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Estimating Radius of Convergence for Modified Newton Method

Ioannis K. Argyros

Cameron University
Department of Mathematical Science
Lawton, OK, 73505, USA

Ştefan Măruşter

West University of Timişoara
Department of Informatics
B-ul V. Parvan 4, Timişoara, Romania

Abstract

The paper investigate the local convergence and local convergence radius of a variant of modified Newton method for solving nonlinear equations. Two (unlike) algorithms for the estimation of local convergence radius are given. The numerical experiments show that the both proposed algorithms give estimations close or very close, even identic, with the maximum convergence radius.

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1 Introduction

Let $F : X \to Y$ be a nonlinear mapping, where X, Y are Banach spaces (we will consider also the particular case $X = Y = \mathcal{H}$, a real Hilbert space). We will investigate the following order three iterative method for solving the equation $F(x) = 0$:

$$\begin{aligned} y_n &= x_n - \Gamma_n F(x_n), \\ x_{n+1} &= x_n - \Gamma_n (F(x_n) + F(y_n)), \end{aligned} \tag{1.1}$$

where $\Gamma_n = F'(x_n)^{-1}$. In fact this is a modified Newton method in which the derivative is re-evaluated periodically after two steps. Often it is called "Potra-Ptak" method; in the case of single equations, (1.1) was considered by Traub [22] (1982). Using non-discrete induction, Potra and Ptak [16] proved the order three of convergence and given sharp a priori and a posteriori error for (1.1). Note that (1.1) is a particular case of a multipoint iterative processes with order three of convergence considered by Ezquerro and Hernandez [6].

The theoretical investigations and numerical experiments on iteration (1.1) show that it has excellent properties of convergence, stability and index of efficiency. Unfortunately the attraction basin (convergence domain) of this iteration, as commonly occurs for high order methods (see Figure 1 below), is an unpredictable and sophisticated set and therefore finding a convergence ball (or a good starting point) for these methods is a very difficult task.

The problem of estimating the local radius of convergence for different iterative methods was considered by numerous authors and several results were obtained particularly for Newton method and its variants. Among the oldest known results on this topic we could mention those given by Vertgeim, Rall, Rheinboldt, Traub and Wozniakowski, Deuflhard and Potra, Smale [23, 18, 19, 22, 5, 21] and this topic is still being studied extensively. However "... effective, computable estimates for convergence radii are rarely available" [19] (1975). A similar remark was made in more recent paper [10] (2015): "The location of starting approximations, from which the iterative methods converge to a solution of the equation, is a difficult problem to solve". It is worth noticing that the larger the radius of convergence is the more initial points become available (see Remark 4(c) and the numerical examples). Moreover, in some cases no knowledge of the fixed point is needed (see Remark 4(c)). The knowledge of the convergence ball is important for example to estimate the region of asymptotic stability of a nonlinear dynamical system in which the family of trajectories is given by difference equations.

Through the paper $U(p,r)$ will denote an open ball and $\overline{U}(p,r)$ its closure, i.e., $U(p,r) = \{x : \|x-p\| < r\}$ and $\overline{U}(p,r) = \{x : \|x-p\| \le r\}$.

In [10] a procedure (formula) is given to estimate the local convergence radius for Ezquerro-Hernandez method [6]. Suppose that p is a solution of the equation $F(x) = 0$, there exists $F'(p)^{-1}$, $\|F'(p)^{-1}\| \le \beta$ and F' is k-Lipschitz continuous on some $\overline{U}(p, r_0)$. Let $\tilde{r} = \min\{r_0, r\}$, where $r = \zeta_0/[(1+\zeta_0)\beta k]$ and ζ_0 is the positive real root of some polynomial equation of order three (in the particular case of (1.1) this equation is $t^3 + 4t^2 - 8 = 0$). Then $\tilde{r}$ is a local convergence radius of Ezquerro-Hernandez method.

In [2] a simple and elegant formula is proposed to estimate the radius of convergence for Picard iteration and the algorithm presumptively gives a sharp value. More precisely, let $G : D \subset \mathcal{R}^m \to D$ be a nonlinear mapping and x^* a fixed point of G. Suppose that G is differentiable on some ball centred in x^*, $\overline{U}(p, r_1)$, and the derivative of G satisfies

$$\begin{aligned} &\|G'(x^*)\| \leq q < 1, \\ &\|G'(x) - G'(y)\| \leq k\|x - y\|^p, \ \forall x \in \overline{U}(p, r_1). \end{aligned}$$

Define

$$r_2 = \left(\frac{(1+p)(1-q)}{k}\right)^{\frac{1}{p}},$$

then $r = \min\{r_1, r_2\}$ is an estimation of local convergence radius. Note that the sequence given by (1.1) can be generated by the following iteration mapping

$$\begin{aligned} &y = x - \Gamma(x)F(x), \\ &T(x) = x - \Gamma(x)(F(x) + F(y)), \end{aligned} \tag{1.2}$$

where $\Gamma(x) = F'(x)^{-1}$, and the Catinas algorithm can be applied [2].

In this paper we are concerned with the local convergence and local convergence radius of (1.1). We give two (unlike) algorithms to estimate the local convergence radius, the both with positive results on their size. The first one (in the setting of real Hilbert spaces) is based on the convergence properties of the Picard iteration for demicontractive mappings. It provides radii of convergence very close to or even identical with the best ones. The second (in the setting of Banach spaces) allows to investigate the local convergence of (1.1) for more general cases, for example, integral equations, boundary value problems, etc.

