Characterization of Continuous Distributions
Conditioned on a Pair of Non-Adjacent Generalized
Order Statistics Using Meijer’s $G$ Function

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Abstract
In this paper, a generalized family of continuous distributions have been characterized through the difference of $p^{th}$, ($p \geq 1$) power of two generalized order statistics ($gos$) conditioned on a pair of two non-adjacent $gos$ using Meijer’s $G$-function. The parallel result for records are also deduced. Further, by examples it has been shown that how these result can be used to characterize a family of distributions.

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1 Introduction

Kamps [17] introduced the concept of generalized order statistics ($gos$) which contains the important models of ordered random variables. i.e. order statis-
order statistics, record values, sequential order statistics, progressively type II censored order statistics and order statistics with non-integral sample size. These models can be effectively applied in reliability theory and survival analysis. Let $n \in \mathbb{N}$, $k \geq 1$, $\tilde{m} = (m_1, m_2, ..., m_{n-1}) \in \mathbb{R}^{n-1}$, $M_r = \sum_{j=r}^{n-1} m_j$, be the parameters such that $\gamma_r = k + n - r + M_r \geq 0$, for all $r \in \{1, ..., n - 1\}$. Then $U(1, n, \tilde{m}, k), U(2, n, \tilde{m}, k), \ldots, U(n, n, \tilde{m}, k)$ are said to be uniform generalized order statistics (gos), if their joint probability density function (pdf) can be written as

$$f_{U(1, n, \tilde{m}, k), \ldots, U(n, n, \tilde{m}, k)}(u_1, ..., u_n) = k \left(\prod_{j=1}^{n-1} \gamma_j\right) \left(\prod_{i=1}^{n-1} (1 - u_i)^{m_i}\right) (1 - u_n)^{k-1}$$

(1.1)
on the cone $0 < u_1 \leq \ldots \leq u_n < 1$

Based on any arbitrary distribution function (df) $F(x)$, generalized order statistics $X(r, n, \tilde{m}, k)$ can be defined by quantile transformation $X(r, n, \tilde{m}, k) = F^{-1}(U(r, n, \tilde{m}, k))$, $1 \leq r \leq n$, where $F^{-1}$ denotes the quantile function of $F$ defined by

$$F^{-1}(u) = \sup\{x \in (\alpha, \beta) : F(x) \leq u\}, u \in (0, 1)$$

where $\alpha = \inf\{x \in \mathbb{R} : F(x) > 0\}$ and $\beta = \sup\{x \in \mathbb{R} : F(x) < 1\}$ are the left and right end points of $X$. Choosing the parameters appropriately, models such as ordinary order statistics ($m = 0, k = 1$ i.e. $\gamma_i = n - i + 1$), $k^{th}$ record value ($m = -1, k \in \mathbb{N}$ i.e. $\gamma_i = k$), sequential order statistics [$\gamma_i = (n - i + 1)\beta_1; \beta_1, \beta_2, ..., \beta_n > 0$], order statistics with non-integral sample size [$\gamma_i = (\beta - i + 1); \beta > 0$], Pfeifer record values ($\gamma_i = \beta_i; \beta_1, \beta_2, ..., \beta_n > 0$) and progressive type II censored order statistics ($m \in \mathbb{N}, k \in \mathbb{N}$) can be obtained as particular cases of gos.

Let $P_F$ stands for the probability measure on $\mathbb{R}$ determined by $F(x)$, then the pdf of $X(r, n, \tilde{m}, k)$ with respect to a measure $P_F$ is given by Cramer and Kamps[7].

$$f_r(x) = c_{r-1}G_r(\bar{F}(x) | \gamma_1, ..., \gamma_r)I_{(\alpha, \beta)}(x)$$

(1.2)
where $\bar{F}(x) = 1 - F(x)$, $c_{r-1} = \prod_{i=1}^{r} \gamma_i$ and $I_A$ denotes the indicator function and

$$G_r(x) = G_{r,r}^0(x | \gamma_1, ..., \gamma_r) = G_{r,r}^0\left(x \mid (\gamma_1, ..., \gamma_r)\right)$$

is the particular Meijer’s G-function defined by

$$G_{r,r}^0\left(x \mid \begin{pmatrix} \gamma_1, ..., \gamma_r \\ \gamma_1 - 1, ..., \gamma_r - 1 \end{pmatrix}\right) = \frac{1}{2\pi i} \int_L \frac{s^z}{\prod_{j=1}^{r} (\gamma_j - 1 - z)} dz$$

(1.3)
Characterization of continuous distributions

and $L$ is an appropriate chosen contour of integration (See chapter 3 Mathai[1]) for the definition of G-function and its numerous properties and applications. For any $x, y < \beta$, denote

$$F_x(y) = \begin{cases} \frac{F(y) - F(x)}{1 - F(x)} & y \geq x \\ 0 & y < x \end{cases}$$  

Here $F_x(y)$ denotes the distribution function obtained from $F$ by truncating on the left at $x$.

