A Splitting Technique for Superposition Type Solutions of Cubic Nonlinear Ordinary Differential Equations

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Abstract

This paper deals with finding new solutions of nonlinear ordinary differential equations with cubic nonlinearity. Few theorems comprising existence of superposition of two Jacobian elliptic functions as its solutions have been proved. Results of these theorems have been used to find some new solutions of the differential equation mentioned above. To exhibit the utility of the new solutions obtained here few of these results have been used to study the new state of motion of quartic oscillator problem. It is observed that the system may have oscillatory solution with positive energy in the neighbourhood of minima of the potential (double) well beyond the oscillation over entire span of the well. The appearance of new solution indicates the existence of more constant of the motion to be explored beyond the Hamiltonian of the system.

Mathematics Subject Classification: 34C15, 34C60, 35Q99, 34L30, 35C05, 35C07

Keywords: Superposition-type solution, Nonlinear ordinary differential equations, Quartic oscillator problem

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1 Introduction

The second order ordinary differential equations (ODE) with various nonlinear terms are playing important role in our understanding of fast challenging world and in the development of modern technologies. In particular, the ODE

\[ \phi''(\xi) + a \phi(\xi) + b \phi^3(\xi) = 0 \]  

with cubic non-linearity appears in mathematical analysis of physical processes in diverse field of science and engineering [1, 2, 6, 7, 8, 9]. The exact solution of Eq.(1) have been found in various form viz., trigonometric, hyperbolic, exponential functions depending on the initial/boundary condition on \( \phi \). However all such solutions can be brought under single roof of twelve Jacobian elliptic functions (JEF) [1, 6, 7, 9]. It is observed recently by Khare and Saxena [3, 4, 5] that apart from the solutions involving single JEF, Eq.(1) admits superposition-type solution (STSol) of two JEFs. Their observation was surprising due to the fact that appearance of STSol violates the classical idea of the lack of superposition principle of the solutions of non-linear ordinary/partial differential equations. In their works [3, 4, 5] it has been pointed out that their finding seem to be mysterious. No rigorous proof of existence of such STSols have been found in the literature.

The objective of this work is to present a proof of existence of STSols and provide a set of new (STSols) solutions of Eq.(1) by using the results of the existence theorem provided here. The results of the theorems proved here disclose the fact that the elements of the STSols are not solutions of Eq.(1) individually. Instead, those are solutions of a system of coupled nonlinear ordinary differential equations each of which are different from Eq.(1). It thus resolves apparent conflict of the classical concept of the non-existence of superposition principle of solutions of nonlinear ordinary or partial differential equations and appearance of their STSols.

The organization of the paper is as follows. The exact solutions of Eq.(1) in terms of individual twelve JEF now available have been summarized in section 2. The dependence of their behaviour (boundedness) on the modulus of JEFs have also been presented there for their use in the subsequent section. Some theorems on existence of STSol of Eq.(1) have been discussed in section 3. Results of sections 3 have been used to obtain some STSols of Eq.(1). Most of these solutions seem to be new. These new solutions and dependence of their behaviour on moduli of JEFs/parameters \( a, b \) involved have been summarized in section 4. Few of these new solutions have been used to study quartic oscillator (QO) problem. Some new solutions of QO and their implications have been discussed in section 5. Our findings have been concluded in section 6.
2 Solutions of Eq.(1)

Solutions of cubic nonlinear ordinary differential equation in terms of twelve JEFs with their moduli \( m \) and amplitude \( A \) as functions of parameters \( a \) and \( b \) involved in Eq.(1) have been presented in Table 1 for their easy access in the subsequent part of the paper. From the data presented in Table 1 it appears that the amplitude \( A \) (column 4) of the solutions (JEF) depends on both the parameters \( a, b \) of Eq. (1) in general, while the modulus \( m \) (column 5) depends solely on the parameter \( a \). On the other hand, the dependence of parameter \( a \) of Eq. (1) on the moduli \( m \) for different JEFs are given in column two. From the relationships between modulus \( m \) of the JEF solution and parameter \( a \) for different JEFs presented columns two and five of Table 1 it appears that a particular JEF with two different moduli are solutions of two different cubic nonlinear equations of the form Eq. (1) (differ through parameters \( a \) and \( b \)). The dependence of nature of the solution (bounded- or unboundedness) on the moduli/parameter \( a \) have also been presented in last two columns of Table-1. These information will be useful in the analysis of behaviour of new solutions to be determined in section 4 by using the result of the following theorems.