2 The first algorithm

2.1 Preliminaries

Let $\mathcal{H}$ be a real Hilbert space with scalar product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$ and C an open subset of $\mathcal{H}$. We recall two basic concepts which are essential for our development, *demicontractivity* and *quasi-expansivity*.

A mapping $T : C \to \mathcal{H}$ is said to be *demicontractive* [13, 8] if the set of fixed points of T is nonempty, $Fix(T) \neq \emptyset$, and

$$\|T(x) - p\|^2 \leq \|x - p\|^2 + L\|x - T(x)\|^2, \ \forall (x, p) \in \mathcal{C} \times Fix(T) \qquad (2.1),$$

where $L > 0$. This condition is equivalent to either of the following two:

$$\langle x - T(x), x - p \rangle \geq \lambda \|x - T(x)\|^2, \ \forall (x, p) \in \mathcal{C} \times Fix(T), \tag{2.1$'$}$$

$$\|T(x) - p\| \leq \|x - p\| + \sqrt{L}\|T(x) - x\|, \ \forall (x, p) \in \mathcal{C} \times Fix(T), \qquad (2.1'')$$

where $\lambda = (1 - L)/2$. Note that (2.1) [or (2.1′)] is often more suitable in Hilbert spaces, allowing easier handling the scalar products and norms. The condition (2.1″) was considered in [17] to prove T-stability of Picard iteration for this class of mappings.

We say that the mapping T is *quasi-expansive* [14] if

$$\|x - p\| \leq \varrho\|x - T(x)\|, \ \forall x \in C, \qquad (2.2)$$

where $\varrho > 0$. If $\varrho < 1$ then $\|x - p\| \leq \frac{\varrho}{1-\varrho}\|T(x) - p\|$ which justifies the terminology. It is also obvious that a mapping T which satisfies (2.2) has a unique fixed point p in C.

Condition (2.2) is similar to the following condition:

$$\|x - T(x)\| \geq \alpha \inf_{p \in Fix(T)} \|x - p\|, \ \forall x \in C,$$

where $0 < \alpha < 1$. This was considered in [20, 15] as an additional condition to prove strong convergence of the Mann iteration for nonexpansive (quasi-nonexpansive) mappings in Banach spaces.

Lemma 2.1 *Let p be a fixed point of T. Suppose that T is Fréchet differentiable at p, that $I - T'(p)$ is invertible, $\delta := \|(I - T'(p))^{-1}\|$ where I is the identity mapping, and that T' is L-Lipschitz continuous on C. Then, for any positive number c satisfying the condition $c\delta < 1$, there holds*

$$\|x - p\| \leq \varrho\|x - T(x)\|, \ \forall x \in \overline{U}(p, r_c),$$

where $r_c = 2c/L$ and $\varrho = \dfrac{\delta}{1-\delta c}$.

Proof.

For any $x \in C$ let R_x be the linear mapping defined by $R_x = T'(p) - \int_0^1 T'(p + t(x - p))dt$. We have for any $x \in \overline{U}(p, r_c)$

$$\begin{aligned} \|R_x\| &\leq \textstyle\int_0^1 \|T'(p) - T'(p + t(x - p))\|dt \\ &\leq L \textstyle\int_0^1 \|t(x - p)\|dt = \frac{1}{2}L\|x - p\| \leq \frac{1}{2}Lr_c = c. \end{aligned}$$

As $c\delta < 1$, from Banach Lemma it results that there exists $(I - T'(p) + R_x)^{-1}$ and

$$\|(I - T'(p) + R_x)^{-1}\| < \frac{\delta}{1 - \delta c}.$$

We have further

$$\begin{aligned} \|x - T(x)\| &= \|(x - p) - (T(x) - T(p))\| \\ &= \| \left(I - \textstyle\int_0^1 T'(p + t(x - p))dt\right)(x - p)\| \\ &= \|(I - T'(p) + R_x)(x - p)\| \geq \|(I - T'(p) + R_x)^{-1}\|^{-1}\|x - p\|, \end{aligned}$$

and

$$\|x-p\| \leq \|(I-T'(p)-R_x)^{-1}\|\|x-T(x)\| < \frac{\delta}{1-\delta c}\|x-T(x)\|, \ \forall x \in \overline{U}(p, r_c). \ \Box$$

2.2 Local convergence

Our approach in studying the convergence of the sequence generated by (1.1) (or by the Picard iteration $x_{n+1} = T(x_n)$, where T is defined by (1.2)) is based on the following Theorem [12]:

Theorem 2.2 *Let $T : C \to \mathcal{H}$ be a (nonlinear) mapping with nonempty set of fixed points, where C is an open subset of a real Hilbert space $\mathcal{H}$. Let p be a fixed point of T and let $r > 0$ be such that $\overline{U}(p, r) \subset C$. Suppose that*
(i) $I - T$ is demiclosed at zero on C,
(ii) T is demicontractive with $\lambda > 0.5$ on $\overline{U}(p, r)$,
then the sequence $\{x_n\}$ given by Picard iteration, $x_{n+1} = T(x_n)$, $x_0 \in \overline{U}(p, r)$ remains in $\overline{U}(p, r)$ and converges weakly to some fixed point of T. If, in addition,
(iii) T is quasi-expansive on $\overline{U}(p, r)$,
then p is the unique fixed point of T in $\overline{U}(p, r)$ and the sequence $\{x_n\}$ converges strongly to p.

