Analytical Approximation of Heat Equation by VIM

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Abstract

In this paper, the variational iteration method is applied to obtain an analytic and approximation solution for wave equations. The results verify that the method is very effective.

Keywords: Variational iteration method, Heat equation.

1 Introduction

The variational iteration method [1] which is a modified general Lagrange multiplier method has been shown to solve effectively, easily and accuracy, large class of nonlinear problems with approximations which convergence rapidly to accurate solution [2-6]. To illustrate the method, consider the following nonlinear equation

\[ Lu(t) + Nu(t) = g(t), \] (1)

where \( L \) is a linear operator, \( N \) is a nonlinear operator and \( g(t) \) is a known analytical function. According to the variational iteration method, we can construct the following correction functional.

\[ u_{n+1}(t) = u_n(t) + \int_0^t \lambda(\xi) (L u_n(\xi) + N u_n(\xi) - g(\xi)) d\xi, \] (2)

where \( \lambda \) is general Lagrange multiplier which can be identified via variational theory, \( u_0(t) \) is an initial approximation with possible unknowns, and \( \tilde{u}_n \) is
Kh. Hosseinzadeh considered as restricted variation, i.e.
\[ \delta \tilde{u}_n = 0. \]
Therefore, we first determine the Lagrange multiplier \( \lambda \) that will be identified optimally via integration by parts. The successive approximations \( u_{n+1}(t) \) of the solution \( u(t) \) will be readily obtained upon using the obtained Lagrange multiplier and by using any selective function \( u_0 \). Consequently, the exact solution may be obtained by using \( u = \lim_{n \to \infty} u_n \).

2 Solution of the heat equation by He’s variational iteration method

To study the solution of heat equation by variational iteration method we consider the heat equation in different dimensions.

Case 1. Consider one-dimensional heat equation

\[ \frac{1}{c(x,t)} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = f(x,t), \quad 0 < x < L, \quad t > 0. \]

To solve this equation using it iteration variational method, we need specify the initial conditions and boundary conditions, which for example may be specified as follows:

Initial conditions:
\[ u(x,0) = f(x), \quad \frac{\partial u}{\partial t}(x,0) = 0, \]

Boundary conditions:
\[ u(0,t) = u(L,t) = 0, \quad t > 0. \]

Hence, its correction functional can be written down as follows:

1) \[ u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda_1 \left( \frac{1}{c(x,\xi)} \frac{\partial u_n}{\partial \xi} - \frac{\partial^2 u_n}{\partial x^2} - f(x,\xi) \right) d\xi, \]
2) \[ u_{n+1}(x,t) = u_n(x,t) + \int_0^x \lambda_2 \left( \frac{1}{c(\xi,t)} \frac{\partial u_n}{\partial \xi} - \frac{\partial^2 u_n}{\partial x^2} - f(\xi,t) \right) d\xi, \]
3) \[ u_{n+1}(x,t) = u_n(x,t) + \int_L^x \lambda_3 \left( \frac{1}{c(\xi,t)} \frac{\partial u_n}{\partial \xi} - \frac{\partial^2 u_n}{\partial x^2} - f(\xi,t) \right) d\xi. \]

Making the above correct functional stationary, notice that \( \delta u_n(0,t) = \delta u_n(L,t) = \delta u_n(x,0) = 0 \), the Lagrange multiplier can be identified:
\[ \lambda_1 = -1, \lambda_2 = x - \xi, \lambda_3 = x - \xi. \]

Example 1. Consider the following equation with initial and boundary conditions:
\[ \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \]
\[ u(x,0) = \sin \pi x, \quad u(0,t) = u(1,t) = 0. \]

The following iteration formula can be obtained:

\[ u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left( \frac{\partial u_n}{\partial \xi} - \frac{\partial^2 u_n}{\partial x^2} \right) d\xi. \]

Beginning with \( u_0(x,t) = \sin \pi x \), by iteration formula (3), we have
Example 3. To make this correct functional stationary, notice that multiplier can be identified, heat equation has the form:

\[ u_1(x, t) = \sin \pi x - \pi^2 t \sin \pi x, \]
\[ u_2(x, t) = \sin \pi x - \pi^2 t \sin \pi x + \pi^4 t^2 \sin \pi x, \]
\[ u_3(x, t) = \sin \pi x - \pi^2 t \sin \pi x + \pi^4 t^2 \sin \pi x - \pi^6 t^3 \sin \pi x, \]

So:

\[ u(x, t) = \sum_{n=0}^{\infty} (-1)^n \pi^{2n} t^n \frac{\pi^n}{n!} \sin \pi x = e^{-\pi^2 t} \sin \pi x \tag{4} \]

which is an exact solution.