The joint $\bigotimes_{i=1}^2 P_{\bar{F}}$ density of $X(r, n, \tilde{m}, k)$ and $X(s, n, \tilde{m}, k)$, $1 \leq r < s \leq n$, is given by

$$f_{r,s}(x, y) = c_{s-1}G_{s-r}(\bar{F}_y | \gamma_{r+1}, \ldots, \gamma_s) \frac{G_r(\bar{F}_x | \gamma_1, \ldots, \gamma_r)I_{(\alpha, \beta)}(x < y)}{F(x)}$$  

and the joint $\bigotimes_{i=1}^3 P_{\bar{F}}$ density of $X(r, n, \tilde{m}, k), X(j, n, \tilde{m}, k)$ and $X(s, n, \tilde{m}, k)$, $1 \leq r < j < s \leq n$ may similarly be given as

$$f_{r,j,s}(x, t, y) = c_{s-1} \frac{1}{F(x)} \frac{1}{F(t)} G_{s-j}(\bar{F}_t | \gamma_{j+1}, \ldots, \gamma_s) G_{j-r}(\bar{F}_s | \gamma_{r+1}, \ldots, \gamma_j) \times G_r(\bar{F}_x | \gamma_1, \ldots, \gamma_r)I_{(\alpha, \beta)}(x < t < y)$$

(1.6)

Hence the conditional $P_{\bar{F}}$ density function of $X(j, n, \tilde{m}, k)$ given $X(r, n, \tilde{m}, k) = x$ and $X(s, n, \tilde{m}, k) = y$, $1 \leq r < j < s \leq n$ is given by

$$f_{j|r,s}(t | x, y) = \frac{1}{F(t)} G_{s-j}(\bar{F}_t | \gamma_{j+1}, \ldots, \gamma_s) G_{j-r}(\bar{F}_s | \gamma_{r+1}, \ldots, \gamma_j) I_{(x,y)}(t)$$

(1.7)

Characterization of continuous distribution conditioned on a single $gos$ was considered by several authors. Based on $gos$, Keseling [5] characterized the continuous distributions by taking the conditional expectations $E[h(X(r + 1, n, \tilde{m}, k)) | X(r, n, \tilde{m}, k) = x]$, where $h(.)$ is a real strictly monotonic function. Keseling[5] also characterized the continuous distributions by taking the conditional expectations $E[X(r, n, \tilde{m}, k)) | X(r + 2, n, \tilde{m}, k) = x]$. Bieniek and Syzna[10] investigated the characterization of the continuous distributions by considering the conditional expectations $E[X(r + l, n, \tilde{m}, k)) | X(r, n, \tilde{m}, k) = x], l \geq 2$. Characterization of continuous distributions conditioned on nonadjacent $gos$ was considered by Cramer et al.[8], Khan and Alzaid[2], Raqab and Abu-Law[11], Ahsanullah and Raqab[12], Khan et al.[3]. In these papers, the authors are mainly concerned in finding the distribution function when the regression lines are linear. Bieniek[14], Khan and Khan[4] have characterized the
continuous distribution functions conditioned on non-adjacent gos using Mei-
jer’s G-function. Later on Khan et al.[16] extended the result of Bieniek[14], Khan and Khan[4] and characterized the continuous distributions conditioned on a pair of nonadjacent gos, i.e. by taking the conditional expectation

\[ E[h\{X(j, n, \tilde{m}, k)\} | X(r, n, \tilde{m}, k) = x, X(s, n, \tilde{m}, k) = y], \quad 1 \leq r < j < s \leq n. \]

Here \( h(x) \) is considered as monotonic and differentiable function of \( x \). Other authors who have worked in this direction and characterized the continuous distributions conditioned on a pair of nonadjacent gos are Ahsanullah and Beg[13] and Ahsanullah et al.[15]. Recently Zubdah e Noor et al.[18] characterized the continuous distributions by taking the conditional expectation

\[ g_{r,s,p} = E[\{\psi(X(s, n, \tilde{m}, k)) - \psi(X(r, n, \tilde{m}, k))\}^p | X(r, n, \tilde{m}, k) = x] \]