Throughout this paper it is assumed that T is Fréchet differentiable on C and that the set of fixed point of T is nonempty, $Fix(T) \neq \emptyset$.

Theorem 2.3 *Suppose that there exists $F'(x)^{-1}$, $\|F'(x)^{-1}\| \leq \beta$, $\forall x \in C$ and that F' is L-Lipschitz continuous on C. Then the set of solutions of the equation $F(x) = 0$ is made of isolated points and the sequence $\{x_n\}$ given by (1.1) converges locally to some solution. The order of convergence is three and the radius of convergence is estimated by $r \leq \alpha/(\beta L)$, where α is the positive solution of the equation $x^3 + 4x^2 - 8\eta$, where $\eta = \sqrt{5} - 2$.*

Proof.
The proof consists in verifying the conditions of Theorem 2.2.
(i) The condition (i) is obviously satisfied.
(ii) Let p be a solution of $F(x) = 0$. We prove then that T is demicontractive with $\lambda > 0.5$ on the ball $\overline{U}(p, r)$.
Using the notations

$$D_x = \int_0^1 F'(p + t(x-p))dt, \ D_y = \int_0^1 F'(p + t(y-p))dt,$$

we have

$$y - p = (x - p) - \Gamma(x)(F(x) - F(p)) = (I - \Gamma(x)D_x)(x - p).$$

We have also that

$$\begin{aligned}\|I-\Gamma(x)D_x\| &= \|\Gamma(x)(F'(x)-D_x)\| \le \beta\|F'(x)-D_x\| \\ &\le \beta \int_0^1 \|F'(x)-F'(p+t(x-p))\|dt \\ &\le \beta L \int_0^1 \|(x-p)-t(x-p)\|dt \\ &= \beta L\|x-p\| \int_0^1 (1-t)dt = \tfrac{1}{2}\beta L\|x-p\| \le \tfrac{1}{2}\beta L r,\end{aligned}$$

and

$$\|y-p\| \le \|I-\Gamma(x)D_x\|\|x-p\| \le \frac{1}{2}\beta L r\|x-p\|.$$

Therefore $\|y-p\| \le \|x-p\| \le r$, sine $\frac{1}{2}\beta L r < \frac{\alpha}{2} < 1$. So $y \in \overline{U}(p,r)$. We can write

$$\begin{aligned}F(x) &= F(x)-F(p) = \left(\int_0^1 F'(p+t(x-p))dt\right)(x-p) = D_x(x-p), \\ F(y) &= F(y)-F(p) = \left(\int_0^1 F'(p+t(y-p))dt\right)(y-p) = D_y(y-p) \\ &= D_y(I-\Gamma(x)D_x)(x-p).\end{aligned}$$

Using $F(x)$ and $F(y)$ in (1.2) we have

$$\begin{aligned}x-T(x) &= \Gamma(x)(D_x(x-p)+D_y(I-\Gamma(x)D_x)(x-p)) \\ &= (\Gamma(x)D_x+\Gamma(x)D_y-\Gamma(x)D_y\Gamma(x)D_x)(x-p) = (I-C_x)(x-p),\end{aligned}$$

where

$$C_x = (I-\Gamma(x)D_y)(I-\Gamma(x)D_x).$$

Now we are searching for a superior bound for $\|C_x\|$ on $\overline{U}(p,r)$. We have

$$\begin{aligned}\|I-\Gamma(x)D_y\| &= \|\Gamma(x)(F'(x)-D_y)\| \le \beta\|F'(x)-D_y\| \\ &\le \beta \int_0^1 \|F'(x)-F'(p+t(y-p))\|dt \\ &\le \beta L \int_0^1 \|(x-p)-t(y-p)\|dt \\ &= \beta L \int_0^1 \|(x-p)-t(I-\Gamma(x)D_x)(x-p)\|dt \\ &\le \beta L\|x-p\| \int_0^1 \|I-t(I-\Gamma(x)D_x)\|dt \\ &\le \beta L\|x-p\|(1+\tfrac{1}{4}\beta L r).\end{aligned}$$

Therefore

$$\|C_x\| \le \frac{1}{2}\beta^2L^2\|x-p\|^2(1+\frac{1}{4}\beta L r) \le \frac{1}{2}\beta^2L^2r^2(1+\frac{1}{4}\beta L r).$$

Let P be the quadratic polynomial

$$P(x) = \frac{1}{2}x^2(1+\frac{1}{4}x)-\eta,$$

where $\eta = \sqrt{5}-2$. The equation $P(x)=0$ (or $x^3+4x^2-8\eta=0$) has a unique positive solution, $\alpha = 0.638106....$ For any $0<x<\alpha$ it results $P(x)<0$. As $\beta L r < \alpha$ we have $\|C_x\| < \eta$.

The rest of the proof follows almost verbatim the proof of Corollary 1 [12]. For completeness we present here a sketch of that proof.