Table 1: Dependence of amplitude, modulus and boundedness property of JEF solution on parameters \( a, b \) of Eq.(1) and vice versa

<table>
<thead>
<tr>
<th>JEF</th>
<th>( a, b ) in Eq.(1)</th>
<th>( A )</th>
<th>( m )</th>
<th>Nature</th>
</tr>
</thead>
<tbody>
<tr>
<td>( cn )</td>
<td>( 1 - 2m^2 )</td>
<td>( b )</td>
<td>( \pm \sqrt{\frac{1 + a}{2}} )</td>
<td>( \pm \sqrt{\frac{1 - a}{2}} )</td>
</tr>
<tr>
<td>( dn )</td>
<td>( m^2 - 2 )</td>
<td>( b )</td>
<td>( \pm \sqrt{\frac{2}{2}} )</td>
<td>( \pm \sqrt{a + 2} )</td>
</tr>
<tr>
<td>( sn )</td>
<td>( m^2 + 1 )</td>
<td>( b )</td>
<td>( \pm \sqrt{\frac{2 - a}{2}} )</td>
<td>( \pm \sqrt{a - 1} )</td>
</tr>
<tr>
<td>( cs )</td>
<td>( m^2 - 2 )</td>
<td>( b )</td>
<td>( \pm \sqrt{-\frac{2}{2}} )</td>
<td>( \pm \sqrt{a + 2} )</td>
</tr>
<tr>
<td>( ds )</td>
<td>( 1 - 2m^2 )</td>
<td>( b )</td>
<td>( \pm \sqrt{-\frac{2}{2}} )</td>
<td>( \pm \sqrt{\frac{1 - a}{2}} )</td>
</tr>
<tr>
<td>( ns )</td>
<td>( m^2 + 1 )</td>
<td>( b )</td>
<td>( \pm \sqrt{\frac{2 - a}{2}} )</td>
<td>( \pm \sqrt{a - 1} )</td>
</tr>
<tr>
<td>( cd )</td>
<td>( m^2 + 1 )</td>
<td>( b )</td>
<td>( \pm \sqrt{\frac{2 - a}{2}} )</td>
<td>( \pm \sqrt{a - 1} )</td>
</tr>
<tr>
<td>( nd )</td>
<td>( m^2 - 2 )</td>
<td>( b )</td>
<td>( \pm \sqrt{\frac{2(a + 1)}{2}} )</td>
<td>( \pm \sqrt{a + 2} )</td>
</tr>
<tr>
<td>( sd )</td>
<td>( 1 - 2m^2 )</td>
<td>( b )</td>
<td>( \pm \sqrt{\frac{1 - a}{2}} )</td>
<td>( \pm \sqrt{\frac{1 - a}{2}} )</td>
</tr>
<tr>
<td>( sc )</td>
<td>( m^2 - 2 )</td>
<td>( b )</td>
<td>( \pm \sqrt{\frac{2(1 + a)}{2}} )</td>
<td>( \pm \sqrt{a + 2} )</td>
</tr>
<tr>
<td>( dc )</td>
<td>( m^2 + 1 )</td>
<td>( b )</td>
<td>( \pm \sqrt{\frac{2 - a}{2}} )</td>
<td>( \pm \sqrt{a - 1} )</td>
</tr>
<tr>
<td>( nc )</td>
<td>( 1 - 2m^2 )</td>
<td>( b )</td>
<td>( \pm \sqrt{\frac{1 - a}{2}} )</td>
<td>( \pm \sqrt{\frac{1 - a}{2}} )</td>
</tr>
</tbody>
</table>

