A Wiener-Hopf Factorization Approach for Pricing

Barrier Options in the Heston Model

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Abstract

We propose an efficient numerical method for pricing barrier options in the Heston model. The method is based on Markov chain approximation for variance and explicit Wiener-Hopf factorization formulae. We prove the advantage of the new method in terms of accuracy and convergence by using numerical experiments.

Keywords: barrier options, Heston model, Wiener-Hopf factorization

1 Introduction

The goal of this article is to propose a new efficient method for pricing European style barrier options in the Heston model, without rebate.

The Heston model [7] is quite popular among practitioners, however the barrier option pricing problem in the model corresponds to solving a complex 3-dimensional PDE. For barrier options, the most commonly used methods are Monte-Carlo methods, tree methods and finite-differences methods. For detailed description see [4, 6] and the bibliography therein.
It is well-known that the convergence rates of standard Monte Carlo methods are rather slow. The recent exception is the Monte Carlo simulation technique developed in [9].

Finite-difference schemes for solving the corresponding equations can be based on implicit, explicit or both implicit and explicit schemes, depending on the values of variables. The detailed description of the schemes could be found in [6]. A hybrid tree and finite difference method was suggested in [4].

We propose a new numerical method which combines a binomial tree approximation for the variance process from [4], and the efficient numerical Wiener-Hopf factorization method from [8]. The advantages of the new method in terms of accuracy and convergence are proved by performing numerical experiments.

2 The problem set-up

The equations describing the Heston model are the diffusion process for the stock price returns (1) and the Cox-Ingersoll-Ross (CIR) process for the variance (2):

\[
\frac{dS_t}{S_t} = r dt + \sqrt{V_t} dZ_t^S,
\]

\[
dV_t = \kappa(\theta - V_t) dt + \sigma \sqrt{V_t} dZ_t^V,
\]

where \( S_t \) stands for the underlying asset price, \( V_t \) is the variance of the process, \( S_0 > 0, V_0 > 0 \). The model parameters also possess an advantage of having the economic interpretations, namely: \( r \geq 0 \) is a riskless interest rate, \( \kappa \) is a mean-reversion rate, \( \theta \) is a long-run variance, \( \sigma \) is a volatility of variance.

We also suppose the condition \( 2\kappa \theta > \sigma^2 r \) to be satisfied. The drivers \( Z_S(t) \) and \( Z_V(t) \) are the Wiener processes. They are correlated with coefficient \( \rho \): \( d < Z_t^S, Z_t^V > (t) = \rho dt \). The “market data” parameters are assumed to be in the following ranges: the strike price \( K > 0 \), the barrier \( H > 0 \), the expiration date \( T > 0 \), \( t \in [0, T] \).

We restrict ourselves here to the European down-and-out option case, keeping in mind that other barrier options could be evaluated analogously.

It is well-known that 0-time price of the down-and-out put option is equal to the following expectation, with initial spot price \( S_0 \) and variance \( V_0 \):

\[
F(S,V) = E \left[ e^{-rT} 1_{S_T \geq H} G(S_T) \mid S_0 = S, V_0 = V \right], S_T = \inf_{0 \leq t \leq T} S_t, \]

where \( G(S) = \max\{0, K - S\} \)

3 The Substitution and Randomization

The effect of correlation between Brownian motions is to be reflected in a pricing procedure. The approach we have chosen is similar to [4]. We construct a suitable substitution for the processes to get rid of the necessity to consider this effect in
A Wiener-Hopf factorization approach for pricing barrier options

further approximations. Let us denote: \( \hat{\rho} = \sqrt{1 - \rho^2}, W = Z^\nu, \rho W + \hat{\rho} Z = Z^\delta \), where \( W_t, Z_t \) are independent Brownian motions. In those terms the equations (1) and (2) would be suitably rewritten.

