Variational Homotopy Perturbation Method of Quadratic Integral Equations

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Abstract

In this paper, a variational homotopy perturbation method (VHPM) is based on variational iteration method and homotopy perturbation method, which is applied to solve nonlinear quadratic integral equations (QIEs), the (VHPM) find the solutions without any discretisation, transformation, or restrictive assumptions, some examples are given indicate the efficiency and simplicity of the proposed method.

Mathematics Subject Classification: 45G10

Keywords: Quadratic Integral Equation, Variational Homotopy Perturbation Method
1 Introduction

Quadratic integral equations (QIEs) have an essential part in the applications of different phenomena from science to engineering. For example, the QIEs are applied in the radiative transformer’s theory, kinetic theory of gases, neutron transport and traffic theory. The QIEs have been studied in several papers and monographs [2, 3, 4, 5, 6, 7, 8, 9, 10, 11].

Recently, various analytical and numerical methods have been applied to gain the approximate solutions of QIEs, since nonlinear quadratic integral equations are difficult to get their exact solutions. It is worth mentioning that up to now only a few papers have dealt with the numerical solutions of QIEs such as Elsayed [10] used the classical method of successive approximations Picard and Adomian decomposition method for solving QIEs, Avazzaden [1] used the radial basis functions to obtain the approximate solutions of QIEs of Urysohn’s type. There are many difficulties for the most of these methods like the calculation of Adomian polynomial, divergent results and huge computational work. To avoid these difficulties and drawbacks, He [13, 16] developed variational iteration method (VIM) for solving linear and nonlinear problems, which arise in various branches of pure and applied science. It is good to mention that the origin of variational iteration method can be traced back to Inokuti et al. [17]. Also, He [12, 14, 15] has presented the homotopy perturbation method (HPM), which is developed by combining the standard homotopy and perturbation method. In the previous methods, the solution is given in an infinite series which usually is converging to an accurate solution.

Variational homotopy perturbation method VHPM is combining the variational iteration method and homotopy perturbation method, which is a process, and it has the positive features of VIM and HPM. Recently, we also notice that some researcher have succeed to apply this method such as Noor and Mohyud-Din [19] used this method for solving higher dimensional initial boundary value problems with variable coefficients, A. Neamaty et al.[18] applied (VHPM) for solving high-order fractional volterra integro-differential equations with Caputo derivative.

The VHPM presents the solution in a rapid convergent series which may lead to the closed from solution. Moreover, the VHPM technique solves nonlinear problems without using the so-called Adomian’s polynomials.

In this work, we have the opportunity to apply the VHPM for the first time to construct the approximate solutions of the following nonlinear quadratic
integral equations.

\[ x(t) = a(t) + g(t, x(t)) \int_0^t f(s, x(s)) ds, \quad (1) \]

where \( a(t) \) is given and \( f(s, x(s)) \) is any nonlinear functions. Many examples are given to verify the reliability and efficiency of the variational homotopy perturbation method.

2 Variational Homotopy Perturbation Method

To express the basic idea of VHPM [18, 19] we consider the QIE (1), VHPM is presented and explained in two steps.

First, we have to apply the VIM and identify the Lagrange multiplier \( \lambda \) by using the variational theory which plays an essential part in applying the VIM method. We should convert QIE (1) to an equivalent integro-differential equation, and this can be done by differentiating both sides of the QIEs, where the Leibniz rule and the chain rule should be used for differentiating the integral at the right side. Differentiating both sides of eq.(1) gives

\[ x'(t) = a'(t) + \left( \frac{\partial g}{\partial t}(t, x(t)) + \frac{\partial g}{\partial x(t)}(t, x(t)) \frac{dx(t)}{dt} \right) \int_0^t f(s, x(s)) ds + g(t, x(t)) f(t, x(t)), \quad (2) \]

we construct the correction functional for eq.(2),

\[
x_{n+1}(x) = x_n(t) + \int_0^t \lambda(\zeta) \left\{ x'_n(\zeta) - a'(\zeta) - \left( \frac{\partial g}{\partial \zeta}(\zeta, x_n(\zeta)) + \frac{\partial g}{\partial x(t)}(\zeta, x_n(\zeta)) \frac{dx_n(\zeta)}{d\zeta} \right) \right. \\
\left. + \int_0^\zeta f(r, x_n(r)) dr - g(\zeta, x_n(\zeta)) f(\zeta, x_n(\zeta)) \right\} d\zeta. \
\]

(3)

The function \( x_n \) is considered as a restricted variation, that is \( \delta x_n = 0 \).