In this paper, motivated by the work of Zubdah e Noor et al.[18], we have investigated the characterization of the continuous distribution functions by considering the conditional expectation

\[ g_{r,s}^p(x, y) = E[\{\psi(X(j, n, \tilde{m}, k)) - \psi(X(r, n, \tilde{m}, k))\}^p | X(r, n, \tilde{m}, k) = x, X(s, n, \tilde{m}, k) = y], \quad (1.8) \]

and

\[ \xi_{r,s}^p(x, y) = E[\{\psi(X(s, n, \tilde{m}, k)) - \psi(X(j, n, \tilde{m}, k))\}^p | X(r, n, \tilde{m}, k) = x, X(s, n, \tilde{m}, k) = y], \quad (1.9) \]

where \( 1 \leq r < j < s \leq n, \quad p \geq 1 \), using Meijer’s G-function. Further throughout the paper, we have assumed that \( \psi : \mathbb{R} \to \mathbb{R} \) is strictly increasing function. Also it is being assumed that \( g_{r,s}^p(x, y) \) and \( \xi_{r,s}^p(x, y) \) are finite and differentiable function of \( x \) and \( y \) respectively.

This paper is divided into three section. In section 2, we have derived the characterization result based on \( g_{r,s}^p(x, y) \) and \( \xi_{r,s}^p(x, y) \) respectively. Section 3 consists the examples based on the results given in section 2.

## 2 Characterization of distributions

In this section, we have characterized the continuous distributions based on the \( p^{th} \), (\( p \geq 1 \)) power of the difference of two gos conditioned on a pair of two non-adjacent gos. The characterization result deduced in this section is more generalized in the sense that at \( p = 1 \), we get the characterization result based on conditional expectation i.e through regression equation, conditioned on a pair of nonadjacent gos, while at \( p = 2 \), we have the characterization result
based on conditional second moments conditioned on a pair of nonadjacent gos. Before the proof of the main results, some auxiliary results are given which are used in the proof of the main results.

**Auxiliary Results**

Various results which are used in the subsequent sections are reproduced here:

(i) \( x^\alpha G_r(x \mid \gamma_1, \ldots, \gamma_r) = G_r(x \mid \gamma_1 + \alpha, \ldots, \gamma_r + \alpha) , a \in R \)

(ii) \( \lim_{x \to 1^-} G_r(x \mid \gamma_1, \ldots, \gamma_r) = \begin{cases} 1, & r = 1 \\ 0, & r \geq 2 \end{cases} \)

and

\[
\lim_{x \to 0^+} G_r(x \mid \gamma_1, \ldots, \gamma_r) = \begin{cases} 0, & \text{if } \gamma_{1:r} > 1 \\ \prod_{j=1}^{r} \frac{1}{(\gamma_j - \gamma_i)}, & \text{if } \gamma_{1:r} = 1 < \gamma_2 \\ \infty, & \text{if } \gamma_{1:r} = \gamma_2 < 1 \end{cases}
\]

where \( \gamma_{1:r} = \min(\gamma_1, \ldots, \gamma_r) \) and \( l = \max(1 \leq j \leq r : \gamma_j = \gamma_{1:r}) \)

(iii) \( \frac{d}{dx} G_r(x \mid \gamma_1, \ldots, \gamma_r) = \frac{1}{x} [(\gamma_r - 1)G_r(x \mid \gamma_1, \ldots, \gamma_r) - G_{r-1}(x \mid \gamma_1, \ldots, \gamma_{r-1})] \)

(iv) \( \frac{d}{dx} G_r(x \mid \gamma_1, \ldots, \gamma_r) = \frac{1}{x} [(\gamma_1 - 1)G_r(x \mid \gamma_1, \ldots, \gamma_r) - G_{r-1}(x \mid \gamma_2, \ldots, \gamma_r)] \)

**Proof:** For the property (i) see Mathai (p. 69)[1]. Property (ii) can easily be deduced from Lemma 2.2 of Cramer et al.[9], whereas property (iii) and property (iv) can be established directly from (1.3).