At this point it is convenient to switch to the log scale using the substitution almost similar to the one in [4] \( Y_t = \log \frac{S_t}{H} - \frac{\rho}{\sigma} V_t, S_t = He^{Y_t + \frac{\sigma}{2}V_t} \). Using this substitution and applying the Itô’s lemma we obtain:

\[
dY_t = \left( r - \frac{1}{2} V_t - \frac{\rho}{\sigma} \kappa(\theta - V_t) \right) dt + \hat{\rho} \sqrt{V_t} dZ_t
\]

\[
dV_t = \kappa(\theta - V_t) dt + \sigma \sqrt{V_t} dW_t
\]

We shall use some shorthand notations for the drifts \( \mu_Y(V), \mu_V(V) \) further on: \( \mu_Y(V_t) = r - \delta - \frac{1}{2} V_t - \frac{\rho}{\sigma} \kappa(\theta - V_t) \), \( \mu_V(V_t) = \kappa(\theta - V_t) \), and denote a new payoff function \( g(Y,V) \), meaning \( g(Y,V) = G(He^{Y + \frac{\sigma}{2}V}) \). The functional (3) would be rewritten as follows:

\[
\hat{F}(y, v) = E \left[ e^{-rT} \mathbb{1}_{Y_T + \frac{\sigma}{2}V_T > 0} g(Y_T, V_T) \mid Y_0 = y, V_0 = v \right]
\]

The next step is to use the Carr randomization, originally introduced in [5], see an extension of the randomization technique to general stochastic control problems in [3]. Let’s define a sequence \( \{\xi^n\}_{n=1}^N \) of i.i.d. exponential random variables independent of \( (Y_t, V_t) \), with \( E(\xi^n) = \Delta t, j = 1, \ldots, N \). It is easy to show that \( T^N := \sum_{n=1}^N \xi^n \rightarrow T, \) as \( N \rightarrow \infty \). \( T^N = \sum_{n=1}^N \xi^n \). Now once we use the independency of \( \xi^n \) from \( (Y_t, V_t) \), we become able, as it is shown in [3], to approximate the expectation (6) with \( \hat{F}^0(y, v) \) which is defined from a recurrent sequence:

\[
\hat{F}^N(y, v) = g(y, v), \quad \text{and for } n = 0, \ldots, N - 1:
\]

\[
\hat{F}^n(y, v) = E \left[ e^{-r\xi^n} \cdot \hat{F}^{n+1}(Y_{\xi^n}, V_{\xi^n}) \cdot \mathbb{1}_{Y_{\xi^n} + \frac{\sigma}{2}V_{\xi^n} > 0} \mid Y_0 = y, V_0 = v \right]
\]

The key role of this procedure is that it becomes possible to operate on small time steps without any artificial construction of the grid, being sure that increasing the number of such steps will lead to increased accuracy of the approximation.

4 Variance process approximation

The expectation (7) would be easier to calculate if we could make \( V \) behave deterministic on \([0, \xi_n]\). If \( \Delta t \) is small enough, we can use the arguments from [4] to make an approximation of \( V \) with a suitable constant specified by a certain Markov Chain state.

Now, we consider the equation (5) for the CIR process. The approximation procedure for the CIR process is, in our case, based on the approach described in
[3]. We should construct a lattice \( \{V_{n,k}\} \), \( n = 0, \ldots, N; \ k = 0, \ldots, n \) of the process with unit variance, and set the transition probabilities to define a Markov chain \( \bar{V} \).

The lattice element is defined by the following formula:

\[
V_{n,k} = \left( \sqrt{V_0} + \frac{\sigma}{2} (2k - n) \sqrt{\Delta t} \right)^2 \left( \sqrt{V_0} + \frac{\sigma}{2} (2k - n) \sqrt{\Delta t} \right)^2 > 0.
\]

The transition probabilities are calculated as follows: for every \( (n,k) \) we first calculate \( k_u \) and \( k_d \):

\[
k_u^\Delta t(n,k) = \max \{ k^* : 0 \leq k^* \leq k \text{ and } V_{n,k} + \mu_v (V_{n,k}) \geq V_{n+1,k} \},
\]

\[
k_d^\Delta t(n,k) = \min \{ k^* : k + 1 \leq k^* \leq n + 1 \text{ and } V_{n,k} + \mu_v (V_{n,k}) h \leq V_{n+1,k} \}.
\]

If the respective defining sets are empty, \( k_u \) is assumed to be \( k + 1 \) and \( k_d \) to be 0. Then we define transition probabilities as:

\[
\rho_{k_u^\Delta t(n,k)} = 0 \sqrt{V_{n+1,k_u^\Delta t(n,k)} - V_{n,k}} \Delta t(n,k) \land 1, \quad \rho_{k_d^\Delta t(n,k)} = 1 - \rho_{k_u^\Delta t(n,k)}.
\]

Here \( a \lor b \) stands for \( \max(a,b) \) and \( a \land b \) for \( \min(a,b) \). It is also to be mentioned that such Markov Chain weakly converges to its parent CIR process.