3 Existence theorems

Theorem 3.1 If \( \phi_1(\xi), \phi_2(\xi) \) be two solutions of Eq.(1), their linear superposition

\[
\phi = \alpha_1 \phi_1 + \alpha_2 \phi_2
\]
will be a solution of Eq.(1) provided the coefficients $\alpha_1$, $\alpha_2$ and functions $\phi_1$, $\phi_2$ satisfy the algebraic or transcendental equation

$$\alpha_1 (\alpha_1^2 - 1) \phi_1^3 + \alpha_2 (\alpha_2^2 - 1) \phi_2^3 + 3 \alpha_1 \alpha_2 \phi_1 \phi_2 \phi = 0. \quad (3)$$

**Proof.** Since $\phi_1$, $\phi_2$ be two solutions of Eq.(1),

$$\phi_i''(\xi) + a_i \phi_i(\xi) + b_i \phi_i^3 = 0, \quad i = 1, 2. \quad (4)$$

Use of Eq.(2) in Eq.(1) gives

$$\alpha_1 \phi_1'' + a \alpha_1 \phi_1 + \alpha_1^3 b \phi_1^3 + \alpha_2 \phi_2'' + a \alpha_2 \phi_2 + \alpha_2^3 b \phi_2^3 + 3 \alpha_1 \alpha_2 b \phi_1 \phi_2 (\alpha_1 \phi_1 + \alpha_2 \phi_2) = 0 \quad (5)$$

This equation can be recast into the form

$$\alpha_1 \{\phi_1'' + a \phi_1 + b \phi_1^3\} + \alpha_2 \{\phi_2'' + a \phi_2 + b \phi_2^3\} + \alpha_1 b (\alpha_1^2 - 1) \phi_1^3 + \alpha_2 b (\alpha_2^2 - 1) \phi_2^3 + 3 \alpha_1 \alpha_2 b \phi_1 \phi_2 \phi = 0.$$

Use of Eqs. in (4) in the last equation leads to Eq.(3). This establishes Theorem 3.1.

**Corollary 3.2** If $\phi_1(\xi)$ and $\phi_2(\xi)$ are two solutions of equation Eq.(1), their sum

$$\phi = \phi_1 + \phi_2 \quad (6)$$

will never be a non-trivial solution of the same Eq.(1).

**Proof.** Comparing Eq.(6) and Eq.(2) we get $\alpha_1 = \alpha_2 = 1$. Use of these values of $\alpha_1$ and $\alpha_2$ in condition (3) gives $3 \phi_1 \phi_2 \phi = 0$. Since $\phi_1$, $\phi_2$ are non zero, above equation reduces to $\phi = 0$. This establishes the non-existence of non-trivial STSol (6) of Eq.(1).

From the results of Theorem 3.1 and its corollary it seems that superposition of solution as it appears in case of linear ODEs is not possible in case of nonlinear Eq.(1). However in the following theorem we will see that it is possible to find superposition-type solution of Eq.(1).

**Theorem 3.3** A linear superposition

$$\phi = \alpha_1 \phi_1 + \alpha_2 \phi_2 \quad (7)$$
A splitting technique for superposition type solutions of \( \phi_1(\xi) \) and \( \phi_2(\xi) \) (none of these are solution of Eq."\( (1) \)) will be a non trivial solution of Eq."\( (1) \) if \( \phi_1 \) and \( \phi_2 \) satisfy a system of non-linear ODEs

\[
\begin{align*}
\phi_1'' + a \phi_1 + b \phi_1 (\alpha_1^2 \phi_1^2 + 3 \alpha_2^2 \phi_2^2) &= 0 \quad (8a) \\
\phi_2'' + a \phi_2 + b \phi_2 (\alpha_2^2 \phi_2^2 + 3 \alpha_1^2 \phi_1^2) &= 0. \quad (8b)
\end{align*}
\]

or,

\[
\begin{align*}
\phi_1'' + a \phi_1 + b \alpha_1 \phi_1 (\alpha_1 \phi_1^2 + 3 \alpha_2 \phi_1 \phi_2) &= 0 \quad (9a) \\
\phi_2'' + a \phi_2 + b \alpha_2 \phi_2 (\alpha_2 \phi_2^2 + 3 \alpha_1 \phi_1 \phi_2) &= 0. \quad (9b)
\end{align*}
\]