As $0 < \eta < \sqrt{5} - 2$ it results $0.5 < (1-\eta)/(1+\eta)^2 < 1$. Let λ be such that

$$0.5 < \lambda < \frac{1-\eta}{(1+\eta)^2}.$$

For this λ and from $\|C_x\| < \eta$, it obtains

$$1 - \|C_x\| \geq \lambda(1 + \|C_x\|)^2, \ \forall x \in \overline{U}(p, r)$$

For any $\|y\| = 1$ we have

$$\langle (I - C_x)y, y\rangle \geq \lambda \|(I - C_x)y\|^2.$$

Taking $y = (x-p)/\|x-p\|$ and $I - C_x = x - T(x)$ we obtain (2.1′) i.e., T is demicontractive with $\lambda > 0.5$ on $\overline{U}(p, r)$, which is the condition (ii) of Theorem 1.

(iii) Observe that δ from Lemma 1 has in this case the value $\delta = 1$. We can suppose that $\beta > \alpha/2 = 0.319...$ and we can take for c in Lemma 1 a number such that $\alpha/(2\beta) < c < 1$ and then $r \leq \alpha/(\beta L) < 2c/L = r_c$. From Lemma 1 we have that T is quasi-expansive on $\overline{U}(p, r)$.

Thus all conditions of Theorem 1 are satisfied on $\overline{U}(p, r)$. The solution p is the unique solution of $F(x) = 0$ in $\overline{U}(p, r)$ and the sequence $\{x_n\}$ converges in norm to p.

To obtain the order of convergence, observe that $T(x) - p = C(x)(x - p)$ and we have

$$\frac{\|T(x) - p\|}{\|x - p\|^3} \leq \frac{1}{2}\beta^2 L^2 (1 + \frac{1}{4}\beta L r). \ \square$$

Remark 2.4 *The radius proposed by Theorem 2.3 ($r = \alpha/(\beta L)$) is similar with that of Hernandez-Romero ($\tilde{r} = \gamma/(\beta_p L)$ where $\gamma = \zeta/(1+\zeta)$). Although $\alpha > \gamma = 0.552773...$, the two resulting radii are not comparable, since $\beta_p \leq \beta$. For example, for the function $F(x) = 0.2x^3 - 0.1x^2 + x$ and $p = 0$ the two radii are $\tilde{r} = 0.767...$, $r = 0.8111...$, while for the function $F(x) = 0.2x^3 - 0.3x^2 + x$ and $p = 0$ the two radii are $\tilde{r} = 0.4732...$, $r = 0.4674...$.*

2.3 The algorithm

In finite dimensional spaces the condition of quasi-expansivity is superfluous, since the first two conditions of Theorem 2.2 are sufficient for the convergence of the Picard iteration. Therefore, in finite dimensional spaces, supposing that the condition (i) of Theorem 2.2 is fulfilled, we can develop the following algorithm to estimate the local radius of convergence:

Find the largest value for r such that

$$m = \min_{x \in \overline{U}(p,r)} \frac{\langle x - T(x), x - p \rangle}{\|x - T(x)\|^2} \qquad (2.3),$$

and $m > 0.5$.

This procedure involves the following main processing:

1. Apply a line search algorithm (for example of the type half-step algorithm) on the positive real axis to find the largest value for r;
2. At every step of 1 solve the constraint optimization problem (2.3) and verify the condition $m > 0.5$.

The main processing of this algorithm is the solution of the constraint optimization problem (2.3). Therefore we need to use a global constrained optimization method. In our experiences we used a population-based metaheuristic method in combination with a local search method.

3 The second algorithm

We consider now our issue in a more general framework, i.e., $F : C \subset X \to Y$ is a nonlinear operator defined on a open convex subset C of a Banach space X with values in a Banach space Y. With some conditions on F, we can investigate the local convergence of (1.1) for more general cases, for example, integral equations, boundary value problems, etc.

Theorem 3.1 *Let $F : C \subseteq X \to Y$ be a Fréchet-differentiable operator. Suppose:*

There exist $p \in C$ and function $w_0 : [0, +\infty] \to [0, +\infty]$ continuous and nondecreasing such that $F(p) = 0$, $F'(p)^{-1} \in L(Y, X)$ and for each $x \in C$

$$\|F'(p)^{-1}(F'(p) - F'(x))\| \leq w_0(\|x - p\|);$$

Equation

$$w_0(t) = 1$$

has positive solutions. Denote by r_0 the smallest such solution. Define $C_0 = C \cap U(p, r_0)$;

There exist functions $w, v : [0, +\infty] \to [0, +\infty]$ continuous and nondecreasing such that for each $x, y \in C_0$

$$\|F'(p)^{-1}(F'(x) - F'(y))\| \leq w(\|x - y\|)$$

and

$$\|F'(p)^{-1}F'(x)\| \leq v(\|x - p\|);$$

Equation

$$\int_0^1 w((1-\theta)t)d\theta\left[1-w_0(t)+\int_0^1 v\left(\frac{\int_0^1 w((1-\theta)t)td\theta}{1-w_0(t)}q\right)dq\right]-(1-w_0(t))^2=0$$

has solutions in the interval $(0, r_0)$. Denote by r^* the smallest such solution;
$\overline{U}(p, r^*) \subseteq C$.