Example 2. Consider the diffusivity equation in radial form:

\[ \frac{\partial u}{\partial t} = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right), \quad u(r_i, 0) = r. \]

The iteration formula can be expressed:

\[ u_{n+1}(r, t) = u_n(r, t) - \int_0^t \left( \frac{\partial u_n}{\partial \xi} - \frac{\partial^2 u_n}{\partial r^2} - \frac{1}{r} \frac{\partial u_n}{\partial r} \right) d\xi \tag{5} \]

Beginning with \( u_0(r, t) = r \), by iteration formula (5), we have:

\[ u_1(r, t) = r + \frac{1}{2} t, \]
\[ u_2(r, t) = r + \frac{1}{2} t + \frac{1}{2} t^2, \]
\[ u_3(r, t) = r + \frac{1}{2} t + \frac{1}{2} t^2 + \frac{1}{2} t^3 t^3 \]

So \( u(r, t) = r + \sum_{n=1}^{\infty} \frac{1}{2^3 \times 3! \times 3!} t^n \).

Case 2. In rectangular Cartesian coordinates, the two-dimensional sourceless heat equation has the form

\[ \frac{1}{c(x,y,t)} \frac{\partial u}{\partial t} - \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = f(x, y, t), \]

\( 0 \leq x \leq L, \quad 0 \leq y \leq L, \quad t \geq 0 \). For example, to solve this equation with initial and boundary conditions as follows:

\( u(0, y, t) = 0, \quad u_y(x, L, t) = 1, \quad u_y(x, 0, t) = 1, \quad u(x, y, 0) = 0 \), its correction functional can be written down as follows:

1) \( u_{n+1}(x, y, t) = u_n(x, y, t) + \int_0^x \lambda_1 \left( \frac{1}{c(\xi,y,t)} \frac{\partial u_n}{\partial \xi} - \left( \frac{\partial^2 u_n}{\partial \xi^2} + \frac{\partial^2 u_n}{\partial y^2} \right) - f(\xi, y, t) \right) d\xi \),

2) \( u_{n+1}(x, y, t) = u_n(x, y, t) + \int_0^y \lambda_2 \left( \frac{1}{c(x,\xi,t)} \frac{\partial u_n}{\partial \xi} - \left( \frac{\partial^2 u_n}{\partial x^2} + \frac{\partial^2 u_n}{\partial y^2} \right) - f(x, \xi, t) \right) d\xi \),

3) \( u_{n+1}(x, y, t) = u_n(x, y, t) + \int_L^y \lambda_3 \left( \frac{1}{c(x,\xi,t)} \frac{\partial u_n}{\partial \xi} - \left( \frac{\partial^2 u_n}{\partial x^2} + \frac{\partial^2 u_n}{\partial y^2} \right) - f(x, \xi, t) \right) d\xi \),

4) \( u_{n+1}(x, y, t) = u_n(x, y, t) + \int_0^t \lambda_4 \left( \frac{1}{c(x,y,t)} \frac{\partial u_n}{\partial \xi} - \left( \frac{\partial^2 u_n}{\partial x^2} + \frac{\partial^2 u_n}{\partial y^2} \right) - f(x, y, \xi) \right) d\xi \).

To make this correct functional stationary, notice that

\( \delta u_n(0, y, t) = \delta u_n(x, 0, t) = \delta u_n(x, L, t) = \delta u_n(x, y, 0) = 0 \), the Lagrange multiplier, can be identified,

\( \lambda_1 = x - \xi, \lambda_2 = y - \xi, \lambda_3 = y - \xi, \text{ and } \lambda_4 = -1 \).