Second, we construct the following iteration formula by using the variational iteration method VIM and homotopy perturbation method HPM:

\[
\sum_{n=0}^{\infty} p^n x_n(t) = x_0(t) + p \left\{ \sum_{n=1}^{\infty} p^n x_n(t) - \int_0^t \left[ \sum_{n=0}^{\infty} p^n x'_n(\zeta) - a'(\zeta) - \left( \sum_{n=0}^{\infty} p^n \frac{\partial g}{\partial \zeta}(\zeta, x_n(\zeta)) \right) \\
+ \sum_{n=0}^{\infty} p^n \left( \frac{\partial g}{\partial x(\zeta)}(\zeta, x_n(\zeta)) \frac{dx_n(\zeta)}{d\zeta} \right) \right] \int_0^\zeta f(r, x_n(r)) dr \\
- \sum_{n=0}^{\infty} p^n \left( g(\zeta, x_n(\zeta)) f(\zeta, x_n(\zeta)) \right) \right\}, \
\]

(4)
which is named as VHPM.
In the formula (4), the zeroth approximation \(x_0(t)\) can be chosen by using the initial value \(x(0)\), and \(p \in [0, 1]\) is embedded parameter.
We can obtain \(x_i(t = 0, 1, 2, \ldots)\) by equating the terms with identical powers of \(p\) in both sides of eq.(4). Finally, according to HPM, when \(p \to 1\), the approximate solution will be as follows

\[
x(t) = \sum_{n=0}^{\infty} x_n(t).
\]

3 Applications and Results

We have to give some examples in order to illustrate the effectiveness of the method which is presented in this paper.

Example 1. Solve the QIE [10]

\[
x(t) = \left(t^2 - \frac{t^{10}}{35}\right) + \frac{t}{5}x(t) \int_0^t s^2x^2(s)ds,
\]

and the exact solution is \(x(t) = t^2\).
we convert the QIE (5) to an equivalent integro-differential equation, and we obtain this equation,

\[
x'(t) = 2t - 10 \left(\frac{t^9}{35}\right) + \frac{t^3}{5}x^3(t) + \left(\frac{1}{5}x(t) + \frac{t}{5}x'(t)\right) \int_0^t s^2x^2(s)ds, \quad x(0) = 0, (6)
\]

first, we construct the correction functional for eq.(6) and calculate the Lagrange multipliers optimally by using variational theory,

\[
x_{n+1}(t) = x_n(t) + \int_0^t \lambda(\zeta) \left(x_n'(\zeta) - 2\zeta + 10 \frac{\zeta^9}{35} - \frac{\zeta^3}{5}x_n^3(\zeta) - \left(\frac{1}{5}x_n(\zeta) + \frac{\zeta}{5}x_n'(\zeta)\right) \int_0^\zeta r^2x_n^2(r)dr\right) d\zeta,
\]

we have substituted \(\lambda(\zeta) = -1\) in the first-order integro-differential equation (7). Also, we can select the zeroth approximation \(x_0(t)\) by using the \(x(0)\) value.
Second, we implement the homotopy perturbation method on the correct functional. Finally, we equate the terms that have the same powers of $p$ then, the result will be

\begin{align*}
    p^0 : x_0(t) &= 0, \\
    p^1 : x_1(t) &= t^2 - \frac{t^{10}}{35}, \\
    p^2 : x_2(t) &= \frac{1}{50} t^{10} - \frac{1}{7288750} t^{34} + \frac{3}{159250} t^{26} - \frac{1}{1050} t^{18}.
\end{align*}