**Theorem 2.1:** Let \( X(i,n,\tilde{m},k) \), \( i = 1, \ldots, n \) be the \( i^{th} \) gos from a continuous population with df \( F(x) \) and the pdf \( f(x) \) over the support \((\alpha, \beta)\). Let \( \psi : \mathbb{R} \to \mathbb{R} \) is strictly increasing function and \( E[|\psi(X(l,n,\tilde{m},k)) - \psi(X(l,n,\tilde{m},k))|^p] < \infty \), for \( l = r, r+1 \). If for two consecutive values \( r \) and \( (r+1) \), \( 1 < r + 1 < j \leq s \leq n \),

\[
E[|\psi(X(j,n,\tilde{m},k)) - \psi(X(l,n,\tilde{m},k))|^p] \mid X(l,n,\tilde{m},k) = x, X(s,n,\tilde{m},k) = y = g_r^p(x,y)
\]

\( l = r, r+1, \) (2.1)

exist, then

\[
(\gamma_{r+1} - 1) f(x) \frac{d}{dx} G_{s-r}(F_x(y) \mid \gamma_{r+1}, \ldots, \gamma_s) = \frac{p\psi'(x)g_{r,s}^{p-1}(x,y) + \frac{d}{dx} g_{r,s}(x,y)}{[g_{r,s}(x,y) - g_{r+1,s}(x,y)]} \]

and

\[
\frac{G_{s-r}(F_x(y) \mid \gamma_{r+1} - \gamma_{r+1} + 1, \ldots, \gamma_s - \gamma_{r+1} + 1)}{G_{s-r}(F(y) \mid \gamma_{r+1} - \gamma_{r+1} + 1, \ldots, \gamma_s - \gamma_{r+1} + 1)} = \exp \left( - \int_{\alpha}^{x} D_1(t,y)dt \right) \tag{2.2}
\]
where \( g_{r,s}^p \) is a finite and differentiable function of \( x \) and

\[
D_1(x, y) = \frac{p \psi'(x) g_{r,s}^{p-1}(x, y) + \frac{d}{dx} g_{r,s}^p(x, y)}{[g_{r,s}^p(x, y) - g_{r+1,s}^p(x, y)]}
\]  

(2.3)

**Proof:** We have

\[
g_{r,s}^p(x, y) G_{s-r}(\bar{F}_x(y) \mid \gamma_{r+1}, \ldots, \gamma_s) = \int_x^y \frac{[\psi(t) - \psi(x)]^p}{F(t)} G_{s-j}(\bar{F}_x(y) \mid \gamma_{j+1}, \ldots, \gamma_s) 
\times G_{j-r}(\bar{F}_x(t) \mid \gamma_{r+1}, \ldots, \gamma_j) f(t) dt
\]  

(2.4)

Differentiating both the sides w.r.to \( x \) and using the property (iv), we have

\[
\frac{d}{dx} g_{r,s}^p(x, y) G_{s-r}(\bar{F}_x(y) \mid \gamma_{r+1}, \ldots, \gamma_s) + g_{r,s}^p(x, y)(\gamma_{r+1} - 1) G_{s-r}(\bar{F}_x(y) \mid \gamma_{r+1}, \ldots, \gamma_s) F(x)
\]

\[- g_{r,s}^p(x, y) G_{s-r-1}(\bar{F}_x(y) \mid \gamma_{r+2}, \ldots, \gamma_s) F(x) = -p \psi'(x) g_{r,s}^{p-1}(x, y) G_{s-r}(\bar{F}_x(y) \mid \gamma_{r+1}, \ldots, \gamma_s)
\]

\[+ g_{r,s}^p(x, y)(\gamma_{r+1} - 1) G_{s-r}(\bar{F}_x(y) \mid \gamma_{r+1}, \ldots, \gamma_s) \frac{f(x)}{F(x)} - g_{r+1,s}^p(x, y) G_{s-r-1}(\bar{F}_x(y) \mid \gamma_{r+2}, \ldots, \gamma_s) \frac{f(x)}{F(x)}
\]  

(2.5)

After rearranging the terms in (2.5), we get

\[
\frac{f(x)}{F(x)} G_{s-r-1}(\bar{F}_x(y) \mid \gamma_{r+2}, \ldots, \gamma_s) \div G_{s-r}(\bar{F}_x(y) \mid \gamma_{r+1}, \ldots, \gamma_s) = \frac{p \psi'(x) g_{r,s}^{p-1}(x, y) + \frac{d}{dx} g_{r,s}^p(x, y)}{[g_{r,s}^p(x, y) - g_{r+1,s}^p(x, y)]}
\]

implying that

\[
(\gamma_{r+1} - 1) \frac{f(x)}{F(x)} \div \frac{d}{dx} G_{s-r}(\bar{F}_x(y) \mid \gamma_{r+1}, \ldots, \gamma_s) \div G_{s-r}(\bar{F}_x(y) \mid \gamma_{r+1}, \ldots, \gamma_s) = \frac{p \psi'(x) g_{r,s}^{p-1}(x, y) + \frac{d}{dx} g_{r,s}^p(x, y)}{[g_{r,s}^p(x, y) - g_{r+1,s}^p(x, y)]}
\]

and hence the result.

