Applied Mathematical Sciences, Vol. 11, 2017, no. 16, 783 - 805
HIKARI Ltd, www.m-hikari.com
https://doi.org/10.12988/ams.2017.7128

A New Approach for the Determination of Publicly Registered Land Values on the Basis of Interval-valued Duality Theory and Regression

Norman Neukel

Department of Mathematics, Applied Mathematics 2
University of Erlangen-Nuremberg, 91058 Erlangen, Germany

Copyright © 2017 Norman Neukel. This article is distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Abstract

In an example for the determination of publicly registered land values in urban building conflict situations an interval value duality and regression are applied. The result is incumbent upon a new interval value optimum – beginning with non-ordered point sets of publicly registered German land values for residential land. These two mathematical approaches with set-valued order relations describe the relation to the socio-economy in an urban building conflict situation.

Keywords: Interval-valued duality, interval-valued regression, socio-economy

1. Introduction

The publicly registered land value in Germany is an average local ground value determined on the basis of purchase prices within a defined area (zone) characterized by essentially the same usage and value relationships. It refers to the square meter plot area of a property with the plot features depicted (so-called publicly registered land plot value).

The publicly registered land plot value in an urban building conflict situation can differ in some ways from a land plot for which the value is to be determined. Deviations of the land plot to be evaluated from the registered land plot value in respect of the properties determining the value, such as the type and extent of building usage, stage of development, land plot layout (particularly plot depth) and

size of the plot result in deviations in its market value. Further important aspects for the determination of the land value are the location, infrastructure and concrete local market situation. The determination of the land values is therefore based upon subjective factors.

In an earlier publication Neukel [11] already investigated order relations of comparable sets for the example of emission influences upon urban building conflict situations with non-ordered as-built properties and their impact upon the land plot market. Point sets, varying according to the consideration of socio-economic properties, are represented as scientifically substantiated sets of minimal and maximal elements. The Federal Association of German Realtors (IVD) [7] analyzes the real estate market over a defined period. Individual data from the residential building structure, purchasing power and real estate price index, without set-valued structure, are given separately and the "best" result always indicated. Neukel [11] points out here the understanding of a "minimizer" in the socio-economy. Furthermore, statistical modellings in scientific applications can be found in Berger and Wong [2]. Nytsch-Geusen, Kaul, Wehage and Färber [13] analyze accommodations on the basis of their energy efficiency.

In this paper the interval-valued duality theory and regression are again applied for the question of land value determination. The application of interval-valued optimization as the solution of a primal and dual problem leads to an interval as a new result; optimal regression polynomials furnish new optimal land values. Both approaches utilize set-valued order relations.

This paper is structured as follows. Section 2 is based upon a duality model of Wu [15] in the area of interval optimization. Here a weak duality is obtained with the concept of the scalar product for interval matrices and the equivalent relation again obtained on the basis of the "set less" order relation between interval-valued and set-valued optimization. The upper and lower representations of the optimization problem are examined in a new way. These two models are socio-economically allocated to the urban building conflict situation in Section 3, resulting in new decision processes for the determination of land values.

2. Duality in interval optimization

In this section it is assumed that the power set

$\mathcal{P}(Y) := \{A \subset Y \,|\, A \text{ is non-empty}\}$,

where Y is represented as an arbitrary real vector space.

Definition 2.1

Let $A, B \in \mathcal{P}(Y)$ be arbitrarily chosen sets.

(a) The *set less* or *Kuroiwa-Nishnianidze-Young (KNY)* – order relation $\preccurlyeq_s$ is defined as

$$A \preccurlyeq_s B :\Leftrightarrow \left(\forall a \in A\ \exists b \in B : a \le b\right) \text{ and } \left(\forall b \in B\ \exists a \in A : a \le b\right)$$

($\le$ refers to a preorder which is compatible with the linear structure).

(b) The *l-type less* order relation $\preccurlyeq_l$ has the form

$$A \preccurlyeq_l B :\Leftrightarrow \left(\forall b \in B\ \exists a \in A :\ a \le b\right).$$

(c) The *u-type less* order relation $\preccurlyeq_u$ is defined as

$$A \preccurlyeq_u B :\Leftrightarrow \left(\forall a \in A\ \exists b \in B :\ a \le b\right).$$

The order relation $\preccurlyeq_s$ was independently introduced by Young [16] in the algebra and by Nishnianidze [12] in the fixed point theory. Chiriaev und Walster [3] use this order relation in interval arithmetic and SUN Microsystems [14] implements this concept in the f95 FORTRAN compiler. Kuroiwa, Tanaka and Ha [8] describe the set less order relation for the first time as a natural criterion.

Definition 2.2

Let $\mathcal{A}$ be a non-empty set with a preorder $\le$ and $\bar{A} \in \mathcal{A}$.
(a) $\bar{A}$ is a *minimal element* of $\mathcal{A}$ when

$$A \le \bar{A} \text{ for } A \in \mathcal{A} \ \Rightarrow\ \bar{A} \le A.$$

(b) $\bar{A}$ is a *maximal element* of $\mathcal{A}$ when

$$\bar{A} \le A \text{ for } A \in \mathcal{A} \ \Rightarrow\ A \le \bar{A}.$$

For simplicity, in this paper abbreviations for the set of minimal and maximal elements of $\mathcal{A}$ are used:

$$\operatorname{Min}(\mathcal{A}) := \left\{\bar{A} \in \mathcal{A} \,\middle|\, \bar{A} \text{ is a minimal element of } \mathcal{A}\right\},$$
$$\operatorname{Max}(\mathcal{A}) := \left\{\bar{A} \in \mathcal{A} \,\middle|\, \bar{A} \text{ is a maximal element of } \mathcal{A}\right\}.$$

If one wants to include minimal and maximal elements, this leads to the definition of the *minmax less* order relation. One then has sets A of $\mathcal{M}$, where

$$\mathcal{M} := \left\{A \in \mathcal{P}(Y) \,\middle|\, \operatorname{Min}(A) \text{ and } \operatorname{Max}(A) \text{ are non-empty}\right\}$$

are assumed to be non-empty.

The following definition derives from Ha and Jahn [6].

Definition 2.3

Let $A, B \in \mathcal{M}$ be arbitrarily chosen sets. The definition of the *minmax less* order relation $\preccurlyeq_m$ then takes the form:

$$A \preccurlyeq_m B :\Leftrightarrow \operatorname{Min}(A) \preccurlyeq_s \operatorname{Min}(B) \text{ and } \operatorname{Max}(A) \preccurlyeq_s \operatorname{Max}(B),$$

where the lower index *m* stands for the term *minmax*.

The algorithms and properties of interval programming are described in the publications of Alefeld and Herzberger [1], Moore, Kearfott and Cloud [9] and Wu [15]. In this paper the optimisation problems are described separately according to "upper" and "lower" criteria. The following algorithms of the interval arithmetic as basic concepts of this theory are also found in the definition of Minkowski addition and multiplication.

Wu [15] emphasizes the difference between set-valued and interval-valued optimization. In this section the set less order relation of set optimization will be related to the interval analysis of Wu [15] and the equivalence of set-valued and interval-valued optimization therefore shown. An optimization problem analogous to that of Wu [15] suggests itself here: a minimization problem for a lower scalar product under the comparative secondary constraint for an upper scalar product as a maximization problem.

Definition 2.4 is taken from Wu [15].

Definition 2.4

Let $\mathcal{I}$ be the family of all closed intervals in $\mathbb{R}$. In addition $A := \left[a^l, a^u\right]$ and $B := \left[b^l, b^u\right]$ with the respective upper and lower limits a^l, b^l and a^u, b^u in $\mathcal{I}$. the following operators are then defined as:

$$A \oplus B := \left\{a + b \middle| a \in A \text{ and } b \in B\right\} = \left[a^l + b^l, a^u + b^u\right],$$

$$A \otimes B := \left\{a \cdot b \middle| a \in A \text{ and } b \in B\right\}$$

$$= \left[\operatorname{Min}\left\{a^l b^l, a^l b^u, a^u b^l, a^u b^u\right\}, \operatorname{Max}\left\{a^l b^l, a^l b^u, a^u b^l, a^u b^u\right\}\right],$$

$$-A := \left\{-a \middle| a \in A\right\} = \left[-a^u, -a^l\right], \quad A \ominus B := A \oplus (-B) = \left[a^l - b^u, a^u - b^l\right]$$

and

$$A \oslash B := \left\{\frac{a}{b} \middle| a \in A \text{ and } b \in B\right\}$$

$$= \left[\operatorname{Min}\left\{\frac{a^l}{b^l}, \frac{a^l}{b^u}, \frac{a^u}{b^l}, \frac{a^u}{b^u}\right\}, \operatorname{Max}\left\{\frac{a^l}{b^l}, \frac{a^l}{b^u}, \frac{a^u}{b^l}, \frac{a^u}{b^u}\right\}\right],$$

where $a^l,\ a^u,\ b^l,\ b^u \neq 0$.