**Proof** Substitution of (7) into the Eq."\( (1) \) leads to

\[
\alpha_1 \phi_1'' + a \alpha_1 \phi_1 + \alpha_3^2 b \phi_1^2 + \alpha_2 \phi_2'' + a \alpha_2 \phi_2 + \alpha_3^2 b \phi_2^2 + 3 \alpha_1 \alpha_2 b \phi_1 \phi_2 (\alpha_1 \phi_1 + \alpha_2 \phi_2) = 0. \quad (10)
\]

This single equation involves two unknown functions \( \phi_1(x) \) and \( \phi_2(x) \). To obtain these two unknown functions splitting of Eq."\( (10) \) is desirable. One can split this equation in many ways. However we opt here to split Eq."\( (10) \) into two in such a way that the resulting system of equations remain symmetric under exchange of \( 1 \leftrightarrow 2 \). This leads us to get system of coupled non-linear ordinary differential equations (8a,b) or (9a,b). This proves the theorem.

The results of this theorem will now be used to find new solutions of the form (7) of Eq."\( (1) \).

### 4 New Solutions

It is demonstrated here that by using solutions of system of coupled equations (8a,b) or (9a,b), it is possible to get several new (STSols) solutions of Eq."\( (1) \) which are different from solutions presented in Table 1 of section 2. As in the case of solutions involving single JEF, the new solutions are also periodic. Some of these solutions are unbounded and rest are bounded. The initial step of getting new STSols of Eq."\( (1) \) is finding solutions of system of coupled equations derived in Theorem 3.2. To solve Eqs."\( (8a)-(8b) \) we have selected various combinations among twelve JEFs as \( \phi_1(\xi) \) and \( \phi_2(\xi) \) a priori with unknown amplitude and modulus. Their substitution into the equations followed by the use of properties of JEFs provide a system of algebraic equations involving unknown amplitudes and moduli of JEFs, the parameters (data) \( a, b \) involved in the equation (1) and \( \alpha_1, \alpha_2 \) appearing in the STSols (7). Solutions of derived algebraic equations give the admissible amplitudes and modulus in terms of \( a, b, \alpha_1, \alpha_2 \). Substitution of their values into the JEFs selected initially for \( \phi_1 \) and \( \phi_2 \) followed by their use in (7) provides the new (STSols) solution of Eq."\( (1) \). Possible choices of \( \phi_1 \) and \( \phi_2 \) among twelve JEFs, dependence of their amplitudes and moduli on parameters \( a, b \) and vice versa, explicit form of the
new solutions have been presented in Table 2.

The results of Table 1 have been used to get information on the boundedness of the new solutions and have been presented in last four columns of the same table. A close observation in the new solutions presented in Table 2 reveals that elements JEFs appear as \( \phi_1(x) \) and \( \phi_2(x) \) of solutions (7) are of distinct nature viz. \( cn \), \( dn \) or \( nd \), \( sd \) so on. None of them are same JEF and its reciprocal. However it is interesting to note that such combinations satisfy the system of coupled equations (9a,b) with appropriate modulus and amplitudes. We have used the similar steps followed in getting solutions of system of Eqs. (8a,b) to obtain solutions of Eqs. (9a,b). The new solutions are presented in Table 3.