**Corollary 2.1:** Using the theorem of residue’s under the condition that \( \gamma_i \neq \gamma_j, \forall\ i, j = r + 1, \ldots, s \) \( i \neq j \), it can be proved that Cramer(p. 36)[6]

\[
G_{s-r}(\bar{F}_x(y) \mid \gamma_{r+1}, \ldots, \gamma_s) = \left( \sum_{i=r+1}^s a_i^{(r)}(s) \bar{F}_x(y)^{\gamma_i-1} \right)
\]

(2.6)

where \( a_i^{(r)}(s) = \prod_{j=r+1}^s \frac{1}{(\gamma_j - \gamma_i)} \), \( \gamma_j \neq \gamma_i, r + 1 \leq i \leq s \leq n \).

Therefore in this case, (2.2) reduces to

\[
\left[ \bar{F}(x) \right]^{\gamma_{r+1}} B_s^r(x, y) = e^{\exp \left( - \int \frac{x}{\alpha} D_1(t, y) dt \right)}.
\]

(2.7)
where
\[ B_i^r(x, y) = \sum_{i=r+1}^{s} a_i^{(r)}(s) [F_x(y)]^{\gamma_i} \]

Also since for \( m_1 = ... = m_{n-1} = m \neq -1, \)
\[ a_i^{(r)}(s) = \frac{1}{\prod_{j=r+1}^{s} (\gamma_j - \gamma_i)} = \frac{(-1)^{s-i}}{(m+1)^{s-r-1} \Gamma(s-r-1)} \left( \frac{s-r-1}{s-i} \right) \]

thus for \( m_1 = ... = m_{n-1} = m \neq -1, \) (2.2) reduces to
\[ \frac{1 - \{\bar{F}(x)\}^{m+1}}{1 - \{\bar{F}(y)\}^{m+1}} = 1 - \exp \left[ -\frac{1}{(s-r-1)} \int_{\alpha}^{x} D_1(t, y) dt \right], \quad m \neq -1 \quad (2.8) \]

and when \( \gamma_{r+1} = \gamma_{r+2} = ... = \gamma_s \) i.e. in case of record statistics see Cramer (p. 35)[6]
\[ G_{s-r}(\bar{F}_x(y) | \gamma_{r+1}, ..., \gamma_s) = \frac{1}{\Gamma(s-r-1)} \left[ -\log \bar{F}(y) + \log \bar{F}(x) \right]^{s-r-1} [F_x(y)]^{\gamma_{r+1}-1} \]

Thus (2.2) reduces to
\[ \frac{\log \{\bar{F}(x)\}}{\log \{\bar{F}(y)\}} = 1 - \exp \left[ -\frac{1}{(s-r-1)} \int_{\alpha}^{x} D_1(t, y) dt \right], \quad m = -1 \quad (2.10) \]

**Theorem 2.2:** Let \( X(i, n, \bar{m}, k), i = 1, 2, \ldots, n \) be the \( i^{th} \) gos from a continuous population with \( df F(x) \) and the pdf \( f(x) \) over the support \( (\alpha, \beta) \). Suppose \( \psi : \mathbb{R} \rightarrow \mathbb{R} \) is strictly increasing function and \( E[|\psi(X(l, n, \bar{m}, k)) - \psi(X(j, n, \bar{m}, k))|] = \infty \), for \( l = s - 1, s \). If for two consecutive values \( s - 1 \) and \( s, 1 \leq r + 1 < j < s - 1 < n, \)
\[ E[|\psi(X(l, n, \bar{m}, k)) - \psi(X(j, n, \bar{m}, k))|] = \xi_{r,s}(x, y) \quad l = s - 1, s \quad (2.11) \]
then
\[ (\gamma_s - 1) \frac{f(y)}{\bar{F}(y)} = \frac{d}{dy} G_{s-r}(\bar{F}_x(y) | \gamma_{r+1}, ..., \gamma_s) \left[ \frac{\xi_{r,s}(x, y) - \psi(y) \xi_{r,s-1}(x, y) - p \psi(y) \xi_{r,s-1}(x, y)}{\xi_{r,s}(x, y) - \xi_{r,s-1}(x, y)} \right] \]
The following two cases will arrive:
**Case (i):** when \( \min(\gamma_{r+1}, \gamma_{r+2}, ..., \gamma_{s-1}) > \gamma_s \), then
\[ G_{s-r}(\bar{F}_x(y) | \gamma_{r+1} - \gamma_s + 1, ..., \gamma_s - \gamma_s + 1) = a_s^{(r)}(s) \exp \left[ -\int_{y}^{\beta} D_2(x, t) dt \right] \quad (2.12) \]
Case (ii): when \( \min(\gamma_{r+1}, \gamma_{r+2}, \ldots, \gamma_{s-1}) \leq \gamma_s \), then the characterizing result is