The *interval-valued scalar product* of A and B in $\mathcal{I}^n = \mathcal{I} \times \ldots \times \mathcal{I}$ is defined as

$$\langle\langle A,B\rangle\rangle := (A_1 \otimes B_1) \oplus \ldots \oplus (A_n \otimes B_n).$$

It follows with $A^l := a^l$, $A^u := a^u$, $B^l := b^l$ and $B^u := b^u$:

$$(A \oplus B)^l = A^l + B^l = a^l + b^l, \ (A \oplus B)^u = A^u + B^u = a^u + b^u,$$

$$(A \otimes B)^l = \text{Min}\{a^l b^l, a^l b^u, a^u b^l, a^u b^u\}, \ (A \otimes B)^u = \text{Max}\{a^l b^l, a^l b^u, a^u b^l, a^u b^u\},$$

$$\langle\langle A,B\rangle\rangle^l = (A_1 \otimes B_1)^l + \ldots + (A_n \otimes B_n)^l$$

$$= \text{Min}\{a_1^l b_1^l, a_1^l b_1^u, a_1^u b_1^l, a_1^u b_1^u\} + \ldots + \text{Min}\{a_n^l b_n^l, a_n^l b_n^u, a_n^u b_n^l, a_n^u b_n^u\},$$

$$\langle\langle A,B\rangle\rangle^u = (A_1 \otimes B_1)^u + \ldots + (A_n \otimes B_n)^u$$

$$= \text{Max}\{a_1^l b_1^l, a_1^l b_1^u, a_1^u b_1^l, a_1^u b_1^u\} + \ldots + \text{Max}\{a_n^l b_n^l, a_n^l b_n^u, a_n^u b_n^l, a_n^u b_n^u\}.$$

In interval arithmetic one uses order relations to compare intervals.

Remark 2.5:

Intervals can be considered in partially ordered vector spaces. Let Z be an arbitrary real vector space with partial order $\leq_Z$. With

$$A = [a^l, a^u] := \{z \in Z \mid a^l \leq_Z z \leq_Z a^u\} \text{ and } B = [b^l, b^u] := \{z \in Z \mid b^l \leq_Z z \leq_Z b^u\}$$

as two closed intervals, we then obtain the setless order relation $\preccurlyeq_s$ for any arbitrary $a^l, a^u, b^l, b^u \in Z$:

$$A \preccurlyeq_s B :\Leftrightarrow a^l \leq_Z b^l \text{ and } a^u \leq_Z b^u.$$

In particular, we can also examine the following relation:

$$A \prec_s B :\Leftrightarrow A \preccurlyeq_s B \text{ with } A \neq B.$$

Accordingly, we write:

$$a^l <_Z b^l :\Leftrightarrow a^l \leq_Z b^l \text{ with } a^l \neq b^l$$

and

$$a^u <_Z b^u :\Leftrightarrow a^u \leq_Z b^u \text{ with } a^u \neq b^u.$$

Equivalently:

$$A \prec_s B \Leftrightarrow \begin{Bmatrix} a^l <_Z b^l \\ a^u = b^u \end{Bmatrix} \vee \begin{Bmatrix} a^l = b^l \\ a^u <_Z b^u \end{Bmatrix} \vee \begin{Bmatrix} a^l <_Z b^l \\ a^u <_Z b^u \end{Bmatrix}.$$

$\preccurlyeq_s$ is constructed from the order relations $\preccurlyeq_l$ and $\preccurlyeq_u$ (see Definition 2.2). $\preccurlyeq_m$ is obtained from the set less comparisons and the minima and maxima and can also be utilized in the interval analysis.

In this paper the following definitions and propositions from Wu [15] will be investigated in regard to upper and lower criteria.

Definition 2.6

A matrix $\mathfrak{A}$ is referred to as an *interval matrix* when each element of $\mathfrak{A}$ is a closed interval $\left[\cdot^l, \cdot^u\right]$.

For $C \in \mathcal{I}^n$, $B \in \mathcal{I}^m$ and $A_{i\cdot}$ (the *i*-th row of the interval matrix $\mathfrak{A}$) we will rename the following problem the *primal interval-valued linear optimization problem*:

We determine a vector $\bar{x} \in \mathbb{R}^n$ which is the minimal solution of the two optimization problems

$$\min_{\substack{\text{with the constraints} \\ \langle\langle C,x\rangle\rangle^u \le \langle\langle C,\bar{x}\rangle\rangle^u \\ \langle\langle A_{i\cdot},x\rangle\rangle \succcurlyeq_s B_i \text{ for } i=1,\ldots,m \\ x \ge 0_n}} \langle\langle C,x\rangle\rangle^l \quad \text{and} \quad \min_{\substack{\text{with the constraints} \\ \langle\langle C,x\rangle\rangle^l \le \langle\langle C,\bar{x}\rangle\rangle^l \\ \langle\langle A_{i\cdot},x\rangle\rangle \succcurlyeq_s B_i \text{ for } i=1,\ldots,m \\ x \ge 0_n}} \langle\langle C,x\rangle\rangle^u \qquad (P\ 1).$$

The minimization problem with the lower property are reintroduced under an upper constraint and vice versa.

Let D_i and E_i be closed intervals for $i = 1,\ldots,n$. It then follows that $D = (D_1,\ldots,D_n) \preccurlyeq_s (E_1,\ldots,E_n) = E$ precisely when $D_i \preccurlyeq_s E_i$ for all $i = 1,\ldots,n.$ The notation $\langle\langle A_{i\cdot}, x\rangle\rangle \succcurlyeq_s B_i$ for $i = 1,\ldots,m$ from $(P\ 1)$ can then be replaced by $\mathfrak{A}x \succcurlyeq_s B$ due to the definition of the interval matrix.

The dual *interval-valued linear optimization problem* is the following problem:

We determine a vector $\bar{y} \in \mathbb{R}^m$, which is the maximal solution of the two optimization problems

$$\max_{\substack{\text{with the constraints} \\ \langle\langle B,\bar{y}\rangle\rangle^u \le \langle\langle B,y\rangle\rangle^u \\ \langle\langle A_j,y\rangle\rangle \preccurlyeq_s C_j \text{ for } j=1,\ldots,n \\ y \ge 0_m}} \langle\langle B,y\rangle\rangle^l \quad \text{and} \quad \max_{\substack{\text{with the constraints} \\ \langle\langle B,\bar{y}\rangle\rangle^l \le \langle\langle B,y\rangle\rangle^l \\ \langle\langle A_j,y\rangle\rangle \preccurlyeq_s C_j \text{ for } j=1,\ldots,n \\ y \ge 0_m}} \langle\langle B,y\rangle\rangle^u \qquad (D\ 1),$$

where A_j describes the *j*-th column of $\mathfrak{A}$.

For the problems under $(D\ 1)$ the dual variables are vectors.

Here also $\langle\langle A_j, y\rangle\rangle \preccurlyeq_s C_j$ for $j = 1,\ldots,n$ can be replaced by $y^T\mathfrak{A} \preccurlyeq_s C$.

$$X := \left\{ x \,\middle|\, \langle\langle A_{i\cdot}, x\rangle\rangle \succcurlyeq_s B_i \text{ for } i = 1,\ldots,m \text{ and } x \ge 0_n \right\}$$

describes the *feasible set* of $(P\ 1)$ and

$$Y := \left\{ y \middle| \langle\langle A_{j\cdot}, y \rangle\rangle \preccurlyeq_s C_j \text{ for } j = 1,\ldots,n \text{ and } y \geq 0_m \right\}$$

the *feasible set* of $(D\ 1)$.