5 Application: quartic oscillator

Here we will explore the existence of new state of motion of the quartic oscillator

\[
\ddot{x}(t) - x(t) + \beta x^3(t) = 0
\]

(11)

beyond the states corresponding to the classical solutions (available in the literature [6, 9] and references there in). This equation may be regarded as the Euler-Lagrange equation with the Lagrangian \( L(x, \dot{x}) = \frac{1}{2} \dot{x}^2 - V(x) \) involved with the double well potential

\[
V(x) = -\frac{1}{2}x^2 + \frac{\beta}{4}x^4
\]

(12)

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{double_well_potential.png}
\caption{Double well potential \( V(x) \) with energy levels corresponding to \( x_d(0) \approx \pm 1.21 \) and \( x_d(0) \approx \pm 0.9 \)}
\end{figure}
Table 2: Coefficients $\alpha_{1,2}$, moduli $m$ of new solutions (7) of Eq.(1) and their bounded/unboundedness properties

<table>
<thead>
<tr>
<th>JEFs</th>
<th>$m^2$</th>
<th>$\alpha_1$</th>
<th>$\alpha_2$</th>
<th>Solution (7) of Eq.(1)</th>
<th>++</th>
<th>--</th>
<th>--</th>
<th>--</th>
<th>--</th>
</tr>
</thead>
<tbody>
<tr>
<td>nc, dc</td>
<td>$2(1-a)$</td>
<td>$\pm \sqrt{m^2 - \frac{1}{a}}$</td>
<td>$\pm \sqrt{\frac{1}{2a}}$</td>
<td>$\pm \left{ \sqrt{m^2 - \frac{1}{a}} \sum_{c,n} \pm \sqrt{-\frac{1}{2a}} \delta_{c}(\xi,m) \right}$</td>
<td>B for $a = \frac{1}{2}$</td>
<td>B $\forall a \geq \frac{1}{2}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>cs, ds</td>
<td>$-(2a+1)$</td>
<td>$\pm \sqrt{-\frac{1}{2a}}$</td>
<td>$\pm \sqrt{-\frac{1}{2a}}$</td>
<td>$\pm \left{ \sqrt{-\frac{1}{2a}} \left{ d_{s}(\xi,m) + m \sqrt{-\frac{1}{2a}} \right} \right}$</td>
<td>UB $\forall a \leq -\frac{1}{2}$</td>
<td>B for $a = -1$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>cs, ns</td>
<td>$a + \frac{1}{2}$</td>
<td>$\pm \sqrt{-\frac{1}{2a}}$</td>
<td>$\pm \sqrt{-\frac{1}{2a}}$</td>
<td>$\pm \left{ \sqrt{-\frac{1}{2a}} \left{ c_{s}(\xi,m) + m \sqrt{-\frac{1}{2a}} \right} \right}$</td>
<td>UB $\forall a \geq -\frac{1}{2}$</td>
<td>B $\forall a \geq \frac{1}{2}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>nd, cd</td>
<td>$a + \frac{1}{2}$</td>
<td>$\pm \sqrt{\frac{1-m^2}{2a}}$</td>
<td>$\pm \sqrt{-\frac{m^2}{2a}}$</td>
<td>$\pm \left{ \sqrt{\frac{1-m^2}{2a}} \left{ c_{d}(\xi,m) + m \sqrt{-\frac{1}{2a}} \right} \right}$</td>
<td>B for $a = \frac{1}{2}$</td>
<td>B $\forall a \geq \frac{1}{2}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>sd, cd</td>
<td>$2(1-a)$</td>
<td>$\pm \sqrt{\frac{m^2-m^4}{2a}}$</td>