Example 3. Let us solve the following partial differential equation:
\[ \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, \]

\[ u_x(0, y, t) = 0, \quad u_x(L, y, t) = 1, \quad u_y(x, 0, t) = 1. \]

The following iteration formula can be obtained:

\[ u_{n+1}(x, y, t) = u_n(x, y, t) + \int_0^x (\xi - x) \left( \frac{\partial u_n}{\partial t} - \left( \frac{\partial^2 u_n}{\partial \xi^2} + \frac{\partial^2 u_n}{\partial y^2} \right) \right) d\xi \quad (6) \]

Now we begin with an arbitrary initial approximation \( u_0(x, y, t) = Ax + By + ct. \)

From (6) we have

\[ u_1(x, y, t) = Ax + By + Ct + C \frac{x^2}{2}, \]

by imposing the boundary conditions yields \( A = 0, \quad B = 1, \quad C = \frac{1}{L}, \)

therefore,

\[ u_1(x, y, t) = \frac{x^2}{2L} + y + \frac{t}{L} \quad (7) \]

which is an exact solution.

**Case 3.** Consider the following general form of heat equation:

\[ \frac{\partial u}{\partial t} = A(x, y, z, t) \frac{\partial^2 u}{\partial x^2} + B(x, y, z, t) \frac{\partial^2 u}{\partial y^2} + C(x, y, z, t) \frac{\partial^2 u}{\partial z^2} + D(x, y, z, t). \quad (8) \]

To solve this equation using the variational iteration method, we need specify the initial or boundary conditions, which for example may be specified as follows

**Initial conditions:**

\( u(x, y, z, 0) = f(x, y, z). \)

**Boundary conditions:**

\( u(L, y, z, t) = f_1(x, y, z), \quad \frac{\partial u}{\partial z}(L, y, z, t) = g_1(y, z, t). \)

For solving (8) by variational iteration method, its correction functional can be written down as:

\[ (a) \quad u_{n+1}(x, y, z, t) = u_n(x, y, z, t) + \int_0^t \lambda_1 \left( \frac{\partial u_n}{\partial \xi} - A \frac{\partial^2 u_n}{\partial x^2} - B \frac{\partial^2 u_n}{\partial y^2} - C \frac{\partial^2 u_n}{\partial z^2} - D \right) d\xi, \]

\[ (b) \quad u_{n+1}(x, y, z, t) = u_n(x, y, z, t) + \int_0^x \lambda_2 \left( \frac{\partial u_n}{\partial t} - A \frac{\partial^2 u_n}{\partial x^2} - B \frac{\partial^2 u_n}{\partial y^2} - C \frac{\partial^2 u_n}{\partial z^2} - D \right) d\xi. \]

To make this correct functional stationary, notice that, \( \delta u_n(x, y, z, 0) = 0 \)

\[ \delta u_{n+1} = \delta u_n + (\lambda \delta u_n) \big|_0^t - \int_0^t \lambda' \delta u_n d\xi = 0. \] Its stationary conditions can be obtained as follows:

\[ \delta u_n : 1 + \lambda(t) = 0, \]

\[ \delta u_n' : \lambda'(\xi) = 0. \]

The Lagrange multiplier can be identified \( \lambda = -1 \) and the following iteration formula can be obtained

\[ u_{n+1}(x, y, z, t) = u_n(x, y, z, t) - \int_0^t \left( \frac{\partial u_n}{\partial \xi} - A \frac{\partial^2 u_n}{\partial x^2} - B \frac{\partial^2 u_n}{\partial y^2} - C \frac{\partial^2 u_n}{\partial z^2} - D \right) d\xi, \quad (9) \]