More precisely:

$$X \supset \begin{cases} \left\{ x \middle| \langle\langle A_{i\cdot}, x \rangle\rangle_{\ u} \succcurlyeq B_i \text{ for } i = 1,\ldots,m \text{ and } x \geq 0_n \right\} \\ \left\{ x \middle| \langle\langle A_{i\cdot}, x \rangle\rangle_{\ l} \succcurlyeq B_i \text{ for } i = 1,\ldots,m \text{ and } x \geq 0_n \right\} \end{cases}$$

and

$$Y \supset \begin{cases} \left\{ y \middle| \langle\langle A_{\cdot j}, y \rangle\rangle \preccurlyeq_u C_j \text{ for } j = 1,\ldots,n \text{ and } y \geq 0_m \right\} \\ \left\{ y \middle| \langle\langle A_{\cdot j}, y \rangle\rangle \preccurlyeq_l C_j \text{ for } j = 1,\ldots,n \text{ and } y \geq 0_m \right\} \end{cases}.$$

Let $F : X \rightrightarrows \mathcal{I}$ be a set-valued mapping with $F(x) := \left[F^l(x), F^u(x) \right]$ and $F^l(x) := \langle\langle C, x \rangle\rangle^l$ and $F^u(x) := \langle\langle C, x \rangle\rangle^u$. We refer to $F(\bar{x})$ as a *non-dominated objective function value* whenever no $x \in X$ exists, so that

$$F^l(x) < F^l(\bar{x}) \text{ and } F^u(x) < F^u(\bar{x}).$$

The *set of all non-dominated points* has the form

$$\begin{aligned} \mathrm{Min}_P(\mathfrak{A}, B, C) := \Big\{ \bar{x} \in \mathbb{R}^n \,\Big|\, & \bar{x} \text{ is the minimal solution of the two} \\ & \text{optimization problems in } (P\ 1), \text{ and there exists no } x \in \mathbb{R}^n, \\ & \text{so that } \langle\langle C, \bar{x} \rangle\rangle^l > \langle\langle C, x \rangle\rangle^l \text{ and } \langle\langle C, \bar{x} \rangle\rangle^u > \langle\langle C, x \rangle\rangle^u \Big\}. \end{aligned}$$

In regard to $(D\ 1)$ we then have

$$\begin{aligned} \mathrm{Max}_D(\mathfrak{A}, B, C) := \Big\{ \bar{y} \in \mathbb{R}^m \,\Big|\, & \bar{y} \text{ is the maximal solution of the two} \\ & \text{optimization problems in } (D\ 1), \text{ and there exists no } y \in \mathbb{R}^m, \\ & \text{so that } \langle\langle B, \bar{y} \rangle\rangle^l < \langle\langle B, y \rangle\rangle^l \text{ and } \langle\langle B, \bar{y} \rangle\rangle^u < \langle\langle B, y \rangle\rangle^u \Big\}, \end{aligned}$$

where the notations *P* and *D* stand for "primal" and "dual".

Describing the interval-valued primal problem according to Wu [15] as follows:

$$\preccurlyeq_s - \min_{\substack{\text{with the constraints} \\ \langle\langle A_{i\cdot}, x \rangle\rangle \succcurlyeq_s B_i \text{ for } i=1,\ldots,m \\ x \geq 0_n}} \langle\langle C, x \rangle\rangle, \qquad (P\ 2)$$

$\mathrm{Min}_P(\mathfrak{A}, B, C)$ then describes the set of minimal solutions of $(P\,2)$ (see Definition 2.2). Here we recall that the objective function of the problem $(P\,2)$ is interval-valued and that the target functions of the problems in $(P\,1)$ and $(D\,1)$ are real-valued.

This results in the following property:

Lemma 2.7

$\bar{x}$ is a minimal solution of $(P\,2)$ precisely when $\bar{x}$ is a minimal solution of the two optimization problems in $(P\,1)$.

Proof:
"$\Rightarrow$": Assume that $\bar{x}$ is a minimal solution of $(P\,2)$. This means: $\langle\langle C, x\rangle\rangle \preccurlyeq_s \langle\langle C, \bar{x}\rangle\rangle,\ x \in X \ \Rightarrow\ \langle\langle C, \bar{x}\rangle\rangle \preccurlyeq_s \langle\langle C, x\rangle\rangle$.
The inequality $\langle\langle C, x\rangle\rangle \preccurlyeq_s \langle\langle C, \bar{x}\rangle\rangle$ is equivalent to
$\langle\langle C, x\rangle\rangle^l \le \langle\langle C, \bar{x}\rangle\rangle^l$ and $\langle\langle C, x\rangle\rangle^u \le \langle\langle C, \bar{x}\rangle\rangle^u$. Then:

$$\left.\begin{array}{l} \langle\langle C, \bar{x}\rangle\rangle^l \le \langle\langle C, x\rangle\rangle^l \text{ and } \langle\langle C, \bar{x}\rangle\rangle^u \le \langle\langle C, x\rangle\rangle^u \\ \Rightarrow\ \langle\langle C, x\rangle\rangle^l = \langle\langle C, \bar{x}\rangle\rangle^l \text{ and } \langle\langle C, x\rangle\rangle^u = \langle\langle C, \bar{x}\rangle\rangle^u. \end{array}\right\} \qquad (2.1)$$

If $\bar{x}$ is not the solution to the first minimization problem of $(P\,1)$, then there exists an $x \in X$, so that

$$\langle\langle C, x\rangle\rangle^l < \langle\langle C, \bar{x}\rangle\rangle^l \text{ and } \langle\langle C, x\rangle\rangle^u \le \langle\langle C, \bar{x}\rangle\rangle^u,$$

contrary to the implication of (2.1). $\bar{x}$ is therefore a minimal solution of the first minimization problem of $(P\,1)$. This proof also follows analogously for the second part of $(P\,1)$.
"$\Leftarrow$": Assume that $\bar{x}$ is a minimal solution of the optimization problems in $(P\,1)$. If $\bar{x}$ is not a minimal solution of $(P\,2)$, then there exists an allowed x, so that $\langle\langle C, x\rangle\rangle \prec_s \langle\langle C, \bar{x}\rangle\rangle$. This inequality is equivalent to

$$\left.\begin{array}{l} \langle\langle C, x\rangle\rangle^l < \langle\langle C, \bar{x}\rangle\rangle^l \\ \langle\langle C, x\rangle\rangle^u = \langle\langle C, \bar{x}\rangle\rangle^u \end{array}\right\} \text{ or}$$

$$\left.\begin{array}{l} \langle\langle C, x\rangle\rangle^l = \langle\langle C, \bar{x}\rangle\rangle^l \\ \langle\langle C, x\rangle\rangle^u < \langle\langle C, \bar{x}\rangle\rangle^u \end{array}\right\} \text{ or}$$

$$\left.\begin{array}{l} \langle\langle C, x\rangle\rangle^l < \langle\langle C, \bar{x}\rangle\rangle^l \\ \langle\langle C, x\rangle\rangle^u < \langle\langle C, \bar{x}\rangle\rangle^u \end{array}\right\},$$

contrary to the premise $\bar{x}$ is a minimal solution of the optimization problems in $(P\ 1)$. □

Lemma 2.7 also applies analogously for $(D\ 1)$ and a related maximization problem.

In the following a duality theory on the basis of interval analysis with order relations is introduced. The propositions advanced are essential in order to characterize a weak duality with the help of interval arithmetic:

Proposition 2.8

Let $\mathfrak{A}$ be an $m \times n$ interval matrix, $x \geq 0_n$ and $y \geq 0_m$. Then

$$\langle\langle y^T \mathfrak{A}, x\rangle\rangle^l = \langle\langle y, \mathfrak{A}x\rangle\rangle^l \text{ and } \langle\langle y^T \mathfrak{A}, x\rangle\rangle^u = \langle\langle y, \mathfrak{A}x\rangle\rangle^u .$$

Proof:
By analogy to Wu [15]. □

The following proposition also applies in the real vector space *Z* for intervals of the form $\left[a^l, a^u\right] = \left\{z \in Z \middle| a^l \leq_Z z \leq_Z a^u\right\}$.