Then, the sequence $\{x_n\}$ generated for $x_0 \in U(p, r^*) - \{p\}$ by method (1.1) is well defined in $U(p, r^*)$, remains in $U(p, r^*)$ for each $n = 0, 1, 2, \ldots$ and converges to p. Moreover, the following estimate holds

$$\|x_{n+1} - p\| \leq g(\|x_n - p\|)\|x_n - p\| \leq \|x_n - p\| < r^*, \tag{3.1}$$

where

$$g(t) = \frac{\int_0^1 w((1-\theta)t)d\theta}{1-w_0(t)}\left[1+\frac{\int_0^1 v\left(\frac{\int_0^1 w((1-\theta)t)d\theta t}{1-w_0(t)}q\right)dq}{1-w_0(t)}\right].$$

Furthermore, if there exists $r_1 \geq r^*$ such that

$$\int_0^1 w_0(\theta r_1)d\theta < 1,$$

then, the point p is the only solution of equation $F(x) = 0$ in $C_1 = C \cap \overline{U}(p, r_1)$.

Proof.

We shall show using mathematical induction that the sequence $\{x_n\}$ is well defined and converges to p so that (3.1) is satisfied. Let $x \in U(p, r^*)$. By the definition of r_0, r^* we get

$$\|F'(p)^{-1}(F'(p) - F'(x))\| \leq w_0(\|x - p\|) \leq w_0(r^*) < w_0(r_0) = 1. \tag{3.2}$$

It follows from (3.2) and the Banach lemma on invertible operators that $F'(x)^{-1} \in L(Y, X)$. In particular $F'(x_0)^{-1} \in L(Y, X)$, so x_1 is well defined from method (1.1) with $n = 0$. We also have by (3.2) that

$$\|F'(x)^{-1}F'(p)\| \leq \frac{1}{1 - w_0(\|x - p\|)}. \tag{3.3}$$

We can write for $z = x - F'(x)^{-1}F(x)$

$$F(z) = F(z) - F(p) = \int_0^1 F'(p + \theta(z - p))(z - p)d\theta. \tag{3.4}$$

Notice that

$$\|p + \theta(z - p) - p\| = \theta\|z - p\| \leq \|z - p\|$$

$$\leq \|F'(x)^{-1}F'(p)\|\|\int_0^1 F'(p)^{-1}(F'(p+\theta(x-p))-F'(x))(x-p)d\theta\| \quad (3.5)$$
$$\leq \frac{\int_0^1 w((1-\theta)\|x-p\|)d\theta}{1-w_0(\|x-p\|)}\|x-p\| \leq \|x-p\| < r^*,$$

so $p+\theta(z-p) \in U(p,r^*)$ for each $\theta \in [0,1]$. Then by (3.4) and (3.5), we get that

$$\begin{aligned} \|F'(p)^{-1}F(z)\| &\leq \int_0^1 v(q\|z-p\|)dq \\ &\leq \int_0^1 v\left(q\frac{\int_0^1 w((1-\theta)\|z-p\|)\|z-p\|}{1-w_0(\|z-p\|)}\right)dq\|z-p\|. \end{aligned} \quad (3.6)$$

Using $(*)$ for $z = x_k - F'(x_k)^{-1}F(x_k)$, (3.3) and (3.6), we get in turn that

$$\begin{aligned} \|x_{k+1}-p\| &\leq \|z-p\| + \|F'(x_k)^{-1}F'(p)\|\|F'(p)^{-1}F(z)\| \\ &\leq g(\|x_k-p\|)\|x_k-p\| \leq \|x_k-p\| < r^* \end{aligned} \quad (3.7)$$

by the definition on r^*, which shows (3.1) and $x_{k+1} \in U(p,r^*)$. Then, from the estimate $\|x_{k+1}-p\| \leq c\|x_k-p\| < r^*$, $c = g(\|x_0-p\|) \in [0,1)$, we deduce that $\lim_{k\to\infty} = p$ and $x_{k+1} \in U(p,r^*)$. Finally, to show the uniqueness part, let $p^* \in C_1$ with $F(p^*) = 0$. Define operator $Q = \int_0^1 F'(p+\theta(p^*-p))d\theta$. Then, we have that

$$\|F'(p)^{-1}(Q-F'(p))\| \leq \int_0^1 w_0(\theta\|p-p^*\|)d\theta \leq \int_0^1 w_0(\theta r_1) < 1.$$

Hence, $Q^{-1} \in L(Y,X)$. Then, from identity $0 = F(p^*) - F(p) = Q(p^*-p)$, we conclude that $p = p^*$. $\square$

Next, we present two more variations of Theorem 3.1 based on the identity

$$\begin{aligned} x_{k+1}-p &= y_k - p - \Gamma_k F'(y_k) = y_k - p - \Gamma_k(F(y_k)-F(p)) \\ &= [I - \Gamma_k \int_0^1 F'(p+\theta(y_k-p))](y_k-p)d\theta \\ &= \Gamma_k[F'(x_k) - \int_0^1 F'(p+\theta(y_k-p))](y_k-p)d\theta. \end{aligned} \quad (3.8)$$

Proposition 3.2 *Let F, p, r_0, C_0, w_0, w and v be as in Theorem 1. Suppose: Equation*

$$\int_0^1 w((1-\theta)t)d\theta w\left[\frac{\frac{1}{2}\int_0^1 w((1-q)t)dq + \int_0^1 v(qt)dq}{1-w_0(t)}t\right] - (1-w_0(t))^2 = 0$$

has solutions in the interval $(0,r_0)$. Denote by r_0^* the smallest such solutin.
$\overline{U}(p,r_0^*) \subseteq C$.