Then, the approximate solution for eq.(5) is obtained by

\begin{equation}
    x(t) = t^2 - \frac{26111}{167641250000} t^{34} + \frac{61}{2113125} t^{26} - \frac{29}{18375} t^{18} + \ldots,
\end{equation}

Table 1: Comparison of the numerical results with the exact solution

<table>
<thead>
<tr>
<th>$t$</th>
<th>Approximate Solution</th>
<th>Exact Solution</th>
<th>Absolute error</th>
</tr>
</thead>
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<td>0.01000000</td>
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</tr>
<tr>
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<td>0.04000000</td>
<td>0.04000000</td>
<td>$4.137 \times 10^{-16}$</td>
</tr>
<tr>
<td>0.30</td>
<td>0.09000000</td>
<td>0.09000000</td>
<td>$6.114 \times 10^{-13}$</td>
</tr>
<tr>
<td>0.40</td>
<td>0.16000000</td>
<td>0.16000000</td>
<td>$1.084 \times 10^{-10}$</td>
</tr>
<tr>
<td>0.50</td>
<td>0.24999999</td>
<td>0.25000000</td>
<td>$6.020 \times 10^{-9}$</td>
</tr>
<tr>
<td>0.60</td>
<td>0.35999984</td>
<td>0.36000000</td>
<td>$1.602 \times 10^{-7}$</td>
</tr>
<tr>
<td>0.70</td>
<td>0.48999743</td>
<td>0.49000000</td>
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</tr>
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<td>0.80</td>
<td>0.63997166</td>
<td>0.64000000</td>
<td>$0.00002834$</td>
</tr>
<tr>
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<td>0.80976498</td>
<td>0.81000000</td>
<td>$0.00023502$</td>
</tr>
<tr>
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<td>0.99845048</td>
<td>1.00000000</td>
<td>$0.00154952$</td>
</tr>
</tbody>
</table>

Figure 1: Comparison of the approximate with exact solution by VHPM
Table 1 and figure 1 express a comparison between the approximate and exact solutions when \( n = 3 \).

**Example 2.** Solve the QIE [10]

\[
x(t) = \left( t^3 - \frac{t^{19}}{100} - \frac{t^{20}}{110} \right) + \frac{t^3}{10} x^2(t) \int_0^t (s + 1) x^3(s) ds,
\]

and the exact solution is \( x(t) = t^3 \).

We convert the QIE (12) to an equivalent integro-differential equations, and we obtain the following,

\[
x'(t) = \left( 3t^2 - 19\frac{t^{18}}{100} - 20\frac{t^{19}}{110} \right) + 3\frac{t^2}{10} x^2(t) \int_0^t (s + 1) x^3(s) ds + \frac{t^3}{10} (2x(t)x'(t)) \int_0^t (s + 1) x^3(s) ds
\]

\[
+ \frac{t^3}{10} (t + 1)x^5(s) ds, \quad x(0) = 0,
\]

first, we construct the correction functional for eq.(13), and calculate the Lagrange multipliers optimally by using variational theory,

\[
x_{n+1}(t) = x_n(t) + \int_0^t \lambda(\zeta) \left( \frac{3}{10} \zeta^2 (\zeta + 1) x_n(\zeta) dx'_{n}(\zeta) \right) d\zeta,
\]

we have substituted \( \lambda(\zeta) = -1 \) in the first-order integro-differential equation (14). Also, we can select the zeroth approximation \( x_0(t) \) by using the \( x(0) \) value. Second, we implement the homotopy perturbation method on the correct functional. Finally, we equate the terms that have the same powers of \( p \) then, the result will be

\[
p^0 : x_0(t) = 0,
\]

\[
p^1 : x_1(t) = t^3 - \frac{1}{100} t^{19} - \frac{1}{110} t^{20},
\]