Proposition 2.9

Let $D = (D_1, \ldots, D_n), E = (E_1, \ldots, E_n) \in \mathcal{I}^n$ $D = \left[D^l, D^u\right]$, $E = \left[E^l, E^u\right]$ and $x \in \mathbb{R}_+^n$. If $D^l \leq E^l$ and $D^u \leq E^u$, it then follows that $\langle\langle D, x\rangle\rangle^l \leq \langle\langle E, x\rangle\rangle^l$ and $\langle\langle D, x\rangle\rangle^u \leq \langle\langle E, x\rangle\rangle^u$.

Proof:
By analogy to Wu [15]. □

With these preliminary remarks we can now prove weak duality.

Proposition 2.10 (weak duality for (*P* 1) and (*D* 1))

Let *x* be feasible with regard to the optimization problems in $(P\ 1)$ and *y* be feasible with regard to the optimization problems in $(D\ 1)$ with $\bar{x} := x$ and $\bar{y} := y$. Then $\langle\langle B, y\rangle\rangle^l \leq \langle\langle C, x\rangle\rangle^l$ and $\langle\langle B, y\rangle\rangle^u \leq \langle\langle C, x\rangle\rangle^u$.

Proof:
By analogy to Wu [15]. □

3. Economic interpretation of the interval-valued duality and regression

The interval-valued duality and regression possess the potential to optimally describe the relationships between urban building functions and usages in a conflict situation. The duality and regression criteria to which the optimality is oriented can differ. The problem consists in having to take the optimality criteria and order relations in urban building structure planning, which in the ideal case complement each other, into consideration. With the help of a duality and regression approach for urban building structure planning conflicts here can be minimized. Complex conflict situations require subdivision into individual "zones" with selective dual properties and order relationships. The functional structure of cities changes continuously. Restrictions are affected by permanent changes, so that the duality and order relations offer a possible solution approach for urban structural optimization. An exact demarcation between urban building, sociological, economic and housing policy aspects in a conflict situation, described by a duality and regression model, is not possible here. All parameters enter into the calculations.

In this section the duality model of Section 2 will be investigated in detail in relation to the solution of a conflict situation.

The publicly registered land value is the average ground value determined on the basis of the majority of land plots within a defined area characterized by essentially the same land plot character and for which essentially the same overall value relationships exist (German Federal Building Code).

Following Proposition 2.10 with interval-valued optimization, the example below gives the result of a solution to a primal and dual problem. As a rule, land values are determined by the public authority every two years on December 31 and refer to the compilation of purchase price data.

Process concept 3.1:

Data: The statistical two-year price increase (according to destatis (German Federal Office of Statistics) [4]) for the definition of intervals, with individual prices for the period between the biennial publication of the reference values.

1st step: Specification of a convex combination of individual price intervals as a objective function with regard to $(P\ 1)$ for each land value zone.

2nd step: Choosing $A_i,\ B_i,\ i=1,\ldots,m$ with fixed minimal solution $\bar{x} \in \mathbb{R}^n$ for the constraints of the optimization problems under $(P\ 1)$.

3rd step: Solving the two optimization problems $(D\,1)$ for a selected fixed maximal solution $\bar{y} \in \mathbb{R}^m$ and application of Proposition 2.10 for the determination of the land values.

Example 3.2:

The biennially determined land values for residential land in an administratively independent city for two land value zones on December 31, 2014 (see Deutscher Städtemarkt (German Urban Price Market) [5]), show the following individual purchase prices (in Euro/m², rounded off):
$\{255,\ 200,\ 235,\ 225,\ 220\}$ and $\{580,\ 310,\ 550,\ 350,\ 340\}$.

The biennial statistical price increase in the amount of 3.8 % (according to destatis [4] price indexes for the building sector published for the period 2012 to 2014) serves for the definition of the land value price intervals. For the individual purchase prices the following rounded intervals are found:

$\{[245,265],\ [192,208],\ [226,244],\ [216,234],\ [212,228]\}$ and
$\{[558,602],\ [298,322],\ [529,571],\ [337,363],\ [327,353]\}$.

As the objective function for the problems under $(P\,1)$, for each land value zone we choose a convex combination of the individual price intervals. For the first reference value zone we then have

$$\langle\langle C, x\rangle\rangle^l = 245x_1 + 192x_2 + 226x_3 + 216x_4 + 212x_5$$

and

$$\langle\langle C, x\rangle\rangle^u = 265x_1 + 208x_2 + 244x_3 + 234x_4 + 228x_5$$

for $\sum_{i=1}^{5} x_i = 1$ and $x_i \geq 0$ for $i = 1,\ldots,5$. Furthermore, we choose $m = 1$ and set

$$A_1 := \left([-1,1],\ldots,[-1,1]\right) \text{ and } B_1 := [-1,1].$$

It then follows that

$$\begin{aligned}
\langle\langle A_1, x\rangle\rangle \succcurlyeq_s B_1 &\Leftrightarrow \left[-x_1 - \ldots - x_n, x_1 + \ldots + x_n\right] \succcurlyeq_s [-1,1] \\
&\Leftrightarrow -\sum_{i=1}^{5} x_i \geq -1,\ \sum_{i=1}^{5} x_i \geq 1 \\
&\Leftrightarrow \sum_{i=1}^{5} x_i = 1.
\end{aligned}$$

We have now described the constraint $\sum_{i=1}^{5} x_i = 1$ in a form as under $(P\ 1)$. With $\bar{x} := (0,1,0,0,0)^T$ the optimization problems under $(P\ 1)$ then assume the concrete form

$$\begin{array}{l}\min \\ \text{with the constraints} \\ 265x_1+208x_2+244x_3+234x_4+228x_5 \leq 208 \\ \langle\langle([-1,1],\ldots,[-1,1]),x\rangle\rangle \succcurlyeq_s [-1,1] \\ x_i \geq 0 \text{ for } i=1,\ldots,5\end{array} \quad 245x_1 + 192x_2 + 226x_3 + 216x_4 + 212x_5$$

$$\begin{array}{l}\min \\ \text{with the constraints} \\ 245x_1+192x_2+226x_3+216x_4+212x_5 \leq 192 \\ \langle\langle([-1,1],\ldots,[-1,1]),x\rangle\rangle \succcurlyeq_s [-1,1] \\ x_i \geq 0 \text{ for } i=1,\ldots,5\end{array} \quad 265x_1 + 208x_2 + 244x_3 + 234x_4 + 228x_5.$$

Analogously, for

$$\{[558,602],\ [298,322],\ [529,571],\ [337,363],\ [327,353]\}$$

$$\begin{array}{l}\min \\ \text{with the constraints} \\ 602x_1+322x_2+571x_3+363x_4+353x_5 \leq 322 \\ \langle\langle([-1,1],\ldots,[-1,1]),x\rangle\rangle \succcurlyeq_s [-1,1] \\ x_i \geq 0 \text{ for } i=1,\ldots,5\end{array} \quad 558x_1 + 298x_2 + 529x_3 + 337x_4 + 327x_5$$

$$\begin{array}{l}\min \\ \text{with the constraints} \\ 558x_1+298x_2+529x_3+337x_4+327x_5 \leq 298 \\ \langle\langle([-1,1],\ldots,[-1,1]),x\rangle\rangle \succcurlyeq_s [-1,1] \\ x_i \geq 0 \text{ for } i=1,\ldots,5\end{array} \quad 602x_1 + 322x_2 + 571x_3 + 363x_4 + 353x_5.$$

As the minimal solution of the problems under $(P\ 1)$ with Maple 17 we obtain, as expected, $\bar{x} = (0,1,0,0,0)^T$ for both reference value zones. Applying the duality theory of Section 2, the problems under $(D\ 1)$ then assume the following form (for $\bar{y} := 0$)

$$\begin{array}{l}\max \\ \text{with the constraints} \\ 0 \leq \langle\langle[-1,1],y\rangle\rangle^u \\ [-1,1]y \preccurlyeq_s [245,265] \\ [-1,1]y \preccurlyeq_s [192,208] \\ [-1,1]y \preccurlyeq_s [226,244] \\ [-1,1]y \preccurlyeq_s [216,234] \\ [-1,1]y \preccurlyeq_s [212,228] \\ y \geq 0\end{array} \quad \langle\langle[-1,1],y\rangle\rangle^l$$

$$\begin{array}{l}\max \\ \text{with the constraints} \\ 0 \leq \langle\langle[-1,1],y\rangle\rangle^l \\ [-1,1]y \preccurlyeq_s [245,265] \\ [-1,1]y \preccurlyeq_s [192,208] \\ [-1,1]y \preccurlyeq_s [226,244] \\ [-1,1]y \preccurlyeq_s [216,234] \\ [-1,1]y \preccurlyeq_s [212,228] \\ y \geq 0\end{array} \quad \langle\langle[-1,1],y\rangle\rangle^u.$$