Let \( (Y_0, V_0) = (y, v), v \in \bar{V} \), be the initial conditions for the Heston model dynamics equations (4) and (5), and \( (Y^{y,v}, V^v) \) – their solutions, respectively. The approximation of \( Y^{y,v} \) with deterministic coefficients defined by \( \bar{V} \) could be written as follows:

\[
Y_t^{y,v} = y + \mu_v (v) t + \hat{\nu} v Z_t \tag{8}
\]

Given sufficiently large \( N \), it is possible to approximate every \( \xi^n \) with its mean, \( \Delta t \). It is easy to show that the expectation in (7) could be rewritten as:

\[
P^n(y, v) = \mathbb{E} \left[ \mathbb{E} \left[ e^{-r \xi^n} \cdot F^{n+1} (Y^{Y^{y,n}, V^{n}}, V^n) \cdot \mathbb{I}_{\{Y^{Y^{y,n}, V^{n}} \geq \xi^n > 0 \}} \right] \right], \tag{9}
\]

where \( \mathbb{I}_{\xi^n} := \{W_t : \xi \leq t\} \). With respect to \( \mathbb{I}_{\xi^n} \) the value of \( V_{n,k}^{Y^{Y^{y,n}, V^{n}}} \) is deterministic and assumed to be equal either to \( V_{n+1,k_u} \) or \( V_{n+1,k_d} \).

Next, we introduce the functions, which approximate the inner expectation in (9) subject to the Markov chain state:

\[
f_{k_u}^{n}(y, V_{n,k}) := \mathbb{E} \left[ e^{-r \xi^n} \cdot \mathbb{I}_{Y^{Y^{y,n}, V^{n}} + \mu_v (V_{n,k}) > 0} \cdot F^{n+1}_{\xi^n} (Y^{Y^{y,n}, V^{n}} + \mu_v (V_{n,k}), V_{n+1,k_u}) \right],
\]

\[
f_{k_d}^{n}(y, V_{n,k}) := \mathbb{E} \left[ e^{-r \xi^n} \cdot \mathbb{I}_{Y^{Y^{y,n}, V^{n}} + \mu_v (V_{n,k}) > 0} \cdot F^{n+1}_{\xi^n} (Y^{Y^{y,n}, V^{n}} + \mu_v (V_{n,k}), V_{n+1,k_d}) \right],
\]

where \( Y^{y,v} \) is approximated by (8)
Thus, we have that $\hat{F}^n(y, V_{n,k})$ could be approximated as follows:

$$\hat{F}^n(y, V_{n,k}) \approx \rho_{k\Delta t}^{\Delta t}(n,k) \cdot f^k_u(y, V_{n,k}) + \rho_{k\Delta t}^{\Delta t}(n,k) \cdot f^k_d(y, V_{n,k}), n = N-1, \ldots, 0, k = 0, \ldots, n.$$ 

### 5 Fast Wiener-Hopf factorization

Let us remind that for a suitable process $X_t$, the commonly used Wiener-Hopf factorization formula (see e.g. [8]) is:

$$\mathbb{E}[e^{i\xi X_T}] = \mathbb{E}[e^{i\xi \bar{X}_T}] \mathbb{E}[e^{i\xi \bar{X}_T}], \forall \xi \in \mathbb{R},$$

where $\bar{X}_T = \sup_{0 \leq t \leq T} X_t$. With the notation:

$$\phi^+_q(\xi) := q \mathbb{E}\left[\int_0^\infty e^{-q t} e^{i\xi \bar{X}_T} dt\right] = \mathbb{E}\left[e^{i\xi \bar{X}_T}\right],$$