<td>$\pm \sqrt{-\frac{m^2}{2a}}$</td>
<td>$\pm \left{ \sqrt{-\frac{m^2}{2a}} \left{ d_{s}(\xi,m) + m \sqrt{-\frac{1}{2a}} \right} \right}$</td>
<td>UB $\forall a \leq 1$</td>
<td>B $\forall a \geq \frac{1}{2}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>ds, ns</td>
<td>$2(1-a)$</td>
<td>$\pm \sqrt{-\frac{1}{2a}}$</td>
<td>$\pm \sqrt{-\frac{1}{2a}}$</td>
<td>$\pm \left{ \sqrt{-\frac{1}{2a}} \left{ d_{s}(\xi,m) + m \sqrt{-\frac{1}{2a}} \right} \right}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>dn, cn</td>
<td>$1 - 2a$</td>
<td>$\pm \sqrt{-\frac{1}{2a}}$</td>
<td>$\pm \sqrt{-\frac{1}{2a}}$</td>
<td>$\pm \left{ \sqrt{-\frac{1}{2a}} \left{ c_{n}(\xi,m) + m \sqrt{-\frac{1}{2a}} \right} \right}$</td>
<td>B $\forall a \leq \frac{1}{2}$</td>
<td></td>
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<tr>
<td>sn, cn</td>
<td>$2(1-a)$</td>
<td>$\pm \sqrt{-\frac{1}{2a}}$</td>
<td>$\pm \sqrt{-\frac{1}{2a}}$</td>
<td>$\pm \left{ \sqrt{-\frac{1}{2a}} \left{ c_{n}(\xi,m) + m \sqrt{-\frac{1}{2a}} \right} \right}$</td>
<td>B $\forall a \leq 1$</td>
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</tr>
<tr>
<td>dn, sn</td>
<td>$a + \frac{1}{2}$</td>
<td>$\pm \sqrt{\frac{m^2}{2a}}$</td>
<td>$\pm \sqrt{\frac{m^2}{2a}}$</td>
<td>$\pm \left{ \sqrt{\frac{m^2}{2a}} \left{ d_{n}(\xi,m) + m \sqrt{\frac{1}{2a}} \right} \right}$</td>
<td>B $\forall a \geq -\frac{1}{2}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>nd, sd</td>
<td>$-(1+2a)$</td>
<td>$\pm \sqrt{\frac{1-m^2}{2a}}$</td>
<td>$\pm \sqrt{-\frac{m^2}{2a}}$</td>
<td>$\pm \left{ \sqrt{\frac{1-m^2}{2a}} \left{ n_{d}(\xi,m) + m \sqrt{-\frac{1}{2a}} \right} \right}$</td>
<td>B $\forall a \geq 1$</td>
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<td></td>
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<td></td>
</tr>
<tr>
<td>sc, nc</td>
<td>$-(1+2a)$</td>
<td>$\pm \sqrt{\frac{m^2-1}{2a}}$</td>
<td>$\pm \sqrt{-\frac{m^2}{2a}}$</td>
<td>$\pm \left{ \sqrt{-\frac{m^2}{2a}} \left{ c_{n}(\xi,m) + m \sqrt{-\frac{1}{2a}} \right} \right}$</td>
<td>B $\forall a \leq 1$</td>
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</tr>
<tr>
<td>dc, sc</td>
<td>$a + \frac{1}{2}$</td>
<td>$\pm \sqrt{-\frac{1}{2a}}$</td>
<td>$\pm \sqrt{-\frac{1}{2a}}$</td>
<td>$\pm \left{ \sqrt{-\frac{1}{2a}} \left{ d_{c}(\xi,m) + m \sqrt{-\frac{1}{2a}} \right} \right}$</td>
<td>B for $a = \frac{1}{2}$</td>
<td></td>
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</tr>
</tbody>
</table>