Then, the sequence $\{x_n\}$ generated for $x_0 \in U(p,r_0^*) - \{p\}$ by the method (1.1) is well defined in $U(p,r_0^*)$, remains in $U(p,r_0^*)$ for each $n = 0,1,2,...$ and converge to p. Moreover, the following estimate holds

$$\|x_{n+1}-p\| \leq \frac{\int_0^1 w((1-\theta)\|x_k-p\|)d\theta\|x_k-p\|}{(1-w_0(\|x_k-p\|))^2}.$$

$$w\left[\frac{\frac{1}{2}\int_0^1 w((1-q)\|x_k-p\|)dq+\int_0^1 v(q\|x_k-p\|)dq}{1-w_0(\|x_k-p\|)}\|x_k-p\|\right]$$
$$\leq g_0(\|x_n-p\|)\|x_n-p\| \leq \|x_n-p\| < r_0^*, \tag{3.9}$$

where

$$g_0(t)=\frac{\int_0^1 w((1-\theta)t)d\theta}{(1-w_0(t))^2}w\left[\frac{\frac{1}{2}\int_0^1 w((1-q)t)dq+\int_0^1 v(qt)dq}{1-w_0(t)}t\right].$$

Furthermore, if there exists $r_2 \geq r_1^*$ *such that*

$$\int_0^1 w_0(\theta r_2)d\theta < 1,$$

then, the point p *is the only solution of equation* $F(x)=0$ *in* $C_2 = C \cap \overline{U}(p, r_2)$.

Proof.
Using (3.8) and following the proof of Theorem 3.1, we obtain in turn that

$$\|x_{k+1}-p\| \leq \|F'(x_k)^{-1}F'(p)\| \cdot$$
$$\|\int_0^1 F'(p)^{-1}[F'(x_k)-F'(p+\theta(x_k-p+F'(x_k)^{-1}F(x_k))]\| \cdot$$
$$\|F'(x_k)^{-1}F'(p)\|\|F'(p)^{-1}[F'(x_k)(x_k-p)-F(x_k)]\|$$
$$\leq \frac{\int_0^1 w((1-\theta)\|x_k-p\|)d\theta\|x_k-p\|}{(1-w_0(\|x_k-p\|))^2}\cdot$$
$$\int_0^1 w(|1-\theta|\|x_k-p-F'(x_k)^{-1}F(x_k\|+\|F'(x_k)^{-1}F(x_k)\|)d\theta$$
$$\leq g_0(\|x_n-p\|)\|x_n-p\| \leq \|x_n-p\| < r_0^*. \ \square$$

It turns out that the conditions on the first derivative and function v can be dropped as follows:

Proposition 3.3 *Let* F, p, r_0, C_0, w_0 *and* w *be as in Theorem 3.1. Suppose: Equation*

$$\int_0^1 w((1-\theta)t)d\theta\left(w_0(t)+\int_0^1 w_0\left(\theta\frac{\int_0^1 w((1-q)t)dqt}{1-w_0(t)}\right)d\theta\right)-(1-w_0(t))^2=0$$

has solutions in the interval $(0, r_0)$. *Denote by* r_1^* *the smallest such solution.*
$\overline{U}(p, r_1^*) \subset C$.

Then, the sequence $\{x_n\}$ *generated for* $x_0 \in U(p, r_1^*) - \{p\}$ *by the method* (1.1) *is well defined in* $U(p, r_1^*)$, *remains in* $U(p, r_1^*)$ *for each* $n = 0, 1, 2, \ldots$ *and converge to* p. *Moreover, the following estimate holds*

$$\|x_{n+1} - p\| \le g_1(\|x_n - p\|)\|x_n - p\| \le \|x_n - p\| < r_1^*,$$

where

$$g_1(t) = \frac{\int_0^1 w((1-\theta)t)d\theta}{(1-w_0(t))^2}\left[w_0(t) + \int_0^1 w_0\left(\theta\frac{\int_0^1 w((1-q)t)dqt}{1-w_0(t)}\right)d\theta\right].$$

Furthermore, if there exists $r_3 \ge r_1^*$ *such that*

$$\int_0^1 w_0(\theta r_3)d\theta < 1,$$

then the point p *is the only solution of equation* $F(x) = 0$ *in* $C_3 = C \cap \overline{U}(p, r_3)$.

Proof.

Using (3.8),(3.9) and following the proof of Proposition 3.2, we get in turn instead of (3.9) that

$$\begin{aligned}\|x_{k+1} - p\| &\le \|F'(x_k)^{-1}F'(p)\|^2\|F'(p)^{-1}[F'(x_k)(x_k - p) - F(x_k)]\|\cdot \\ &\cdot\|F'(p)^{-1}(F'(x_k) - F'(p))\| + \\ &+\|\int_0^1 F'(p)^{-1}[F'(p) - F'(p + \theta(x_k + p - F'(x_k)^{-1}F(x_k))]d\theta\|] \\ &\le \frac{\int_0^1 w((1-\theta)\|x_k - p\|)d\theta\|x_k - p\|}{(1 - w_0(\|x_k - p\|))^2}\cdot \\ &\cdot\left[w_0(\|x_k - p\|) + \int_0^1 w_0\left(\theta\frac{\int_0^1 w((1-q)\|x_k - p\|)dq\|x_k - p\|}{1 - w_0(\|x_k - p\|)}\right)d\theta\right] \\ &= g_1(\|x_k - p\|)\|x_k = p\| \le \|x_k - p\| < r_1^*. \ \square\end{aligned}$$

