>
<td>n=150</td>
<td>MLE</td>
<td>2.1454 (0.1257)</td>
</tr>
<tr>
<td></td>
<td>IFM</td>
<td>2.0520 (0.2007)</td>
</tr>
<tr>
<td></td>
<td>CML</td>
<td>src:</td>
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</table>
From Table 3 and 4, the ML estimates of parameters for two models were better in most of cases compared with the corresponding estimates by IFM, knowing that $\alpha_1 < \alpha_2$. For copula parameter, it is clear that the estimates of FGM copula parameter $\theta_F$ by MLE were better than the estimates by IFM and CML methods. It is also clear that IFM and CML estimates were closed together. The IFM and CML estimates of Plackett copula parameter $\theta_P$ were closed and efficient compared with the corresponding ML estimates.

Table 5. The average estimates of parameters and the mean squared errors (in brackets) by simulation study for bivariate generalized exponential distribution based on FGM copula at $\alpha_1=3$, $\lambda_1=1.8$, $\alpha_2=3$, $\lambda_2=2.5$ and $\theta_F=0.80$.

<table>
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<th>$\lambda_1 = 1.8$</th>
<th>$\alpha_2 = 3$</th>
<th>$\lambda_2 = 2.5$</th>
<th>$\theta_F = 0.80$</th>
</tr>
</thead>
<tbody>
<tr>
<td>n=25</td>
<td>MLE</td>
<td>3.4550 (0.207)</td>
<td>1.9217 (0.0148)</td>
<td>3.4402 (0.1938)</td>
<td>2.6895 (0.0359)</td>
<td>0.6661 (0.0179)</td>
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<tr>
<td></td>
<td>IFM</td>
<td>3.6388 (0.4081)</td>
<td>1.9410 (0.0199)</td>
<td>3.6259 (0.3918)</td>
<td>2.7200 (0.0484)</td>
<td>0.6369 (0.0266)</td>
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<tr>
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<td>CML</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>0.6353 (0.0271)</td>
</tr>
<tr>
<td>n=50</td>
<td>MLE</td>
<td>3.1988 (0.0395)</td>
<td>1.8568 (0.0032)</td>
<td>3.2214 (0.049)</td>
<td>2.5855 (0.0073)</td>
<td>0.7489 (0.0026)</td>
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<tr>
<td></td>
<td>IFM</td>
<td>3.2410 (0.0581)</td>
<td>1.8631 (0.0040)</td>
<td>3.3106 (0.0965)</td>
<td>2.6110 (0.0123)</td>
<td>0.7187 (0.0066)</td>
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<td>CML</td>
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<td>-</td>
<td>-</td>
<td>-</td>
<td>0.7202 (0.0064)</td>
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<tr>
<td>n=75</td>
<td>MLE</td>
<td>3.1392 (0.0194)</td>
<td>1.8357 (0.0013)</td>
<td>3.1222 (0.0149)</td>
<td>2.5436 (0.0019)</td>
<td>0.7726 (0.0007)</td>
</tr>
<tr>
<td></td>
<td>IFM</td>
<td>3.1773 (0.0314)</td>
<td>1.8453 (0.0021)</td>
<td>3.1839 (0.0338)</td>
<td>2.5684 (0.0047)</td>
<td>0.7423 (0.0033)</td>
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<td>CML</td>
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<td>-</td>
<td>-</td>
<td>-</td>
<td>0.7420 (0.0033)</td>
</tr>
<tr>
<td>n=100</td>
<td>MLE</td>
<td>3.0861 (0.0074)</td>
<td>1.8195 (0.0004)</td>
<td>3.0806 (0.0065)</td>
<td>2.5259 (0.0007)</td>
<td>0.7930 (4.9×10^{-5})</td>
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<tr>
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<td>IFM</td>
<td>3.1111 (0.0123)</td>
<td>1.8277 (0.0008)</td>
<td>3.1342 (0.018)</td>
<td>2.5514 (0.0026)</td>
<td>0.7645 (0.0013)</td>
</tr>
<tr>
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<td>CML</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>0.7641 (0.0013)</td>
</tr>
<tr>
<td>n=125</td>
<td>MLE</td>
<td>3.0607 (0.0037)</td>
<td>1.8121 (0.0001)</td>
<td>3.0527 (0.0028)</td>
<td>2.5155 (0.0002)</td>
<td>0.8027 (7.3×10^{-6})</td>
</tr>
</tbody>
</table>
Table 5. (Continued): The average estimates of parameters and the mean squared errors (in brackets) by simulation study for bivariate generalized exponential distribution based on FGM copula at $\alpha_1=3$, $\lambda_1=1.8$, $\alpha_2=3$, $\lambda_2=2.5$ and $\theta_F=0.80$.