\[
G_{s-r}(\tilde{F}_x(y) \mid \gamma_{r+1} - \gamma_s + 1, \ldots, \gamma_s - \gamma_s + 1) = \exp \left[ - \int_y^q D_2(x, t)dt \right],
\]

(2.13)

where \( q \) is defined as

\[
q = \inf \{ z \in (\alpha, \beta) : G_{s-r}(F_x(z) \mid \gamma_{r+1} - \gamma_s + 1, \ldots, \gamma_s - \gamma_s + 1) \geq 1 \},
\]

(2.14)

and

\[
D_2(x, y) = \frac{d}{dy} \xi_{r,s}^p(x, y) - p\psi'(y)\xi_{r,s}^{p-1}(x, y) - \frac{\xi_{r,s}^p(x, y) - \xi_{r,s}^{p-1}(x, y)}{\xi_{r,s}^p(x, y) - \xi_{r,s}^{p-1}(x, y)}.
\]

(2.15)

**Proof:** We have

\[
G_{s-r}(\tilde{F}_x(y) \mid \gamma_{r+1}, \ldots, \gamma_s) = \int_x^y \left[ \psi(y) - \psi(t) \right]^p \frac{G_{s-j}(F_t(y) \mid \gamma_{j+1}, \ldots, \gamma_s)}{F(t)} G_{s-r}(\tilde{F}_x(y) \mid \gamma_{r+1}, \ldots, \gamma_s) \times G_{j-r}(F_t(t) \mid \gamma_{r+1}, \gamma_j) f(t)dt
\]

(2.16)

Similarly we proceed like theorem (2.1) on differentiating both the sides w.r.t. to \( y \) and using the property, we get implying that

\[
(\gamma_s - 1) \frac{f(y)}{F(y)} - \frac{d}{dy} \frac{G_{s-r}(\tilde{F}_x(y) \mid \gamma_{r+1}, \ldots, \gamma_s)}{G_{s-r}(\tilde{F}_x(y) \mid \gamma_{r+1}, \ldots, \gamma_s)} = \frac{d}{dy} \frac{\xi_{r,s}^p(x, y) - p\psi'(y)\xi_{r,s}^{p-1}(x, y)}{\xi_{r,s}^p(x, y) - \xi_{r,s}^{p-1}(x, y)}.
\]

It can be seen that when \( \min(\gamma_{r+1}, \gamma_{r+2}, \ldots, \gamma_{s-1}) > \gamma_s \),

\[
G_{s-r}(x \mid \gamma_{r+1} - \gamma_s + 1, \ldots, \gamma_s - \gamma_s + 1) \to a_s^{(r)}(s) \text{ as } x \to 0,
\]

and therefore

\[
G_{s-r}(\tilde{F}_x(y) \mid \gamma_{r+1} - \gamma_s + 1, \ldots, \gamma_s - \gamma_s + 1) = a_s^{(r)}(s) \exp \left[ - \int_y^\beta D_2(x, t)dt \right].
\]

Further when \( \min(\gamma_{r+1}, \gamma_{r+2}, \ldots, \gamma_{s-1}) \leq \gamma_s \), in this case

\[
G_{s-r}(x \mid \gamma_{r+1} - \gamma_s + 1, \ldots, \gamma_s - \gamma_s + 1) \to \infty \text{ as } x \to 0 \text{ (see lemma 2.2: Cramer et al. [9]).}
\]

Thus in this case, the characterization result is

\[
G_{s-r}(\tilde{F}_x(y) \mid \gamma_{r+1} - \gamma_s + 1, \ldots, \gamma_s - \gamma_s + 1) = \exp \left[ - \int_y^q D_2(x, t)dt \right], \text{ where } q \text{ is defined in (2.14).}
\]