Starting with \( u_0 \), which can be choose arbitrary. The iteration formula (9) will be used to drive iterative approximations to \( u \). In this case, considering
initial conditions, we use $u_0$ as, $u_0(x, y, z, t) = f(x, y, z)$. To make this correct functional stationary, notice that, $\delta u_n(L, y, z, t) = 0$,
\[
\delta u_{n+1} = \delta u_n - \left( \lambda_2 A \left( \delta u_n \right)' \right)^x + \left( (\lambda_2 A)' \delta u_n \right)^x - \int_L^x (\lambda_2 A)'' \delta u_n d\xi = 0.
\]
Its stationary conditions can be obtained as follows:
\[
\begin{align*}
\delta u_n : & \quad 1 + (\lambda(x)A(x, y, z, t))' = 0, \\
\delta u_n' : & \quad \lambda(x)A(x, y, z, t) = 0, \\
\delta u_n : & \quad (\lambda(\xi)A(\xi, y, z, t))'' = 0.
\end{align*}
\]
From which Lagrange multiplier can be identified as $\lambda_2 = \frac{1}{A(\xi, y, z, t)} (x - \xi)$, and the following iteration formula will be obtained:
\[
u_{n+1}(x, y, z, t) = u_n(x, y, z, t)
+ \int_x^L \frac{1}{A(\xi, y, z, t)} (x - \xi) \left( \frac{\partial u_n}{\partial \xi} - A \frac{\partial^2 u_n}{\partial x^2} - B \frac{\partial^2 u_n}{\partial y^2} - C \frac{\partial^2 u_n}{\partial z^2} - D \right) d\xi.
\]
In this case, considering boundary conditions, we use $u_0$ as,
\[
u_0(x, y, z, t) = f_1(y, z, t) + g_1(y, z, t) x.
\]

**Example 4.** Let us solve the following partial differential equations:
\[
\begin{align*}
\frac{\partial u}{\partial t} &= \frac{1}{6} \left( x^2 \frac{\partial^2 u}{\partial x^2} + y^2 \frac{\partial^2 u}{\partial y^2} + z^2 \frac{\partial^2 u}{\partial z^2} \right), \\
u(x, y, z, 0) &= x^2 y^2 z^2.
\end{align*}
\]
From (9) the variational iteration formula will be obtained as:
\[
u_{n+1}(x, y, z, t) = \nu_n(x, y, z, t) - \int_0^t \left( \frac{\partial \nu_n}{\partial \xi} - \frac{1}{6} \left( x^2 \frac{\partial^2 \nu_n}{\partial x^2} + y^2 \frac{\partial^2 \nu_n}{\partial y^2} + z^2 \frac{\partial^2 \nu_n}{\partial z^2} \right) \right) d\xi.
\]
By imposing $\nu_0(x, y, z, t) = x^2 y^2 z^2$,
we have the following approximate solutions:
\[
\begin{align*}
u_1(x, y, z, t) &= x^2 y^2 z^2 + x^2 y^2 z^2 t, \\
u_2(x, y, z, t) &= x^2 y^2 z^2 + x^2 y^2 z^2 t + x^2 y^2 z^2 t^2 2t, \\
&\vdots \\
u_n(x, y, z, t) &= x^2 y^2 z^2 \sum_{k=0}^n t^k. \\
\end{align*}
\]
Thus we have
\[
u = \lim_{n \to \infty} \nu_n(x, y, z, t) = x^2 y^2 z^2 e^t,
\]
which is an exact solution.

**Example 5.** Consider the following heat equation with the following boundary conditions:
\[
\begin{align*}
\frac{\partial \nu}{\partial t} &= x \frac{\partial^2 \nu}{\partial x^2} + y \frac{\partial^2 \nu}{\partial y^2} + z \frac{\partial^2 \nu}{\partial z^2}, \\
u(1, y, z, t) &= y + z + t,
\end{align*}
\]
From (10), we have
\[
\begin{align*}
\frac{\partial \nu}{\partial \xi}(1, y, z, t) &= - (y + z + t).
\end{align*}
\]
\[
u_{n+1}(x, y, z, t) = \nu_n(x, y, z, t) + \int_1^x \frac{1}{\xi} (x - \xi) \left( \frac{\partial \nu_n}{\partial t} - x \frac{\partial^2 \nu_n}{\partial x^2} - y \frac{\partial^2 \nu_n}{\partial y^2} - z \frac{\partial^2 \nu_n}{\partial z^2} \right) d\xi.
\]
(11)
Beginning with $u_0 = 2(y + z + t) - x(y + z + t)$ by iteration formula (12), the exact solution as follows

$$u_1 = 2(y + z + t) - x(y + z + t) + 2x \ln x - x - \frac{x^2}{2} + \frac{3}{2}.$$ 

3 Conclusion

In this work, we presented an analytical approximation to the solution of wave equations, in the three different cases. We have achieved this goal by applying He’s variational iteration method. The small size of computations in comparison with the computational size required in numerical methods and the rapid convergence shows that the variational iteration method is reliable and introduces a significant improvement in solving the wave equation over existing methods. The computations associated with the examples in this paper were performed using Maple 10.

References


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