<table>
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<th>$\alpha_2 = 3$</th>
<th>$\lambda_2 = 2.5$</th>
<th>$\theta_F = 0.80$</th>
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</thead>
<tbody>
<tr>
<td>n=150</td>
<td>IFM</td>
<td>3.0879 (0.0077)</td>
<td>1.8221 (0.0005)</td>
<td>3.1055 (0.0111)</td>
<td>2.5425 (0.0018)</td>
<td>0.7743 (0.0007)</td>
</tr>
<tr>
<td></td>
<td>CML</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>0.7738 (0.0007)</td>
</tr>
<tr>
<td></td>
<td>MLE</td>
<td>3.0467 (0.0022)</td>
<td>1.8076 (5.8×10^{-5})</td>
<td>3.0417 (0.0017)</td>
<td>2.5074 (0.0001)</td>
<td>0.8049 (2.4×10^{-5})</td>
</tr>
<tr>
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<td>IFM</td>
<td>3.0766 (0.0059)</td>
<td>1.8186 (0.0003)</td>
<td>3.0983 (0.0097)</td>
<td>2.5370 (0.0014)</td>
<td>0.7749 (0.0006)</td>
</tr>
<tr>
<td></td>
<td>CML</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>0.7738 (0.0007)</td>
</tr>
</tbody>
</table>

Table 6. The average estimates of parameters and the mean squared errors (in brackets) by simulation study for bivariate generalized exponential distribution based on Plackett copula at $\alpha_1=3$, $\lambda_1=1.8$, $\alpha_2=3$, $\lambda_2=2.5$ and $\theta_P=2.27$.

<table>
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<tr>
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<th>Estimation Method</th>
<th>Estimates of Parameter</th>
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<td>MLE</td>
<td>$\alpha_1 = 3$</td>
</tr>
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<td>IFM</td>
<td>3.4787 (0.2292)</td>
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<td>CML</td>
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</tr>
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<td>n=50</td>
<td>MLE</td>
<td>3.6310 (0.3982)</td>
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<tr>
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<td>IFM</td>
<td>3.2767 (0.0766)</td>
</tr>
<tr>
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<td>CML</td>
<td>-</td>
</tr>
<tr>
<td>n=75</td>
<td>MLE</td>
<td>3.1790 (0.0320)</td>
</tr>
<tr>
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<td>IFM</td>
<td>3.1790 (0.0320)</td>
</tr>
<tr>
<td></td>
<td>CML</td>
<td>-</td>
</tr>
</tbody>
</table>
Table 6. (Continued): The average estimates of parameters and the mean squared errors (in brackets) by simulation study for bivariate generalized exponential distribution based on Plackett copula at $\alpha_1=3$, $\lambda_1=1.8$, $\alpha_2=3$, $\lambda_2=2.5$ and $\theta_P=2.27$.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>n=100 MLE</th>
<th></th>
<th>IFM</th>
<th></th>
<th>CML</th>
<th></th>
<th>n=125 MLE</th>
<th></th>
<th>IFM</th>
<th></th>
<th>CML</th>
<th></th>
<th>n=150 MLE</th>
<th></th>
<th>IFM</th>
<th></th>
<th>CML</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>3.1548</td>
<td>(0.0240)</td>
<td>1.8461</td>
<td>(0.0021)</td>
<td>3.1295</td>
<td>(0.0168)</td>
<td>2.5573</td>
<td>(0.0033)</td>
<td>2.4788</td>
<td>(0.0436)</td>
<td>3.1333</td>
<td>(0.0178)</td>
<td>1.8378</td>
<td>(0.0014)</td>
<td>3.1051</td>
<td>(0.0110)</td>
<td>2.5370</td>
<td>(0.0014)</td>
</tr>
</tbody>
</table>

