Nonsmooth Lyapunov Stability of Differential Equations

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Abstract

In this work we focus on nondifferentiable Lyapunov functions where we derive conditions which ensure the existence of time-dependent upper bounds for such functions. The aforementioned conditions are based on the notion of upper right-hand Dini derivative of Lyapunov functions. As an application we study the attractivity and the asymptotic stability of the time–varying class of systems $\dot{x}(t) = f(t, x(t))$. The results are illustrated by a numerical simulation.

Keywords: nonsmooth Lyapunov function, upper right-hand Dini derivative, attractivity

1 Introduction

Lyapunov functions play an essential role for the stability analysis of nonlinear systems and therefore have been attracting attention of scientists and engineers [9, 10, 11, 15]. They are usually represent mapping of states of systems to solutions of one-dimensional systems that are referred to as the original systems. Furthermore, they can provide sufficient conditions to ensure the stability of a dynamic system without the need to have an explicit expression for the state. The classical Lyapunov approach to stability of the system $\dot{x}(t) = f(t, x(t))$ requires the existence of a positive definite Lyapunov function $V(t, x)$ with negative derivative along the trajectory of the system [8, 2]. Positive definite
Lyapunov functions can be also used to prove the so called “attractivity” of systems in which all trajectories converge to an equilibrium as \( t \to \infty \) [2].

A major problem in the approach of Lyapunov stability is that they; the Lyapunov functions, are often difficult to obtain analytically or maybe non-constructive in many cases. Therefore there is a critical need to extend the standard continuously differentiable Lyapunov function theory to the nonsmooth case [1, 4, 7, 14]. In practice, many systems in physics, engineering and biology exhibit generally nonsmooth energy functions, which are usually typical candidates for Lyapunov functions.

This paper deals with continuous nondifferentiable Lyapunov functions \( z(t) := V(t, x(t)) \) of differential equations of the form \( \dot{x}(t) = f(t, x(t)) \). More precisely, we consider the upper right-hand Dini derivative of the Lyapunov function \( z(t) \) and investigate the existence of a nonegative continuously differentiable real function \( \phi(t) \) such that \( z(t) \leq \phi(t) \). As a result, if \( \lim_{t \to \infty} \dot{\phi}(t) = 0 \), then the attractivity of the system is guaranteed and the origin \( x = 0 \) is asymptotically stable whenever \( \dot{z} \leq 0 \).

This paper is organized as follows. In Section 2, we derive generalized sufficient conditions for the inequality \( z(t) \leq \phi(t) \). We then apply these conditions in Section 3 to investigate the attractivity and the asymptotic stability for the origin of the class of systems \( \dot{x}(t) = f(t, x(t)) \). Furthermore, we provide an example to clarify the derived results.

## 2 Stability theorem

The main result of the present section is Theorem 2.1 which gives time-dependent upper bound for nonsmooth Lyapunov functions. Let \( C^0(\mathbb{R}, \mathbb{R}^n) \) denotes the Banach space of continuous functions \( z : \mathbb{R} \to \mathbb{R}^n \) endowed with the norm \( \| \cdot \| \). For a function \( z \in C^0(\mathbb{R}, \mathbb{R}) \), the upper right-hand Dini derivative is denoted and defined by \( D^+z(t) := \limsup_{h \to 0^+} \frac{z(t+h)-z(t)}{h} \). On the extended reals, the upper right-hand Dini derivative always exist [3]. If \( z \) is locally Lipschitz, then \( D^+z(t) \) is finite. If \( z \) is differentiable, then \( D^+z = \dot{z} \) [12].

**Theorem 2.1.** Consider a function \( z \in C^0([t_0, \infty), \mathbb{R}_+) \) where \( t_0 \in \mathbb{R} \) and \( t_0 < \omega \leq \infty \). Assume that there exists a function \( \phi \in C^1(\mathbb{R}, \mathbb{R}_+) \) such that \( z(t_0) < \phi(t_0) \) and

\[
D^+z(t) < \dot{\phi}(t), \text{ for all } t \in (t_0, \omega)
\]

that satisfy \( z(t) = \phi(t) \).

\[ (1) \]
Then

(i) \( z(t) \leq \phi(t), \forall t \in [t_0, \omega) \) and \( \lim_{t \to \infty} z(t) = 0 \) whenever \( \lim_{t \to \infty} \phi(t) = 0 \).