Table 3: Coefficients $\alpha_{1,2}$, moduli $m$ of new solutions (7) of Eq.(1) and their bounded/unboundedness properties

<table>
<thead>
<tr>
<th>JEFs</th>
<th>$m^2$</th>
<th>$\alpha_1$</th>
<th>$\alpha_2$</th>
<th>Solution (7) of Eq.(1)</th>
<th>++</th>
<th>--</th>
<th>--</th>
<th>--</th>
<th>--</th>
</tr>
</thead>
<tbody>
<tr>
<td>cn, nc</td>
<td>$\frac{1}{10} \left( 8 + a + 3 \sqrt{8 + a^2} \right)$</td>
<td>$\pm \sqrt{2m^2 \frac{a}{b}}$</td>
<td>$\pm \sqrt{2m^2 \frac{a}{b}}$</td>
<td>$\pm \left{ \sqrt{\frac{2m^2}{b}} \left{ c_{n}(\xi,m) + m \sqrt{-\frac{1}{2a}} \right} \right}$</td>
<td>B $\forall a \in R$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>dn, nd</td>
<td>$-16 + a \pm 6 \sqrt{8 - a}$</td>
<td>$\pm \sqrt{\frac{a}{2}}$</td>
<td>$\pm \sqrt{\frac{2m^2}{b}}$</td>
<td>$\pm \left{ \sqrt{\frac{2m^2}{b}} \left{ d_{n}(\xi,m) + m \sqrt{-\frac{1}{2a}} \right} \right}$</td>
<td>B $\forall a \leq 8$</td>
<td></td>
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</tr>
<tr>
<td>sn, ns</td>
<td>$17 + a \pm 6 \sqrt{8 + a}$</td>
<td>$\pm \sqrt{-\frac{2m^2}{b}}$</td>
<td>$\pm \sqrt{-\frac{2m^2}{b}}$</td>
<td>$\pm \left{ \sqrt{-\frac{2m^2}{b}} \left{ m_{s}(\xi,m) + \sqrt{-\frac{1}{2a}} \right} \right}$</td>
<td>UB $\forall a \in R$</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>cs, sc</td>
<td>$a \pm 16 \pm 6 \sqrt{8 - a}$</td>
<td>$\pm \sqrt{-\frac{2m^2}{b}}$</td>
<td>$\pm \sqrt{-\frac{2m^2}{b}}$</td>
<td>$\pm \left{ \sqrt{-\frac{2m^2}{b}} \left{ c_{s}(\xi,m) + m \sqrt{-\frac{1}{2a}} \right} \right}$</td>
<td>UB $\forall a \leq 8$</td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>ds, sd</td>
<td>$\frac{1}{10} \left( 8 + a \pm 3 \sqrt{8 + a^2} \right)$</td>
<td>$\pm \sqrt{-\frac{2m^2}{b}}$</td>
<td>$\pm \sqrt{-\frac{2m^2}{b}}$</td>
<td>$\pm \left{ \sqrt{-\frac{2m^2}{b}} \left{ d_{s}(\xi,m) + m \sqrt{-\frac{1}{2a}} \right} \right}$</td>
<td>UB $\forall a \in R$</td>
<td></td>
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</tr>
<tr>
<td>dc, cd</td>
<td>$17 + a - 6 \sqrt{8 + a}$</td>
<td>$\pm \sqrt{-\frac{2m^2}{b}}$</td>
<td>$\pm \sqrt{-\frac{2m^2}{b}}$</td>
<td>$\pm \left{ \sqrt{-\frac{2m^2}{b}} \left{ d_{c}(\xi,m) + m \sqrt{-\frac{1}{2a}} \right} \right}$</td>
<td>B for $a = -4.8$</td>
<td></td>
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</tr>
</tbody>
</table>
as evident from Fig.1 (for $\beta = 2$). Accordingly, the hamiltonian of the system is given by $H(p, x) = \frac{p^2}{2} + V(x)$, $p \equiv \dot{x}$. To obtain new solutions of Eq.(11), we compare it with (1) and get

$$a = -1, \ b = \beta. \quad (13)$$

One of the nontrivial bounded STSols of Eq. (11) can be selected from Table 2 (involving $dn$ & $sn$ functions ) as

$$x_{STS}^\pm(t) = \pm \sqrt{\frac{1}{2\beta}} dn\left(t, \pm \frac{i}{\sqrt{2}}\right) \mp \frac{1}{2\sqrt{\beta}} sn\left(t, \pm \frac{i}{\sqrt{2}}\right). \quad (14)$$

One can use formula (22.17.1-8)(cf. [9], p.563) to recast it into the form

$$x_{STS}^\pm(t) = \pm \sqrt{\frac{1}{2\beta}} \left\{ nd\left(\sqrt{\frac{3}{2}}, \frac{1}{\sqrt{3}}\right) - \sqrt{\frac{1}{3}} sd\left(\sqrt{\frac{3}{2}}, \frac{1}{\sqrt{3}}\right)\right\}. \quad (15)$$