Remark 3.4 *In practice after computing* $r^*, r_1^*, r_2^*, g, g_0$ *and* g_1, *we shall use the optimum choice.*

Remark 3.5 *(a) It follows from the condition on* w_0 *that condition on* v *can be dropped and be replaced by*

$$v(t) = 1 + w_0(t) \text{ or } v(t) = w_0(r_0)$$

since

$$\|F'(p)^{-1}[(F(x) - F(p)) + F'(p)\| = 1 + \|F'(p)^{-1}(F'(x) - F'(p))\|$$

$$\leq 1 + w_0(\|x - p\|) = 1 + w_0(t0)$$

for $\|x - p\| \leq r_0$.

(b) If the function w_o *is strictly increasing, then we can choose*

$$r_0 = w_0^{-1}(1)$$

instead of $w_0(t) = 1$.

(c) If w_0, w, v *are constant functions, then*

$$r_1 = \frac{2}{2w_0 + w} \text{ and } r < r_1.$$

Therefore, the radius of convergence r *cannot be larger than the radius of convergence* r_1 *for Newton's method*

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n).$$

Notice also that the earlier radius of convergence given independently by Rheinboldt [19] and Traub [22] is

$$r_{RT} = \frac{2}{3w_1}$$

and Argyros [1]

$$r_A = \frac{2}{2w_0 + w_1},$$

where w_1 *is the Lipschitz constant on* C. *But, we have*

$$w \leq w_1, \; w_0 \leq w_1,$$

so

$$r_{RT} \leq r_A \leq r_1$$

and

$$\frac{r_{RT}}{r_A} \to \frac{1}{3} \text{ and } \frac{w_0}{w} \to 0.$$

The radius of convergence a_1 *used in [3] is smaller than* r_{DS} *given by Dennis and Schnabel [4],*

$$a_1 < r_{DS} = \frac{1}{2w_1} < r_{RT}.$$

However, a_1 can not be computed using the Lipschitz constants.

(d) We have only used hypotheses on the first-order derivative of F. The order of convergence can be determined by using the following formula for the computational order of convergence (COC) given by

$$\xi = \frac{ln\left\|\frac{x_{n+2}-p}{x_{n+1}-p}\right\|}{ln\left\|\frac{x_{n+1}-p}{x_{n}-p}\right\|},\ for\ each\ n = 1, 2, ...$$

or the approximate computational order order of convergence (ACOC) [11], given by

$$\xi^* = \frac{ln\left\|\frac{x_{n+2}-x_{n+1}}{x_{n+1}-x_n}\right\|}{ln\left\|\frac{x_{n+1}-x_n}{x_{n}-x_{n-1}}\right\|},\ for\ each\ n = 1, 2, ...$$

that do not require derivatives. Notice also that ξ^* does not even require the knowledge of exact root p.

(e) The results obtained here can be used for operator F satisfying the autonomous differential equation [1] of the form

$$F'(x) = P(F(x)),$$

where P is a known continuous operators. Since $F'(p) = P(F(p)) = P(0)$, we can apply the results without actually knowing the solution p. Let as an example $F(x) = e^x - 1$. Then, we can choose $P(x) = x + 1$.

4 Numerical experiments

This section is devoted to numerical experiments in order to evaluate the efficiency of the proposed procedures.

4.1 Algorithm 1

Experiment 1

We have computed the local radius of convergence using the Algorithm 1 for the method (1.1) and for a number of real functions. In the most these examples the estimated radii are close to (or even coincide with) the maximum radii. For example, in the case of the function $f(x) = x^5 - 2x^2 + x$ and $p = 1$ the estimate and the best radius (computed with 15 decimal digits) are identical, $r = 0.080959069788847$.

Experiment 2

We applied Algorithm 1 to estimate the local radius of convergence for the method (1.1) and for a number of mappings in several variables. For the following three test mappings (we will refer to them as Example 1-3):

$$F_1(x) = \begin{pmatrix} x_1 - x_1^2 + 0.2x_2^3 \\ 0.3x_1^2 + x_1x_2 + 0.5x_1 \end{pmatrix},$$

$$F_2(x) = \begin{pmatrix} x_1 + x_1x_2 - sin(x_2) \\ x_1^4 + 0.6x_2 \end{pmatrix}, \; F_3(x) = \begin{pmatrix} x_1x_2^3 - x_1 + 2x_2^2 \\ x_1^2 + sin(x_2 \end{pmatrix},$$

the results are given in Figure 1.

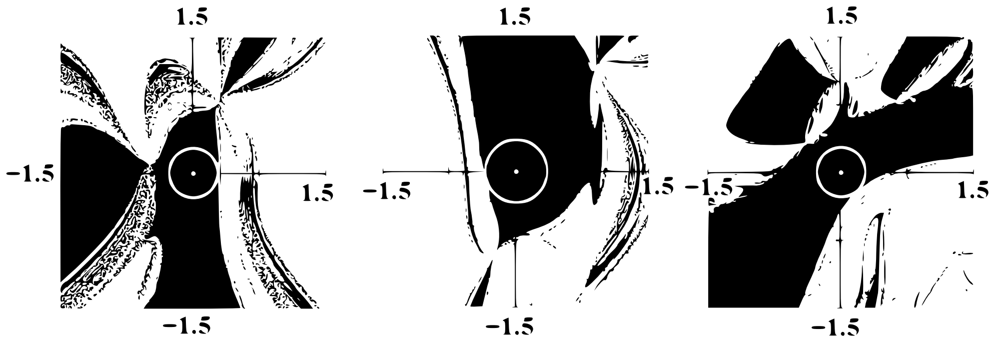

Figure 1: Attraction basins and convergence ball obtained by first algorithm.