(ii) If \( \phi \) is a positive constant function; say \( \phi(t) = c > 0, \forall t \geq t_0 \), and that \( z \) is absolutely continuous; then \( z(t) \leq z(t_0), \forall t \geq t_0 \).

Proof. To prove (i) suppose that there exists \( t_1 \in [t_0, \omega) \) such that \( z(t_1) > \phi(t_1) \), then \( t_0 < t_1 \) because it is assumed that \( z(t_0) < \phi(t_0) \). Consider the nonempty set \( A = \{ \tau \in [t_0, t_1)/z(t) \leq \phi(t), \forall t \in [t_0, \tau] \} \). We have \( t_0 < t_2 < t_1 \) because \( z(t_1) > \phi(t_1) \). By the definition of \( t_2 \), there exist sequences \( \{t_n < t_2\}_{n=1}^{\infty} \) and \( \{t'_n > t_2\}_{n=1}^{\infty} \) such that \( \lim_{n \to \infty} t_n = \lim_{n \to \infty} t'_n = t_2, z(t_n) \leq \phi(t_n), \forall n \in \mathbb{N} \) and \( z(t'_n) > \phi(t'_n), \forall n \in \mathbb{N} \). Thus the continuity of \( z \) and \( \phi \) imply that

\[
\lim_{n \to \infty} z(t_n) = z(t_2) = \lim_{n \to \infty} \phi(t_n) = \phi(t_2) = \lim_{n \to \infty} \phi(t'_n) \leq \lim_{n \to \infty} z(t'_n) = z(t_2).
\]

so that \( z(t_2) = \phi(t_2) \). Let \( B = \{ t \in [t_2, t_1)/z(t) = \phi(t) \} \). The set \( B \) is nonempty because \( t_2 \in B \). Let \( t_3 := \sup B \). By the definition of \( t_3 \), there exist two sequences \( \{t_n < t_3\}_{n=1}^{\infty} \) and \( \{t'_n > t_3\}_{n=1}^{\infty} \) such that \( \lim_{n \to \infty} t_n = \lim_{n \to \infty} t'_n = t_3, z(t_n) < \phi(t_n), \forall n \in \mathbb{N} \) and \( z(t'_n) > \phi(t'_n), \forall n \in \mathbb{N} \). Hence we deduce by the continuity of \( z \) and \( \phi \) that

\[
\lim_{n \to \infty} z(t_n) = z(t_3) = \lim_{n \to \infty} \phi(t_n) = \phi(t_2) = \lim_{n \to \infty} \phi(t'_n) \leq \lim_{n \to \infty} z(t'_n) = z(t_3).
\]

Hence \( z(t_3) = \phi(t_3) \) so that \( t_3 \in B \) and \( t_2 < t_3 < t_1 \).

Claim 2.2. \( z(t) > \phi(t), \forall t \in (t_3, t_1) \).

Proof. Assume not, that is there exists some \( T \in (t_3, t_1) \) such that \( z(T) \leq \phi(T) \). By the definition of \( t_3 \) we have \( z(T) < \phi(T) \). Consider the continuous function \( z_1(\cdot) = z(\cdot) - \phi(\cdot) \). One has \( z_1(T) < 0 < z_1(t_1) \). Therefore by applying the Intermediate Value Theorem on the function \( z_1 \) and the interval \([T, t_1]\) we deduce that there must exist \( T' \in (T, t_1) \) such that \( z(T') = \phi(T') \) which contradicts the fact that \( t_3 = \sup B \). This completes the proof of the claim.