\[
p^2 : x_2(t) = \frac{1}{190} t^{19} + \frac{1}{200} t^{20} + \frac{17}{61600000000} t^{84} + \frac{1}{16600000000} t^{83} - \frac{1}{94501000} t^{71}
\]

\[
- \frac{43}{9317000000} t^{70} - \frac{21}{2783000000} t^{69} + \frac{41}{7480000000} t^{68} + \ldots,
\]

then, the approximate solution for eq.(5) is given by

\[
x(t) = t^3 - \frac{3}{950} t^{19} - \frac{3}{1100} t^{20} - \frac{46741}{4101343400000} t^{71} + \ldots,
\]
Table 2: Comparison of the numerical results with exact solution $x(t)$

<table>
<thead>
<tr>
<th>$x$</th>
<th>Approximate Solution</th>
<th>Exact Solution</th>
<th>Absolute error</th>
</tr>
</thead>
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<td>0.00100000</td>
<td>0.00100000</td>
<td>$3.4306 \times 10^{-22}$</td>
</tr>
<tr>
<td>0.20</td>
<td>0.00800000</td>
<td>0.00800000</td>
<td>$1.9416 \times 10^{-16}$</td>
</tr>
<tr>
<td>0.30</td>
<td>0.02700000</td>
<td>0.02700000</td>
<td>$4.6212 \times 10^{-13}$</td>
</tr>
<tr>
<td>0.40</td>
<td>0.06400000</td>
<td>0.06400000</td>
<td>$1.1679 \times 10^{-10}$</td>
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<td>0.21599971</td>
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<td>$2.9214 \times 10^{-7}$</td>
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<td>0.34299422</td>
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<tr>
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<td>1.00</td>
<td>0.99994142</td>
<td>1.00000000</td>
<td>$0.00649704$</td>
</tr>
</tbody>
</table>

Figure 2: Comparison of the approximate with exact solution by VHPM

Table 2 and figure 2 express a comparison between the approximate and exact solutions when $n = 3$.

**Example 3.** Solve the QIE [2]

$$x(t) = t^3 + \left( \frac{1}{4} x(t) + \frac{1}{4} \right) \int_0^t \left( t + \cos \left( \frac{x(s)}{1 + x^2(s)} \right) \right) ds,$$

we convert the QIE (19) to an equivalent integro-differential equation, and we
obtain the following,

\[ x'(t) = 3t^2 + \frac{1}{2}tx(t) + \frac{1}{4}t^2x'(t) + \frac{1}{4}x(t)\cos\left(\frac{x(t)}{1+x^2(t)}\right) + \frac{1}{4}x'(t) \int_0^t \cos\left(\frac{x(s)}{1+x^2(s)}\right) ds \\
+ \frac{1}{2}t + \frac{1}{4}\cos\left(\frac{x(t)}{1+x^2(t)}\right), \quad x(0) = 0, \quad (20) \]

first, we construct the correction functional of eq.(21), and we calculate the Lagrange multipliers optimally by using variational theory,

\[
x_{n+1}(t) = x_n(t) + \int_0^t \lambda(\zeta) \left( x'_n(\zeta) - 3\zeta^2 - \frac{1}{2}\zeta x_n(\zeta) - \frac{1}{4}\zeta^2 x'_n(\zeta) - \frac{1}{4}x_n(\zeta)\cos\left(\frac{x_n(\zeta)}{1+x_n^2(\zeta)}\right) \\
- \frac{1}{4}x'_n(\zeta) \int_0^\zeta \cos\left(\frac{x_n(r)}{1+x_n^2(r)}\right) dr - \frac{1}{2}\zeta - \frac{1}{4}\cos\left(\frac{x_n(\zeta)}{1+x_n^2(\zeta)}\right) \right) d\zeta, \quad (21)\]

we have substituted \( \lambda(\zeta) = -1 \) in the first-order integro-differential equation (21). Also, we can select the zeroth approximation \( x_0(t) \) by using the \( x(0) \) value. Second, we implement the homotopy perturbation method on the correct functional. Finally, we equate the terms that have the same powers of \( p \) then, the result will be

\[
p^0 : x_0(t) = 0, \quad (22) \\
p^1 : x_1(t) = \frac{1}{4}t + t^3 + \frac{1}{4}t^2, \quad (23) \\
p^2 : x_2(t) = \frac{47}{384}t^3 + \frac{1}{16}t^2 - \frac{1}{80}t^{10} - \frac{1}{96}t^9 - \frac{15}{1024}t^8 - \frac{89}{3584}t^7 - \frac{47}{3072}t^6 \\
+ \frac{601}{2560}t^5 + \frac{631}{2048}t^4. \quad (24)\]