These optimization problems can be simplified. Namely, we know that

$$\begin{aligned}
[-1,1]\, y \preccurlyeq_s [245,265] &\Leftrightarrow y \in [-245,265] \\
[-1,1]\, y \preccurlyeq_s [192,208] &\Leftrightarrow y \in [-192,208] \\
[-1,1]\, y \preccurlyeq_s [226,244] &\Leftrightarrow y \in [-226,244] \\
[-1,1]\, y \preccurlyeq_s [216,234] &\Leftrightarrow y \in [-216,234] \\
[-1,1]\, y \preccurlyeq_s [212,228] &\Leftrightarrow y \in [-212,228],
\end{aligned}$$

and for the variable y this means that

$$y \in [-245,265] \cap [-192,208] \cap [-226,244] \cap [-216,234] \cap [-212,228] = [-192,208].$$

The two maximization problems therefore assume the form

$$\max_{\substack{\text{with the constraints} \\ 0 \le y \\ y \in [-192,208] \\ y \ge 0}} -y \quad \Leftrightarrow y \in [0,208]$$

$$\max_{\substack{\text{with the constraints} \\ y \le 0 \\ y \in [-192,208] \\ y \ge 0}} y. \quad \Leftrightarrow y \in [0,208]$$

As only one variable exists, these problems can be easily solved. The same is true by analogy for $\{[558,602],\ [298,322],\ [529,571],\ [337,363],\ [327,353]\}$.

$$\max_{\substack{\text{with the constraints} \\ 0 \le \langle\langle [-1,1], y \rangle\rangle^u \\ [-1,1]\, y \preccurlyeq_s [558,602] \\ [-1,1]\, y \preccurlyeq_s [298,322] \\ [-1,1]\, y \preccurlyeq_s [529,571] \\ [-1,1]\, y \preccurlyeq_s [337,363] \\ [-1,1]\, y \preccurlyeq_s [327,353] \\ y \ge 0}} \langle\langle [-1,1], y \rangle\rangle^l$$

$$\max_{\substack{\text{with the constraints} \\ 0 \le \langle\langle [-1,1], y \rangle\rangle^l \\ [-1,1]\, y \preccurlyeq_s [558,602] \\ [-1,1]\, y \preccurlyeq_s [298,322] \\ [-1,1]\, y \preccurlyeq_s [529,571] \\ [-1,1]\, y \preccurlyeq_s [337,363] \\ [-1,1]\, y \preccurlyeq_s [327,353] \\ y \ge 0}} \langle\langle [-1,1], y \rangle\rangle^u.$$

These optimization problems can be simplified and, as only one variable exists, they can also be easily solved.

$$\left.\begin{array}{l}\max\limits_{\substack{\text{with the constraints}\\ 0\le y\\ y\in[-298,322]\\ y\ge 0}} -y\end{array}\right\} \Leftrightarrow y\in[0,322]$$

$$\left.\begin{array}{l}\max\limits_{\substack{\text{with the constraints}\\ y\le 0\\ y\in[-298,322]\\ y\ge 0}} y.\end{array}\right\} \Leftrightarrow y\in[0,322]$$

As the maximal solution for the four problems under $(D\,1)$ we obtain, as expected, $\bar{y}=0$ for both reference value zones.

This example illustrates that the interval-valued duality theory can be applied to questions dealing with the determination of land values.

Weak duality in the interval arithmetic can be viewed as a "socio-economic prerequisite" for the regression treatment.

The dependence between land value features will now be investigated with regression analysis. Even if previous theoretical considerations clearly indicate that the features are interrelated, the regression analysis furnishes information about the type and extent of this interrelationship. If a linear relationship exists, we again speak of linear regression with regard to a set-valued mapping $Y: X \rightrightarrows \mathcal{I}$ with $Y(x) := \left[Y^l(x), Y^u(x)\right]$ and $X \subset \mathbb{R}$:

$$Y^l(x) := (a_0 + a_1 x)^l \text{ and } Y^u(x) := (a_0 + a_1 x)^u;\ a_i, x \in \mathbb{R},\ i \in \mathbb{N}.$$

The parameters a_i are calculated from the feature data according to the method of least squares.

For the regression lines which follow an Excel tool was used:

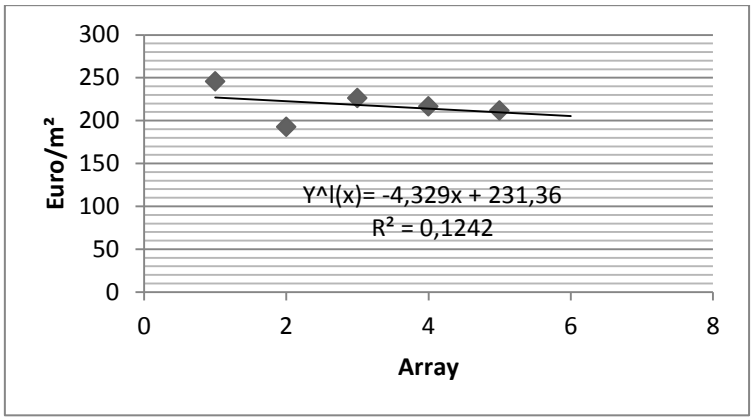
I. Data input
The data pairs $(a_1, b_1), \ldots, (a_n, b_n),\ a_i, b_i \in \mathbb{R},\ i = 1, \ldots, n$ are entered. a_i describes the sequence and b_i the land values for residential land in an administratively independent city in Euro/m² with value adjustment.

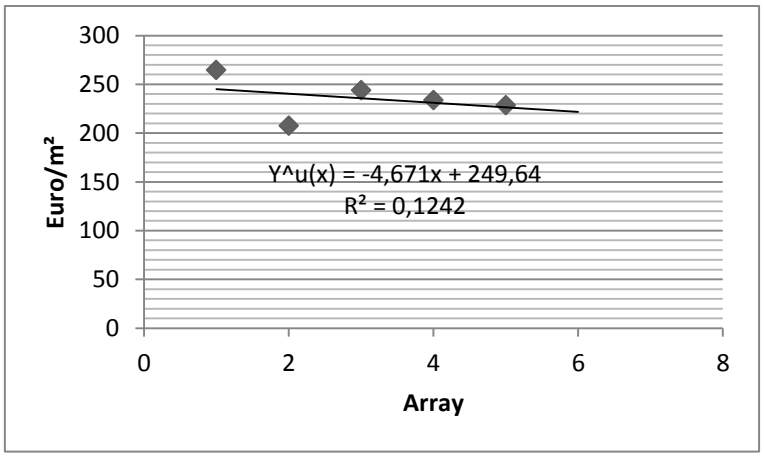
II. Graphical representation
(a) The data region is marked.
(b) The "diagram assistant" is accessible under ENTER/DIAGRAM.

III. The regression line and the corresponding equation
Upon marking the diagram we obtain this with the menu item ADD DIAGRAM → TRENDLINE → TYPE.

This gives us the regression lines and regression polynomials for both the upper and the lower values with the sequence allocated.