The black areas represent the domain of convergence corresponding to the fixed point $p = (0,0)^T$ (for all three examples) and the white circles the local convergence balls. It can be seen that the estimates are satisfactory close to the best possible ones.

Experiment 3

Suppose that the motion of an object in three dimensions is governed by the system of differential equations

$$\begin{aligned} & f_1'(x) - f_1(x) = 0, \\ & f_2'(y) - (e-1)y - 1 = 0, \\ & f_3'(z) - 1 = 0, \end{aligned}$$

with $x, y, z \in C$ for $f_1(0) = f_2(0) = f_3(0) = 0$. Then, the solution $\Gamma = (0,0,0)^T$ of the system is given for $w = (x,y,z)^T$ by considering the function $F := (f_1, f_2, f_3) : C \to \mathcal{R}^3$ defined by

$$F(v) = \left(e^x - 1, \frac{e-1}{2}y^2 + y, z\right)^T.$$

The Fréchet derivative is given by

$$F'(v) = \begin{bmatrix} e & 0 & 0 \\ 0 \; (e-1 &)y + 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The estimate radius is close to the maximum convergence radius; the six decimal digits of the two radii are identical, $r = 0.321708$.

4.2 Algorithm 2

Example 1. Let $X = Y = C[0,1]$ and consider the nonlinear integral equationof the mixed Hammerstein-type [7, 9] defined by

$$x(s) = \int_0^1 H(s,t)\left(x(t)^{3/2} + \frac{x(t)^2}{2}\right) dt,$$

where the kernel H is the Green's function defined on the interval $[0,1]\times[0,1]$ by

$$H(s,t) = \begin{cases} (1-s)t, & t \le s, \\ s(1-t), & s \le t. \end{cases}$$

The solution $\Gamma(s) = 0$ is same as the solution of the equation where $F : C[0,1] \to C[0,1]$ is defined by

$$F(x)(s) = x(s) - \int_0^1 H(s,t)\left(x(t)^{3/2} + \frac{x(t)^2}{2}\right) dt = 0.$$

Notice that

$$\left\|\int_0^1 H(s,t)dt\right\| \le \frac{1}{8}.$$

Then, we have that

$$F'(x)y(s) = y(s) - \int_0^1 H(s,t)\left(x(t)^{3/2} + \frac{x(t)^2}{2}\right) dt,$$

since $F'(\Gamma(s)) = I$, so

$$\|F'(x^*)^{-1}(F'(x) - F'(y))\| \le \frac{1}{8}\left(\frac{3}{2}\|x-y\|^{1/2} + \|x-y\|\right).$$

Therefore, we can choose

$$w_0(t) = w(t) = \frac{1}{8}\left(\frac{3}{2}t^{1/2} + 1\right)$$

and by Remark 2

$$v(t) = 1 + w_0(t).$$

The results in [6] can not be used to solve this problem, since F' is not Lipschitz. However, our results can apply. Indeed, using the above choices of functions

v, w_0, w, we get $r^* = 1.300124302$, $r_0^* = 1.776107913$, $r_1^* = 2.037193935$. Hence, we choose $r^* = r_0^* = r_1^* = 1$ since we work on $C[0,1]$.

Example 2. Identical with Example 3, Experiment 2.

(a) Let $C = \overline{U}(0.1)$. Then, we have that $w_0 = (e-1)t,\ w(t) = e^{\frac{1}{e-1}}t,\ v(t) = \frac{1}{e-1}$ and $w_= e^{\frac{1}{e-1}}$ and $r^* = 0.198328385,\ r_0^* = 0.248557761,\ r_1^* = 0.301877345$

(b)Let $D = \overline{U}(0.0.5)$. Chose $w_0(t) = 2(\sqrt{e}-1)t,\ w(t) = \sqrt{e}t,\ v(t) = \sqrt{e}$ and then we have $r^* = 0.169662849,\ r_0^* = 0.204989342,\ r_1^* = 0.234425291$.

Example 3. Let $X = Y = C[0,1]$, be the space of continuous functions defined on the interval $[0,1]$ and be equipped with max norm. Let $C = \overline{U}(0,1)$. Define

$$F(\varphi)(x) = \varphi(x) - 5\int_0^1 x\theta\varphi(\theta)^3 d\theta.$$

We have that $\Gamma(x) = 0$ and

$$F'(\varphi(\xi))(x) = \xi(x) - 15\int_0^1 x\theta\varphi(\theta)^2\xi(\theta)d\theta,\ for\ each\ \xi \in C.$$

Thus we have $w_0(t) = 7.5t,\ w(t) = 15t,\ v(t) = 2$. We obtain $r^* = 0.029229812,\ r_0^* = 0.038572676,\ r_1^* = 0.060841465$.

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