**Corollary 2.2:** It may be noted that at \( \gamma_i \neq \gamma_j \) but \( m_1 = \ldots = m_{n-1} = m > -1 \)

\[
\frac{G_{s-r}(\tilde{F}_x(y) \mid \gamma_{r+1} - \gamma_s + 1, \ldots, \gamma_s - \gamma_s + 1)}{a_s^{(r)}(s)} = \sum_{i=r+1}^s a_i^{(r)}(s)(\tilde{F}_x(y))^{\gamma_i - \gamma_j} = [1 - (\tilde{F}_x(y))^{m+1}]^{s-r-1}
\]

Therefore it reduces to

\[
[\tilde{F}_x(y)]^{m+1} = 1 - \exp \left[ - \frac{1}{(s - r - 1)} \int_y^\beta D_2(x, t)dt \right], m > -1.
\]

(2.17)
Also for $\gamma_{r+1} = \ldots = \gamma_s$, i.e. in case of records and using (2.9), (2.2) reduces to

$$\frac{1}{1 + \log \bar{F}(y)} = 1 - \exp \left[ - \frac{1}{(s - r - 1)} \int_y^q D_2(x, t) dt \right]$$

(2.18)

where $q$ is as defined as $q = \inf \{ z \in (\alpha, \beta) : z \geq F^{-1} \left( \frac{e-1}{e} \right) \}$.

**Corollary 2.3:** Under the assumptions given in Corollary 2.1 and Corollary 2.2,

$$\bar{F}(x) = \left[ \frac{e^{I_1} I_1}{e^{I_1} + e^{I_2} - 1} \right] \frac{1}{m+1}, m > -1$$

(2.19)

and

$$\bar{F}(y) = \left[ \frac{e^{I_2} - 1}{e^{I_1} + e^{I_2} - 1} \right] \frac{1}{m+1}, m > -1$$

(2.20)

where $I_1 = \int_\alpha^x A_1(t, y) dt$, $I_2 = \int_y^\beta A_2(x, t) dt$ and $A_1(x, y) = \frac{D_1(x, y)}{(s - r - 1)}$, $A_2(x, y) = \frac{D_2(x, y)}{(s - r - 1)}$.

Similarly for records, we have

$$\bar{F}(x) = \exp \left[ - \frac{e^{I_1} - 1}{e^{I_1} + e^{I_2} - 1} \right], m = -1$$

(2.21)

and

$$\bar{F}(y) = \exp \left[ - \frac{e^{I_1}}{e^{I_1} + e^{I_2} - 1} \right], m = -1$$

(2.22)

### 3 Examples

In this section, a continuous distribution is characterized under the condition stated in Corollary 2.1 and Corollary 2.2 respectively. Based on $g_{p, r,s}(x, y)$ and $\psi_{p, r,s}(x, y)$, these result can be utilized to get the answer that from which distribution the sample is being obtained. Also any intermediate $gos$ can be predicted if we have information about $X(r, n, m, k)$ and $X(s, n, m, k), \ r < s$ for a family of distribution given in (3.2) and (3.8) respectively.

(i): For $m_1 = \ldots = m_{n-1} = m \geq -1$

$$g_{r,s}(x, y) = [\psi(y) - \psi(x)]^p \frac{\Gamma(s - r) \Gamma(p + j - r)}{\Gamma(j - r) \Gamma(p + s - r)}$$

(3.1)

If and only if

$$1 - \{ \bar{F}(x) \}^{m+1} = a\psi(x) + b, \ m > -1$$

(3.2)
where $F(x)$ is so chosen that $a\psi(\beta) + b = 1$. And for record values
\[ F(x) = 1 - \exp^{[a\psi(x) + b]}, \quad m = -1 \quad (3.3) \]
Provided that there exist a $q \in (\alpha, \beta)$ such that $ahq + b = 1$.