For the intervals $\{[245,265],\ [192,208],\ [226,244],\ [216,234],\ [212,228]\}$ Excel gives the following regression lines:

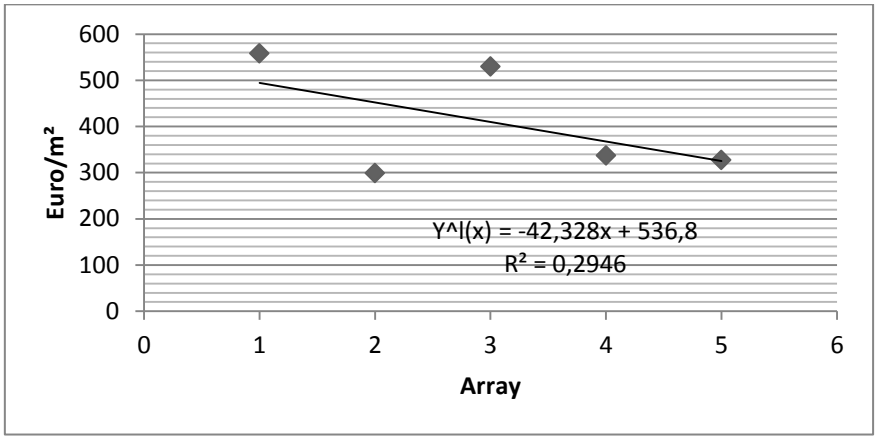


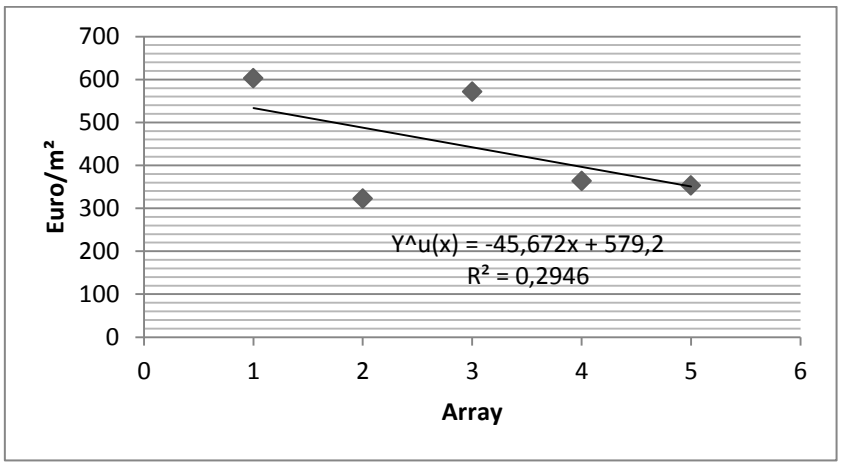


In place of the Pearson correlation coefficient R the coefficient of determination R^2 is specified. Here, the closer the coefficient of determination R^2 is to 1, the greater the probability of a linear relationship. If $R^2 = 0$ no correlation exists. The coefficient of determination therefore constitutes a measure of the fit quality.

The optimal land values (in €/m²) are $Y^l(1) = 227$, $Y^l(2) = 223$, $Y^l(3) = 218$, $Y^l(4) = 214$, $Y^l(5) = 210$ and $Y^u(1) = 245$, $Y^u(2) = 240$, $Y^u(3) = 236$, $Y^u(4) = 231$, $Y^u(5) = 226$.

For the intervals $\{[558,602],\ [298,322],\ [529,571],\ [337,363],\ [327,353]\}$ the Excel program yields the following regression lines:





The optimal land values (in €/m²) are $Y^l(1)=495$, $Y^l(2)=452$, $Y^l(3)=410$, $Y^l(4)=367$, $Y^l(5)=325$ and $Y^u(1)=534$, $Y^u(2)=488$, $Y^u(3)=442$, $Y^u(4)=397$, $Y^u(5)=351$.

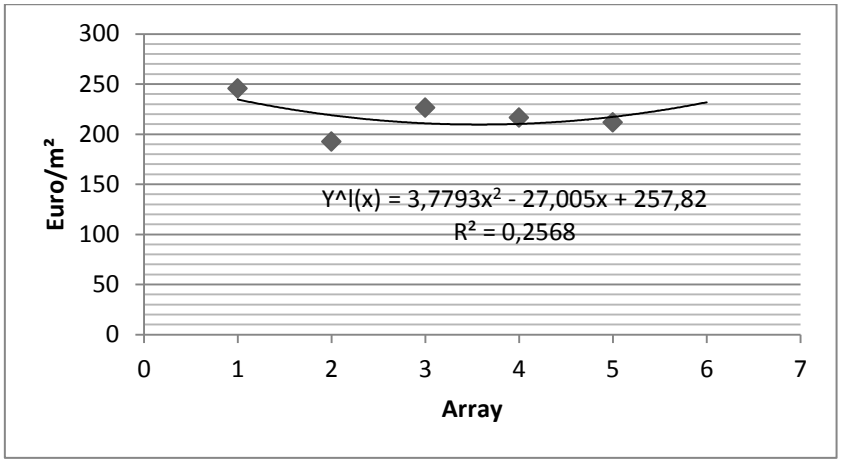
Altogether $\{[245,265],\ [192,208],\ [226,244],\ [216,234],\ [212,228]\} \preccurlyeq_m \{[558,602],\ [298,322],\ [529,571],\ [337,363],\ [327,353]\}$.

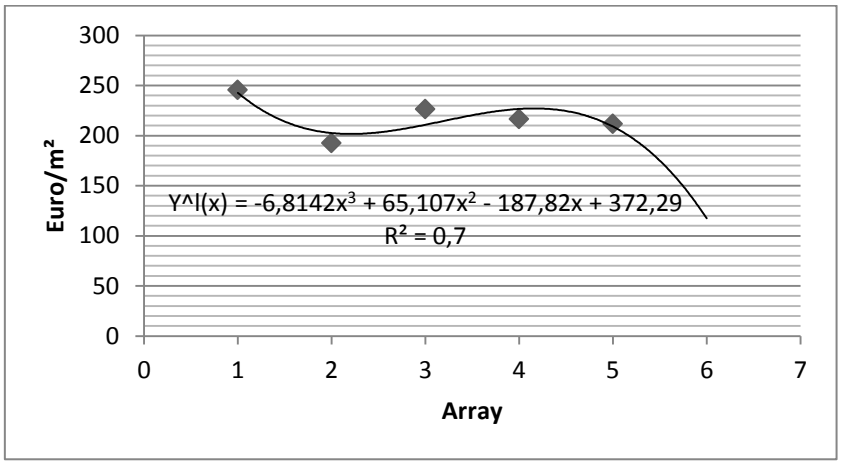
For a set-valued mapping $Y : X \rightrightarrows \mathcal{I}$ with $Y(x) := \left[Y^l(x), Y^u(x) \right]$ and $X \subset \mathbb{R}$:

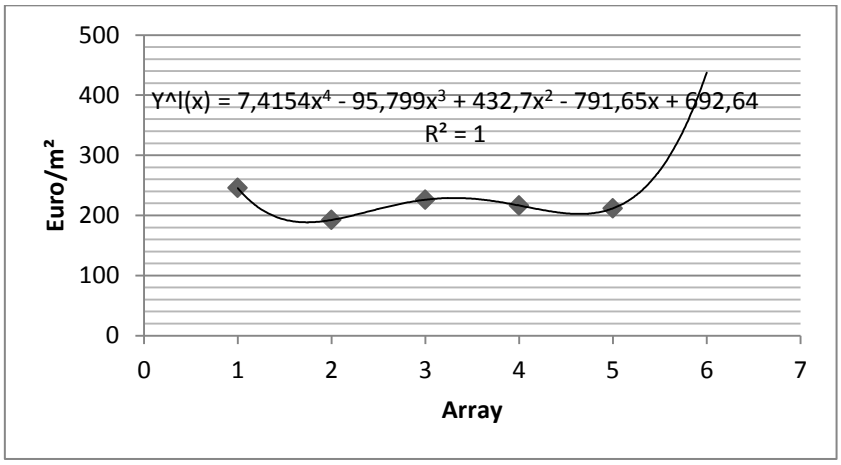
$$Y^l(x) := \left(a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n \right)^l \text{ and } Y^u(x) := \left(a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n \right)^u;$$
$$a_i, x \in \mathbb{R},\ i, n \in \mathbb{N}$$

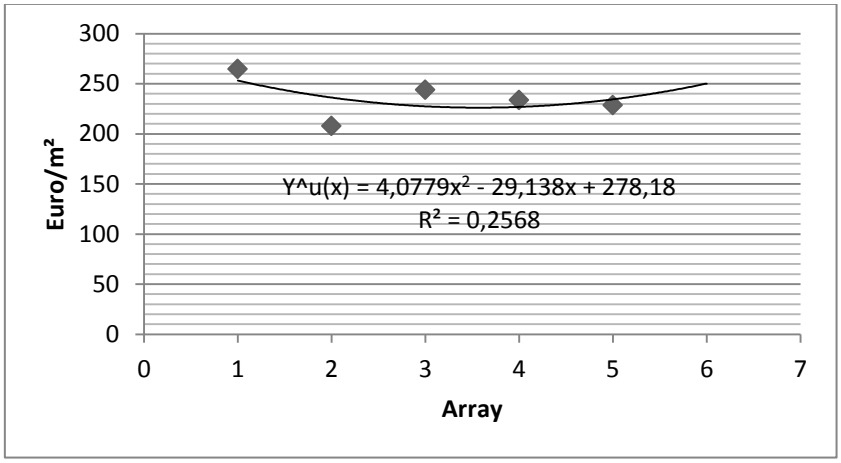
the following results are found.