We deduce by Claim 2.2 that \( z(t_3 + h) > \phi(t_3 + h), \forall h \in (0, t_1 - t_3) \) and hence the fact that \( t_3 \in B \) implies

\[
\frac{z(t_3 + h) - z(t_3)}{h} > \frac{\phi(t_3 + h) - \phi(t_3)}{h}, \forall h \in (0, t_1 - t_3).
\]

This implies that \( D^+z(t_3) \geq \dot{\phi}(t_3) \) which contradicts Inequality (1) (see [6, pp.34]). Thus, the fact that \( z \) is nonnegative implies that \( \lim_{t \to \infty} z(t) = 0 \) whenever \( \lim_{t \to \infty} \phi(t) = 0 \). Therefore, Result (i) has been proved. To prove
Result (ii), assume that Inequality (1) is satisfied with \( \phi(t) = c > 0, \forall t \geq t_0 \), then the upper Dini derivative of \( z \) is nonpositive almost everywhere. Thus the fact that \( z \) is absolutely continuous implies that \( z \) is nonincreasing so that \( z(t) \leq z(t_0), \forall t \geq t_0. \)

\[ \text{Comment 2.3. If all assumptions of Theorem 2.1 are satisfied with (1) being weakened to be} \]
\[ D^+ z(t) \leq \dot{\phi}(t), \text{ for all } t \in (t_0, \omega) \]
\[ \text{that satisfy } z(t) = \phi(t), \text{ then the result } z(\cdot) \leq \phi(\cdot) \text{ cannot be guaranteed. To see this, take } t_0 = -0.5, \]
\[ \omega = \infty, z(t) = t^3 + 1, \text{ and } \phi(t) = 1 \text{ for all } t \geq -0.5 = t_0. \]
\[ \text{Observe that } z(t_0) < \phi(t_0). \text{ The functions } z \text{ and } \phi \text{ are differentiable and the only point of} \]
\[ \text{intersection between them occurs when } t = 0. \text{ At this point, we have } D^+ z(t) = 3t^2 = 0 \leq 0 = \dot{\phi}(t) \text{ which makes (2) satisfied. However, one can easily see} \]
\[ \text{that } z > \phi \text{ on the interval } (0, \infty). \]

\[ \text{3 System under study} \]

Consider the class of systems
\[ \dot{x}(t) = f(t, x(t)), \quad t \in \mathbb{R}_+, \]
\[ x(t_0) = x_0, \]
where \((t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^m\), state \( x(t) \) takes values in \( \mathbb{R}^m \) for some strictly positive integer \( m \), and a well-defined function \( f : \mathbb{R} \times \mathbb{R}^m \rightarrow \mathbb{R}^m \). In this paper we assume that the function \( f \) satisfies the following local Carathéodory conditions (see [2, §1.1] and [5]).

\[ (A_1) \text{ The function } f(t, x) \text{ is locally essentially bounded on } \mathbb{R} \times \mathbb{R}^m, \]
\[ (A_2) \text{ for each } x \in \mathbb{R}^m, \text{ the function } t \mapsto f(t, x) \text{ is measurable,} \]
\[ (A_3) \text{ for almost all } t \in \mathbb{R}_+, \text{ the function } x \mapsto f(t, x) \text{ is continuous.} \]

Thus an absolutely continuous Carathéodory solution of (3)–(4) exists that is defined on some interval \( I \). If the function \( f \) is continuous on \( \mathbb{R} \times \mathbb{R}^m \) then Assumptions \((A_1)–(A_3)\) are satisfied. By Peano’s Theorem it follows that a classical solution \( x \) of (3)–(4) exists on some interval \( I \subset \mathbb{R}_+ \) [13].
3.1 Sufficient conditions for the attractivity

In this subsection we apply the results of section 2 on System (3)-(4).

**Definition 3.1.** [8] A function $\beta \in C^0(\mathbb{R}_+, \mathbb{R}_+)$ belongs to class $K_{\infty}$ if $\beta(0) = 0$ and $\beta$ is strictly increasing with $\lim_{v \to \infty} \beta(v) = \infty$.