**Proof:** For $m_1 = \ldots = m_{n-1} = m \neq -1$, the value of $g^p_{r,s}(x, y)$ is given by
\[
g^p_{r,s}(x, y) = C_{r,j,s}(m+1) \int_x^y \frac{[\psi(t) - \psi(x)]^p}{[F(x)^{m+1} - F(y)^{m+1}]} \left[ 1 - \frac{[\bar{F}(x)^{m+1} - \bar{F}(t)^{m+1}]}{[\bar{F}(x)^{m+1} - \bar{F}(y)^{m+1}]} \right]^{s-j-1} \times \left[ \frac{[\bar{F}(x)^{m+1} - \bar{F}(t)^{m+1}]}{[\bar{F}(x)^{m+1} - \bar{F}(y)^{m+1}]} \right]^{j-r-1} \psi'(t)dt, \quad (3.4)\]
Now to prove the necessary part i.e. (3.2) implies (3.1), we have
\[
g^p_{r,s}(x, y) = C_{r,j,s}(m+1) \int_x^y \frac{[\psi(t) - \psi(x)]^p}{[\psi(y) - \psi(x)]} \left[ 1 - \frac{[\psi(t) - \psi(x)]}{[\psi(y) - \psi(x)]} \right]^{s-j-1} \times \left[ \frac{[\psi(t) - \psi(x)]}{[\psi(y) - \psi(x)]} \right]^{j-r-1} \psi'(t)dt \quad (3.5)\]
implying that
\[
g^p_{r,s}(x, y) = [\psi(y) - \psi(x)]^p \frac{\Gamma(s-r)\Gamma(p+j-r)}{\Gamma(j-r)\Gamma(p+s-r)} \]
Now to prove the necessary part i.e. (3.2) implies (3.1), we have
\[
A_1(x, y) = \frac{p\psi'(x)g^p_{r,s-1}(x, y) + \frac{d}{dx}g^p_{r,s}(x, y)}{(s-r-1) \left[ g^p_{r,s}(x, y) - g^p_{r+1,s}(x, y) \right]} = \frac{\psi'(x)}{[\psi(y) - \psi(x)]},
\]
Now in view of (2.8), we get
\[
1 - \{\bar{F}(x)^{m+1} - \bar{F}(y)^{m+1}\} = \frac{a\psi(x) + b}{a\psi(y) + b}, \quad m \neq -1 \quad (3.6)
\]
The solution of (3.6) is
\[
1 - \{\bar{F}(x)^{m+1} = K[a\psi(x) + b],
\]
where $K$ is constant of integration. Thus as $x \to \beta$, $a\psi(x) + b \to 1$, and the value of $K$ is one and hence the sufficient part. Similarly we can prove the result for records.
(ii): For \( m_1 = \ldots = m_{n-1} = m \geq -1 \),
\[
\xi_{r,s}^p(x,y) = [\psi(y) - \psi(x)]^p \frac{\Gamma(s - r) \Gamma(p + s - j)}{\Gamma(s - j) \Gamma(p + s - r)}
\] (3.7)

If and only if
\[
1 - \{\bar{F}(x)\}^{m+1} = a\psi(x) + b, \ m > -1
\] (3.8)

here \( F(x) \) is so chosen that \( a\psi(\alpha) + b = 1 \).

And for records
\[
F(x) = 1 - \exp[a^h(x) + b], \ m = -1
\] (3.9)

Provided that there exists a \( q \in (\alpha, \beta) \) such that \( a\psi(q) + b = 1 \).

**Proof:** We similarly proceed like example(i) Now in view of (2.17), we get
\[
\frac{\{\bar{F}(y)\}^{m+1}}{\{F(x)\}^{m+1}} = \frac{a\psi(y) + b}{a\psi(x) + b}, \ m \neq -1
\] (3.10)

The solution of (3.10) is
\[
\{\bar{F}(x)\}^{m+1} = K[a\psi(x) + b],
\]
where \( K \) is constant of integration. Thus as \( x \to \alpha, \ a\psi(x) + b \to 1 \), and the value of \( K \) is one and hence the sufficient part. Similarly we can prove the result for records.

## 4 Conclusion

In real problem, a statistician is often interested in guessing the distribution from which the true data is obtained. Characterization problem is theoretical approach to obtain the distribution function if certain condition is fulfilled. A probability distribution can be characterized in many ways and the method under study here is one of them. We have used here the conditional expectation of generalized order statistics conditioned on a pair of non-adjacent generalized order statistics, i.e. We have used the regression equation which is truncated at both sides, left side at point \( x \) and right side at point \( y \). In many real problems, we find the data which is truncated from both ends. In those cases, we can use the result obtained in this paper to get the answer that from which population the sample is drawn. Keeping this in view, we have characterized the probability distributions through the difference of \( p^{th}, \ (p \geq 1) \) power of two generalized order statistics (gos) conditioned on a pair of two non-adjacent gos using Meijer’s G-function. Further this result can be utilized
in case of ordinary order statistics, record values, sequential order statistics, order statistics with non integral sample size, Pfeifer record values and progressive type II censored order statistics.

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