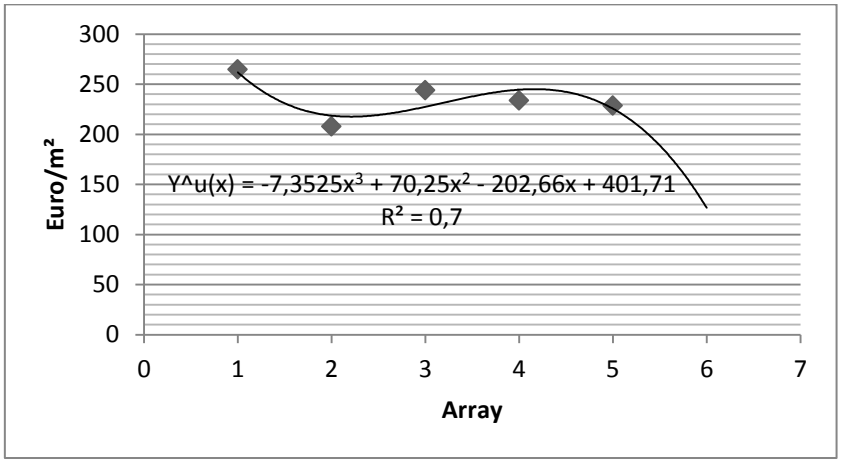
With regard to $\{[245, 265],\ [192, 208],\ [226, 244],\ [216, 234],\ [212, 228]\}$:

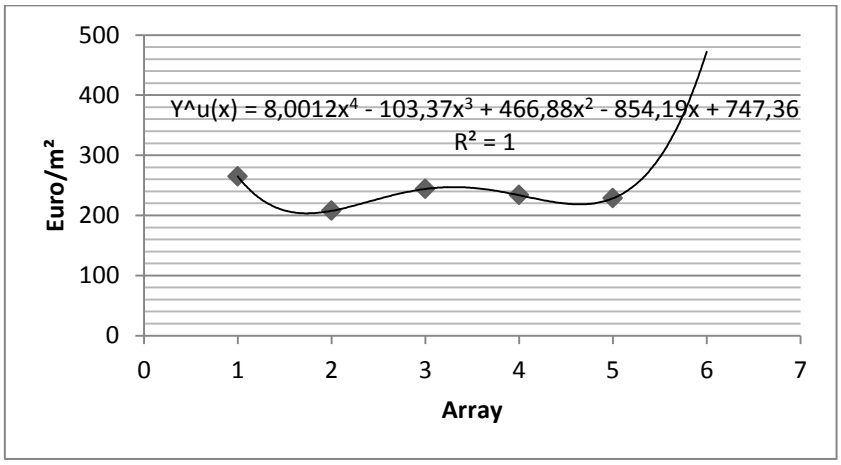




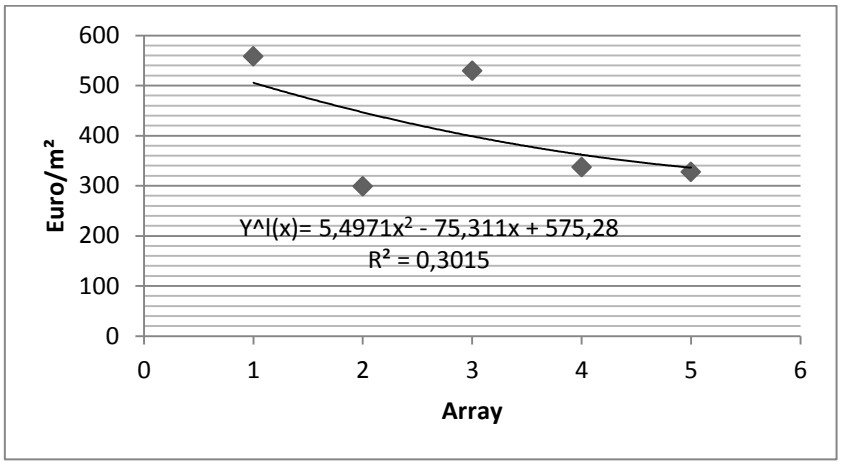

Y^l(x) = 7,4154x⁴ - 95,799x³ + 432,7x² - 791,65x + 692,64
R² = 1
Euro/m²
Array
0
100
200
300
400
500
0
1
2
3
4
5
6
7

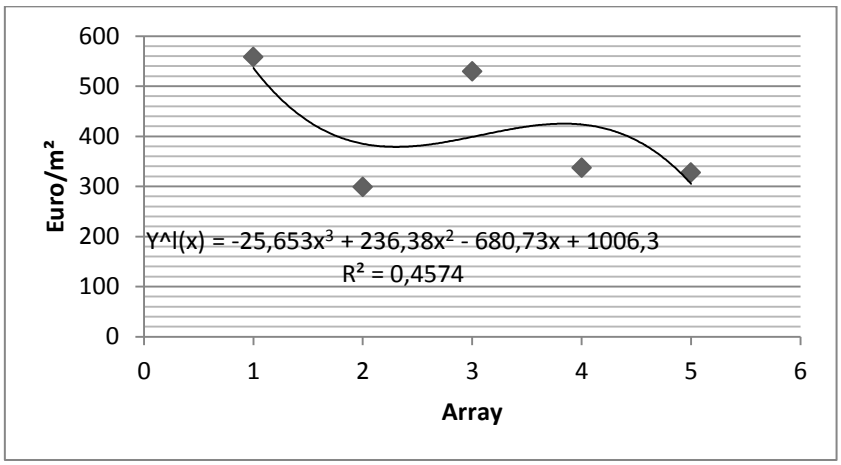

Y^u(x) = 4,0779x² - 29,138x + 278,18
R² = 0,2568
Euro/m²
Array
0
50
100
150
200
250
300
0
1
2
3
4
5
6
7

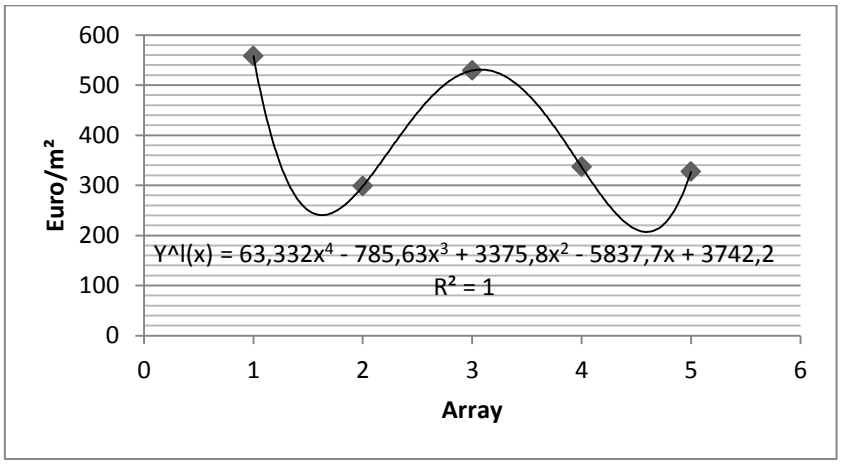

Y^u(x) = -7,3525x³ + 70,25x² - 202,66x + 401,71
R² = 0,7
Euro/m²
Array
0
50
100
150
200
250
300
0
1
2
3
4
5
6
7