It can be checked that for $\beta = 2$, solution $x_{STS}^+(t) > 0$, $x_{STS}^-(t) < 0 \ \forall \ t \in \mathbb{R}$. Their optimum values can be found (Mathematica Ver. 10.0) as

$$x_{STS}^{Max}^+ \approx 1.21 (= d, \text{say}) \ \text{at} \ t \approx 4.71, \cdots, \quad (16a)$$

$$x_{STS}^{Min}^+ \approx .90 (= c, \text{say}) \ \text{at} \ t \approx 1.57, \cdots, \quad (16b)$$

$$x_{STS}^{Max}^- \approx -c, \quad (16c)$$

$$x_{STS}^{Min}^- \approx -d. \quad (16d)$$

The classical solutions of Eq. (11) satisfying the initial conditions

$$x(t^I) = d, \ \dot{x}(t^I) = 0 \quad (17a)$$

and

$$x(t^I) = c, \ \dot{x}(t^I) = 0 \quad (17b)$$

can be found as (cf. [9], §22.19.9, p.565)

$$x_{cd}^\pm(t) = \pm d \ cn\left(\sqrt{2d^2 - 1}(t - t^I), \sqrt{\frac{d^2}{2d^2 - 1}}\right) \quad (18a)$$

and

$$x_{cd}^\pm(t) = \pm c \ dn\left(\pm c(t - t^I), \sqrt{\frac{2c^2 - 1}{c^2}}\right) \quad (18b)$$

respectively. We have presented the graph of solutions $x_{STS}^\pm(t)$, $x_{cd}^\pm(t)$ in Fig. 2(a) and $x_{STS}^\pm(t)$, $x_{cd}^\pm(t)$ in Fig. 2(b) to visualize their fundamental differences.
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6 Conclusions

This work is an attempt to develop a systematic procedure to obtain new solutions of nonlinear ordinary/partial differential equations. We have considered here Eq. (1) as our initial step. It is proved that superposition principle of solutions for linear equations does not hold for this equation, thus resolves the ambiguity arises in the studies of Khare and Saxena [3, 4, 5]. Subsequently, the existence of superposition-type solutions of Eq. (1) have been established. It is proved that each functions $\phi_1$, $\phi_2$ involved in the STSols (7) are solution of system of coupled equations (3.8a,b) or (3.9a,b). Each of the equation contains two parts. The first one is the linear part of Eq. (1) while the rest is the self interaction coupled with the interaction with the other part. Equations (3.8a,b)/(3.9a,b) have been solved separately. Their solutions are found in
terms of twelve Jacobian elliptic functions. These solutions have been used in (7) to obtain new solutions of (1). Such new solutions are presented in Table 2 and Table 3. It is found that STSols are periodic, bounded and unbounded as in the case of Jacobian elliptic functions. To exhibit the usefulness of the STSols obtained here, new solutions presented in section 4 have been used to study the motion of quartic oscillator. It is found in case of double well potential that apart from the oscillation in between $-d$ to $d$ corresponding to the classical solution involving single JEF, STSols corresponds the state of oscillation in the neighbourhood of the bottoms of the potential wells at an energy higher than the depth of the well. To the best of our knowledge this observation is new. It may be regarded as an important application of STSols in the mathematical analysis of a physical process. We believe the existence of STSol $x_{STS}^\pm(t)$ of quartic oscillator with double well potential has deep rooted impact in the understanding of underlying theory of classical mechanics, transition between classical to quantum mechanics, in particular. In our study it is observed that the scheme developed here for ordinary differential equation (1) with cubic nonlinearity can be extended to other nonlinear problems viz, quadratic nonlinearity, combination of different nonlinear terms, partial differential equations, system of ordinary and partial differential equations and so on. Works in these directions as well as further studies on impact of existence of new solution of quartic oscillator problem are in progress and will be reported shortly.

Acknowledgements. This work is supported partly by the SERB (DST) funded research project (Project No.SR/S4/MS:821 dt. 24-4-2014) and the UGC assisted SAP program (DRS Phase II F.510/4/DRS/2009 (SAP-I)) through the Department of Mathematics, Visva-Bharati.
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Received: February 12, 2017; Published: March 11, 2017