Based on the results in Table 5 and 6, knowing that $\alpha_1 = \alpha_2$, the ML estimates for parameters of bivariate generalized exponential distribution based on FGM copula were more efficient than estimates by other estimation methods. The estimates of parameters of bivariate generalized exponential distribution based on Plackett copula by IFM method were more efficient in most of cases compared with other estimation methods. It is observed that the IFM and CML estimates for FGM copula parameter $\theta_F$ and Plackett copula parameter $\theta_P$ were closed together. In most of cases in Table 1, 2, 3, 4, 5 and 6, it is clear that as sample size increases, the mean square errors decrease and the parameters estimates become better.

### 5.2 Real Data

The data set from McGilchrist and Aisbett in [15]. This data represents the recurrence time of infection for kidney patients. Let $T_1$ refers to first recurrence time and $T_2$ to second recurrence time. The data for 30 patients are reported in Table 7.
Table 7. Kidney infection data 30 patients.

<table>
<thead>
<tr>
<th>Patient</th>
<th>$T_1$</th>
<th>$T_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>8</td>
<td>16</td>
</tr>
<tr>
<td>2</td>
<td>23</td>
<td>13</td>
</tr>
<tr>
<td>3</td>
<td>22</td>
<td>28</td>
</tr>
<tr>
<td>4</td>
<td>447</td>
<td>318</td>
</tr>
<tr>
<td>5</td>
<td>30</td>
<td>12</td>
</tr>
<tr>
<td>6</td>
<td>24</td>
<td>245</td>
</tr>
<tr>
<td>7</td>
<td>7</td>
<td>9</td>
</tr>
<tr>
<td>8</td>
<td>511</td>
<td>30</td>
</tr>
<tr>
<td>9</td>
<td>53</td>
<td>196</td>
</tr>
<tr>
<td>10</td>
<td>15</td>
<td>154</td>
</tr>
<tr>
<td>11</td>
<td>7</td>
<td>333</td>
</tr>
<tr>
<td>12</td>
<td>141</td>
<td>8</td>
</tr>
<tr>
<td>13</td>
<td>96</td>
<td>38</td>
</tr>
<tr>
<td>14</td>
<td>149</td>
<td>70</td>
</tr>
<tr>
<td>15</td>
<td>536</td>
<td>25</td>
</tr>
<tr>
<td>16</td>
<td>17</td>
<td>4</td>
</tr>
<tr>
<td>17</td>
<td>185</td>
<td>117</td>
</tr>
<tr>
<td>18</td>
<td>292</td>
<td>114</td>
</tr>
<tr>
<td>19</td>
<td>22</td>
<td>159</td>
</tr>
<tr>
<td>20</td>
<td>15</td>
<td>108</td>
</tr>
<tr>
<td>21</td>
<td>152</td>
<td>362</td>
</tr>
<tr>
<td>22</td>
<td>402</td>
<td>24</td>
</tr>
<tr>
<td>23</td>
<td>13</td>
<td>66</td>
</tr>
<tr>
<td>24</td>
<td>39</td>
<td>46</td>
</tr>
<tr>
<td>25</td>
<td>12</td>
<td>40</td>
</tr>
<tr>
<td>26</td>
<td>113</td>
<td>201</td>
</tr>
<tr>
<td>27</td>
<td>132</td>
<td>156</td>
</tr>
<tr>
<td>28</td>
<td>34</td>
<td>30</td>
</tr>
<tr>
<td>29</td>
<td>2</td>
<td>25</td>
</tr>
<tr>
<td>30</td>
<td>130</td>
<td>26</td>
</tr>
</tbody>
</table>