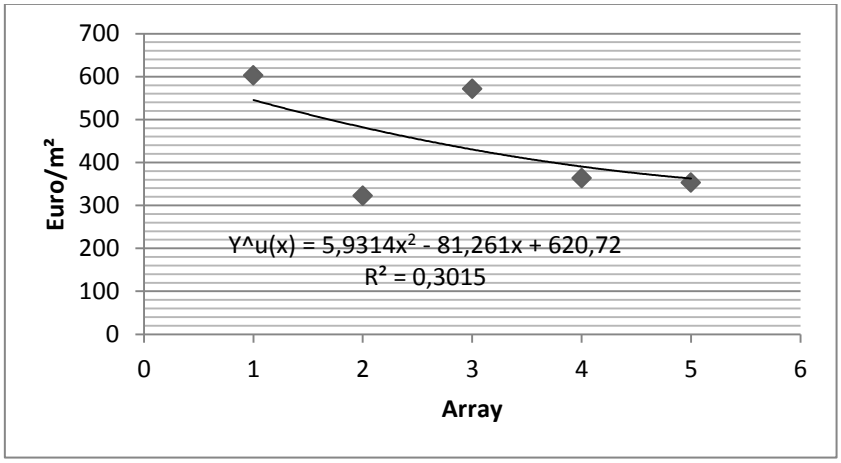


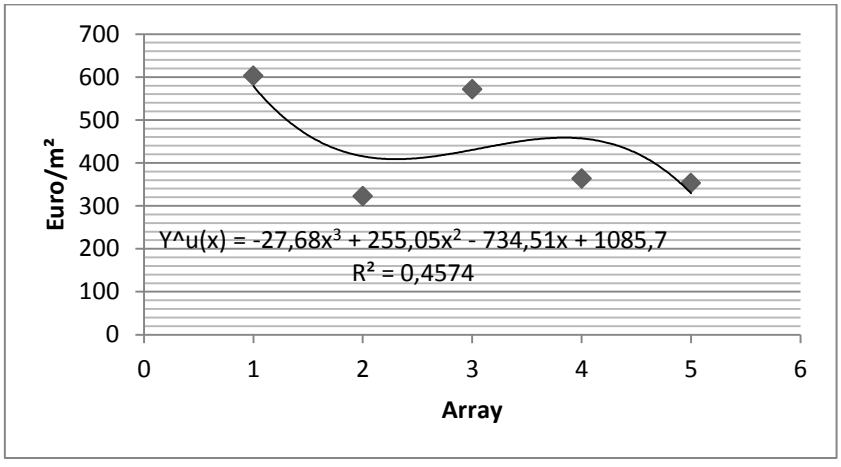
With regard to $\{[558,602],\ [298,322],\ [529,571],\ [337,363],\ [327,353]\}$:

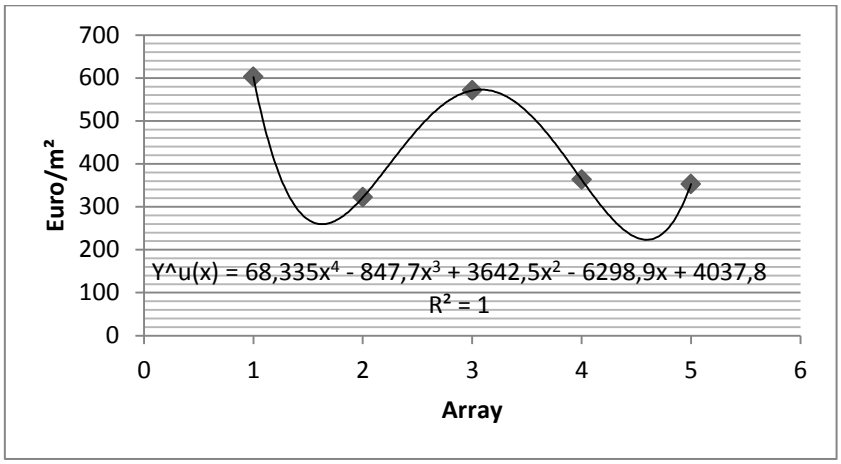





Euro/m²
Array
Y^l(x) = 63,332x⁴ - 785,63x³ + 3375,8x² - 5837,7x + 3742,2
R² = 1


Euro/m²
Array
Y^u(x) = 5,9314x² - 81,261x + 620,72
R² = 0,3015


Euro/m²
Array
Y^u(x) = -27,68x³ + 255,05x² - 734,51x + 1085,7
R² = 0,4574



3. Résumé and outlook

In this paper minimal elements with reference to primal problems are examined for the set less order relation. This leads to the set-valued and interval-valued duality and regression model. This combines minimization and maximization problems with follow-on regression and is explained here as an example of a decision-making strategy for the determination of land values in the Federal Republic of Germany. The application of regression requires that the land values belong to a dual structure with a set less order relation.

This paper refutes the current usual practice of public administrations in Germany of determining land values in land value zones (conflict situations with socio-economic influences) in the sense of a deduction. Instead, the examination of a weak duality with regression taking into account set-valued order relations in order to extract a result as an interval is shown to be the correct approach (see upper and lower regression polynomials). Together, these describe an optimal area (zone).

References

[1] G. Alefeld, J. Herzberger, *Introduction to Interval Computations*, Academic Press, 1983.

[2] M.P.F. Berger, W.-K. Wong, *An Introduction to Optimal Designs for Social and Biomedical Research*, John Wiley & Sons, 2009. https://doi.org/10.1002/9780470746912

[3] A. Chiriaev, G.W. Walster, *Interval Arithmetic Specification*, Technical Report, 1998.

[4] Destatis.de, called up on September 16, 2016.

[5] Deutscher Städtemarkt, Immobilienmarkt in großen deutschen Städten, Arbeitskreis der Gutachterausschüsse in der Bundesrepublik Deutschland, 2009.

[6] T.X.D. Ha, J. Jahn, New Order Relations in Set Optimization, *J. Optim. Theory Appl.*, **148** (2011), 209-236. https://doi.org/10.1007/s10957-010-9752-8

[7] IVD Bundesverband, IVD-Wohn-Preisspiegel 2008/2009, IMMO Ideen GmbH für Immobilienverband Deutschland IVD Bundesverband der Immobilienberater, 2009.

[8] D. Kuroiwa, T. Tanaka, T.X.D. Ha, On Cone Convexity of Set-Valued Maps, *Nonlinear Analysis: Theory, Methods & Applications,* **30** (1997), 1487-1496. https://doi.org/10.1016/s0362-546x(97)00213-7

[9] R.E. Moore, R.B. Kearfott, M.J. Cloud, *Introduction to Interval Analysis*, SIAM, 2009. https://doi.org/10.1137/1.9780898717716

[10] N. Neukel, *Dualitätskonzepte und Sozio-ökonomische Paradigmen mit Ordnungsrelationen zur Mengenoptimierung*, Dissertation, FAU Erlangen-Nuremberg, 2016.

[11] N. Neukel, Order Relations of Sets and its Application in Socio-Economics, *Appl. Math. Sciences*, **7** (2013), no. 115, 5711-5739. https://doi.org/10.12988/ams.2013.37419

[12] Z.G. Nishnianidze, Fixed Points of Monotonic Multiple-Valued Operators, *Bull. Georgian Acad. Sci.*, **114** (1984), 489-491.

[13] C. Nytsch-Geusen, W. Kaul, P. Wehage, M. Färber, *Nachhaltige und Energieeffiziente Bestandsentwicklung von Stadtquartieren – Entwicklung Eines Analyse- und Planungstools zur Energetischen Bewertung und Sanierung von Stadtquartieren (Open eQuarter),* Forschungsinitiative Zukunft Bau, 2016.

[14] SUN Microsystems, Inc., Interval Arithmetic Programming Reference, 2000.

[15] H.-C. Wu, Duality Theory in Interval-Valued Linear Programming Problems, *J. Optim. Theory Appl.*, **150** (2011), 298-316. https://doi.org/10.1007/s10957-011-9842-2

[16] R.C. Young, The Algebra of Many-Valued Quantities, *Math. Ann.*, **104** (1931), 260-290. https://doi.org/10.1007/bf01457934

Received: February 3, 2017; Published: March 19, 2017