**Lemma 3.2.** Let $x(t)$ be a solution of (3)-(4) with maximal interval of existence $[t_0, \omega)$. Suppose that there exist a function $\phi \in C^1(\mathbb{R}, \mathbb{R}_+)$ and a class $K_{\infty}$ function $\beta \in C^0(\mathbb{R}_+, \mathbb{R}_+)$; that are independent of $t_0$ and $x_0$, such that Inequality (1) of Theorem 1 is satisfied with $z(\cdot) = \beta(|x(\cdot)|)$. Then for each initial condition $|x_0| < \beta^{-1}(\phi(t_0))$, one has $\omega = \infty$ and $|x(t)| \leq \beta^{-1}(\phi(t)), \forall t \geq t_0$. (Observe that if further one has $\lim_{t \to \infty} \phi(t) = 0$, then $\lim_{t \to \infty} \phi(t) = 0$).

**Proof.** Let $x_0 \in \mathbb{R}^m$ be such that $|x_0| < \beta^{-1}(\phi(t_0))$, then the monotonicity of $\beta$ implies $z(t_0) < \phi(t_0)$. Thus all conditions of Theorem 2.1 are satisfied. This implies that $z(t) \leq \phi(t), \forall t \in [t_0, \omega]$ so that $|x(t)| \leq \beta^{-1}(\phi(t)), \forall t \in [t_0, \omega)$. Thus, the boundedness of the function $\phi$ guarantees the boundedness of the state $x$. Hence $\omega = \infty$. Moreover, one has $\lim_{t \to \infty} x(t) = 0$ whenever $\lim_{t \to \infty} \phi(t) = 0$ because the function $\beta$ belongs to class $K_{\infty}$.

**Comment 3.3.** If all conditions of Lemma 3.2 are satisfied with $\lim_{t \to \infty} \phi(t) = 0$ and $\dot{z} \leq 0$, then $x(t) = 0$ is asymptotically stable [8].

**Comment 3.4.** Lemma 3.2 can be used to investigate the stability of nonlinear systems at which explicit expressions for solutions are hard (or maybe impossible) to derive. As a result, it is a good alternative to the Comparison lemma [8, Lemma 3.4]. This is illustrated in the following subsection.

3.2 Simulation

In the comparison principle [8, Lemma 3.4], one derives a one-dimensional differential inequality that contains a Lyapunov function. This inequality is related to some resulting one-dimensional differential equation. The comparison lemma works when one can find the solution of the aforementioned differential equation explicitly. For instance, Example 3.9 in [8] has studied the attractivity of the system $\dot{x} = -(1 + x^2) x + e^t$ using the Lyapunov function $v(t) = |x(t)|$. This yielded a linear differential equation and thus it could be easily solved and the attractivity has been proved. What happens if one cannot find the solution? Let us consider the following system as an example

\[ \dot{x}(t) = -2t^4 \left(1 + x^4(t)\right) x^3(t) + t, \quad (5) \]
\[ x(t_0) = x_0, \quad (6) \]
where \( t \geq t_0 > 1 \) and both initial condition \( x_0 \) and output \( x(t) \) belong to the set \( \mathbb{R} \). Due to the continuity of the right-hand side of (5), a continuously differentiable solution \( x(t) \) exists and is defined on an interval of the form \([t_0, \omega)\) [13]. Let \( \phi \in C^1(\mathbb{R}, \mathbb{R}_+) \) be such that \( \phi(t) = \frac{1}{t}, \forall t \geq t_0 \). Observe that \( \lim_{t \to \infty} \phi(t) = 0 \). Let \( \beta \in C^0(\mathbb{R}_+, \mathbb{R}_+) \) be the identity function. Then \( \beta \) belongs to class \( \mathcal{K}_\infty \) and function \( z \) in Lemma 3.2 is defined as \( z(\cdot) = \beta(\|x(\cdot)\|) = \|x(\cdot)\| \). Observe that the Lyapunov function \( z \) may be nondifferentiable.