Then, the approximate solution for eq.(19) is given by this equation

\[
x(t) = \frac{1}{4}t + \frac{437}{384}t^3 + \frac{5}{16}t^2 - \frac{6860712707}{1033476505600}t^{11} - \frac{704929}{2617245696000}t^{26} \\
- \frac{3042881761}{2367600721920000}t^{25} - \frac{1094354801}{378816115507200}t^{24} + ..., \quad (25)\]
Table 3: Approximate solution $x(t)$ by VHPM

<table>
<thead>
<tr>
<th>$t$</th>
<th>Approximate Solution</th>
</tr>
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<tbody>
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<tr>
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<td>2.510075563</td>
</tr>
</tbody>
</table>

Figure 3: Approximate solution for eq.(19) by VHPM

Table 3 and figure 3 show the approximate solution when $n = 3$.

**Example 4.** Solve the QIE [4]

$$x(t) = e^{-t} + x(t) \int_0^t \frac{t^2 \ln(1 + s|x(s)|)}{2e^{(t+s)}} ds, \quad 0 < t \leq 2. \quad (26)$$

We convert the QIE (26) to an equivalent integro-differential equation, and we obtain the following,

$$x'(t) = -e^{-t} + x'(t) \int_0^t \frac{t^2 \ln(1 + s|x(s)|)}{2e^{(t+s)}} ds + x(t) \left( \frac{t^2}{2e^{2t}} \ln(1 + t|x(t)|) ight. 
+ \left. \int_0^t \frac{-2t^2e^{(t+s)} + 4te^{(t+s)}}{4e^{(t+s)^2}} \ln(1 + s|x(s)|) ds \right), \quad x(0) = 1. \quad (27)$$
First, we construct the correction functional of eq.(27) and calculate the Lagrange multipliers optimally by using variational theory,

\begin{align*}
x_{n+1} &= x_n(t) - \int_0^t \left( x'_n(\zeta) + e^{-\zeta} - x'_n(\zeta) \right) \int_0^\zeta \frac{\zeta^2}{2e(\zeta + r)} \ln(1 + r|x_n(r)|) dr \\
&\quad - x_n(\zeta) \left[ \frac{\zeta^2}{2e^2} \ln(1 + \zeta|x_n(\zeta)|) + \int_0^\zeta -2\zeta e^{(\zeta + r)} + 4\zeta e^{(\zeta + r)} \ln(1 + r|x_n(r)|) dr \right] d\zeta,
\end{align*}

we can select the zeroth approximation \( x_0(t) \) by using the \( x(0) \) value. Second, we implement the homotopy perturbation method on the correct functional. Finally, we equate the terms that have the same powers of \( p \) then, the result will be

Table 4: Approximate solution \( x(t) \) by VHPM

<table>
<thead>
<tr>
<th>( t )</th>
<th>Approximate Solution</th>
</tr>
</thead>
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</table>

Figure 4: Approximate solution for eq.(26) by VHPM

Table 4 and figure 4 show the approximate solution when \( n = 1 \).
4 Conclusion

We have been successfully applied the variational homotopy perturbation method (VHPM) to find the approximate solutions for nonlinear quadratic integral equations. We have found out that the method are applicable and efficient technique. We have been observing that the accuracy can be improved by computing more n-terms of approximate solutions or by taking more terms in the Taylor expansion of the nonlinear terms. To find the calculations we have used the Maple package (2015).

References


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