First, the generalized exponential distribution is fitted to $T_1$ and $T_2$ separately. The ML estimates of the shape and scale parameters of the generalized exponential distribution for $T_1$ and $T_2$ are (0.6638, 0.0062) and (0.9244, 0.0096) respectively. The Kolmogorov-Smirnov distances between the fitted distribution and the empirical distribution function for $T_1$ and $T_2$ are 0.1731 and 0.1555 respectively. The corresponding p-value are 0.3298 and 0.4630 respectively. Based on the p-
value, the generalized exponential distribution can be fit to the marginal. The Kendall’s tau and the Spearman’s rho for data are 0.1110 and 0.1531, respectively. It is noted that there is a weak positive dependence between data. A copula goodness-of-fit test is applied to check the relevance of copulas for data. Furthermore, popular criteria for model selection are used which are the Akaike information criterion (AIC) and the Bayesian information criterion (BIC). The AIC and BIC are defined respectively as

\[
AIC = -2 \log \text{likelihood} + 2k, \\
BIC = -2 \log \text{likelihood} + k \ln(N),
\]

where \( k \) equals the number of parameters in the model and \( N \) equals the number of observations. The lowest values of AIC and BIC indicate the best fit for a model. The results of copula goodness-of-fit test, AIC and BIC are reported in Table 8.

<table>
<thead>
<tr>
<th>Copula</th>
<th>p-value</th>
<th>Estimated dependence parameter</th>
<th>AIC</th>
<th>BIC</th>
</tr>
</thead>
<tbody>
<tr>
<td>FGM copula</td>
<td>0.7338</td>
<td>0.4670</td>
<td>689.0881</td>
<td>696.0940</td>
</tr>
<tr>
<td>Plackett copula</td>
<td>0.7877</td>
<td>1.6630</td>
<td>689.0495</td>
<td>696.0555</td>
</tr>
</tbody>
</table>

The resulting p-values of copula goodness-of-fit test were 0.7338 for FGM copula and 0.7877 for Plackett copula, which confirms that both copulas are suitable for the data set. The resulting estimated dependence parameter was 0.467 for FGM copula and 1.663 for Plackett copula. The results of AIC were 689.0881 for FGM copula and 689.0495 for Plackett copula. The results of BIC were 696.0940 for FGM copula and 696.0555 for Plackett copula. From the goodness of fit test, the AIC and BIC results, it seems that the Plackett copula is fitting better than FGM copula.

Finally, the bivariate generalized exponential distribution based on FGM and Plackett copulas are fitted to the data separately. The proposed models are estimated with different estimation methods. Table 9 displayed the estimates and the corresponding mean square errors (in brackets) of parameters.
Table 9. The estimates and the corresponding mean squared errors (in brackets) of parameters of bivariate generalized exponential distribution based on FGM and Plackett copulas respectively.

<table>
<thead>
<tr>
<th>Copula</th>
<th>Estimation method</th>
<th>Estimates of parameters</th>
<th>Copula Parameter</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$\alpha_1$</td>
<td>$\lambda_1$</td>
</tr>
<tr>
<td>FGM</td>
<td>MLE</td>
<td>0.6645</td>
<td>0.0063</td>
</tr>
<tr>
<td></td>
<td>IFM</td>
<td>0.6603</td>
<td>0.0062</td>
</tr>
<tr>
<td></td>
<td>CML</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Plackett</td>
<td>MLE</td>
<td>0.6676</td>
<td>0.0063</td>
</tr>
<tr>
<td></td>
<td>IFM</td>
<td>0.6603</td>
<td>0.0062</td>
</tr>
<tr>
<td></td>
<td>CML</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

It is observed that the efficient estimators of marginal parameters of two models differ according to the parameters. It seems that ML estimates of shape parameters $\alpha_1$ and $\alpha_2$ of two models are better than the corresponding IFM estimates. The IFM estimates of scale parameters $\lambda_1$ and $\lambda_2$ of two models are better than the corresponding ML estimates. For copula parameter, the CML estimation method provided efficient estimates for both FGM and Plackett copula parameters compared to MLE or IFM methods. It is also noted that the MLE and IFM estimates for copula parameter are closed together.

6 Conclusion

In this paper, a bivariate generalized exponential distribution is studied basing on FGM and Plackett copulas. Three different estimation methods are applied on the proposed models, and it is observed that the efficient estimation method for marginal differs according to the nature of data and values of parameters. For FGM copula parameter, the efficient estimator also differs according to the nature of data and value of parameter while the CML estimation method is efficient for estimating the Plackett copula parameter in most of cases. A real data set is analyzed and fitted to the proposed models. It is observed that a bivariate generalized exponential distribution based on Plackett copula is fitting best for data than a bivariate generalized exponential distribution based on FGM copula.
References


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