**Claim 3.5.** \( D^+ z(t) \leq -2t^4 z^2(t) + t \), for all \( t \in (t_0, \omega) \).

**Proof.** Let \( t \in (t_0, \omega) \). We discuss the following two cases:

If \( x(t) \neq 0 \), then \( z \) is differentiable at \( t \) so that \( D^+ z(t) = \dot{z}(t) \). Thus the fact that \( z(t) = \|x(t)\| = \sqrt{x^2(t)} \) leads to \( \dot{z}(t) = \frac{x(t)\dot{x}(t)}{|x(t)|} \). Therefore the fact that \( 1 + x^4(\cdot) > 1 \) and Equation (5) imply that \( D^+ z(t) = \dot{z}(t) \leq -2t^4 z^2(t) + t \).

If \( x(t) = 0 \), then we cannot be sure about the differentiability \( z \) at \( t \). Hence \( z(t) = \|x(t)\| = 0 \). Thus the Fundamental Theorem of Calculus leads to

\[
\frac{z(t+h) - z(t)}{h} = \frac{|x(t+h)|}{h} = \frac{|\int_t^{t+h} \dot{x} (\tau) \, d\tau|}{h}, \forall h > 0.
\]

Therefore, Equation (5) implies that

\[
\frac{z(t+h) - z(t)}{h} = \frac{\left|\int_t^{t+h} (-2\tau^4 (1 + x^4(\tau)) x^3(\tau) + \tau \, d\tau)\right|}{h}, \forall h > 0,
\]

so that

\[
\frac{z(t+h) - z(t)}{h} \leq t + \frac{\left|\int_t^{t+h} (-2\tau^4 (1 + x^4(\tau)) x^3(\tau) + \tau - t \, d\tau)\right|}{h}, \forall h > 0. \tag{7}
\]

Using the fact that \( x(t) = 0 \) and by applying l'Hôpital's rule, one easily verify that

\[
\lim_{h \to 0^+} \frac{\int_t^{t+h} (-2\tau^4 (1 + x^4(\tau)) x^3(\tau) + \tau - t \, d\tau)}{h} = 0.
\]

Thus we deduce by (7) that \( D^+ z(t) \leq t \) so that \( D^+ z(t) (t) \leq -2t^4 z^2(t) + t \) because \( z(t) = 0 \). This completes the proof of the claim. \( \Box \)

It not an easy task to solve the differential equation \( \dot{u}(t) = -2t^4 u^3(t) + t \), for all \( t \in (t_0, \omega) \). As a result, the Comparison lemma is not a good option to
use. However, Theorem 2.1 can be used to prove the attractivity. To see this, observe that one can verify that

$$-2t^4z^3(t) + t < \dot{\phi}(t) \text{ for all } t \in (t_0, \omega) \text{ that satisfy } z(t) = \phi(t) = \frac{1}{t}.$$ 

Thus we conclude by Claim 3.5 that Equation (1) in Theorem 2.1 is satisfied. Therefore, all assumptions of Lemma 3.2 are satisfied. Hence \( \omega = \infty, |x(t)| \leq \phi(t) = \frac{1}{t}, \forall t \geq t_0 \) and \( \lim_{t \to \infty} x(t) = 0 \) whenever \( |x_0| < \phi(t_0) = \frac{1}{2} \). (Observe that \( \lim_{t \to \infty} \phi(t) = 0 \)). This is illustrated in Figure 1 when \( t_0 = 2 \) and \( x_0 = -0.1 \).

![Graph](image.png)

Figure 1: Observe that \( |x(\cdot)| \leq \phi(\cdot) \) and \( \lim_{t \to \infty} x(t) = 0 \).

Conclusions

This paper has dealt with a class of continuous nondifferentiable Lyapunov functions \( V(t, x(t)) \) that enables us to study some stability properties of the class of systems \( \dot{x}(t) = f(t, x(t)) \). We have proved that when the upper right-hand Dini derivative of \( V(t, x(t)) \) is strictly less than the derivative of some \( C^1 \) function \( \phi(t) \) on every intersection point, the function \( V(t, x(t)) \) is less or equal \( \phi(t) \). We have then studied the attractivity and the asymptotic stability of the system under study that satisfies local Carathéodory conditions. An example with a numerical simulation is also mentioned.

References

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