$$\phi^-_q(\xi) := q \mathbb{E}\left[\int_0^\infty e^{-q t} e^{-i\xi \bar{X}_T} dt\right] = \mathbb{E}\left[e^{i\xi \bar{X}_T}\right],$$

one can write (10) as:

$$\frac{q}{q + \psi(\xi)} = \phi^+_q(\xi) \phi^-_q(\xi).$$

This equation is a special case of the W-H factorization of the symbol of a Pseudo-Differential Operator (PDO). In this case, the symbol is $\frac{q}{q + \psi(\xi)}$, and the PDO is

$$\varepsilon := \frac{q}{q-L} = q\left(q + \psi(D)\right)^{-1}.$$ 

For a stream $g(X_t) : \varepsilon g(x) = q \mathbb{E} \left[ \int_0^\infty e^{-q t} g(t) dt \mid X_0 = x \right].$

The operator $\varepsilon$ is referred to as the normalized resolvent or the expected present value operator (EPV).

The factors $\phi^\pm_q(\xi)$ could also be interpreted as the symbols of the EPV operators $\varepsilon^+_q = \phi^+_q(D)$:

$$\varepsilon^+ g(x) = q \mathbb{E} \left[ \int_0^\infty e^{-q t} g(X_t) dt \mid X_0 = x \right].$$

$$\varepsilon^- g(x) = q \mathbb{E} \left[ \int_0^\infty e^{-q t} g(X_t) dt \mid X_0 = x \right].$$

In our case it is also true that $\varepsilon = \varepsilon^- \varepsilon^+$

Using the approach described in [8] one can show that $f^k_u$ could be found as the solution to the problem for $n = N - 1, N - 2, \ldots, 0$:

$$\left\{ \begin{array}{l}
(q - L_{n,k}) f^k_u(y) = \Delta t^{-1} f^k_{n+1}(y), \quad y > -\frac{\rho}{\sigma} V_{n,k} u \\
q f^k_u(y) = 0, \quad y \leq -\frac{\rho}{\sigma} V_{n,k} u
\end{array} \right.$$ 

where $q = \Delta t^{-1} + r$ and the infinitesimal generator
\[ L_{n,k} = \mu_y(V_{n,k}) \partial_y + \frac{1}{2} \beta^2 V_{n,k} \partial^2_{yy} \]

Notice, that \( f^k_N(y) = G(He^{y\frac{\Delta t}{\sigma}}^{N,k}), \ k = 0, 1, ..., n. \)

The problem for \( f^k_n \) could be written analogously as:

\[
\begin{aligned}
(q - L_{n,k})f^k_d(y) &= \Delta t^{-1}f^k_{n+1}(y), \ y > -\frac{\rho}{\sigma} V_{n,k} \\
\int f^k_d(y) &= 0, \ y \leq -\frac{\rho}{\sigma} V_{n,k}
\end{aligned}
\]

To solve the arising problems, we suggest to use the instrument of Fast Wiener-Hopf Factorization (FWHF) described in [8].

The solution to each problem could be found in terms of \( \varepsilon^+_q \) and \( \varepsilon^-_q \) - the Wiener-Hopf factors and written as follows:

\[
\begin{aligned}
f^k_u(y) &= (q\Delta t)^{-1}\varepsilon^-_q(-\frac{\rho}{\sigma} V_{n,k} + \infty)(y)\varepsilon^+_q f^k_{n+1}(y) \\
\int f^k_d(y) &= (q\Delta t)^{-1}\varepsilon^-_q(-\frac{\rho}{\sigma} V_{n,k} + \infty)(y)\varepsilon^+_q f^k_{n+1}(y)
\end{aligned}
\]

In the case of Heston model, the process \( Y \), with \( V(t) \) fixed as \( V_{n,k} \), has a known explicit formula for its characteristic exponent:

\[
\psi(\xi) = \frac{\sigma^2_{(n,k)}}{2} \xi^2 - i\gamma_{(n,k)} \xi, \ \sigma_{(n,k)} = \hat{\rho} \sqrt{V_{n,k}}, \gamma_{(n,k)} = \mu_y(V_{n,k})
\]

We now can obtain the closed formulae for WH operators:

\[
\begin{aligned}
\varepsilon^+_q u(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\xi x} \frac{\beta^*}{\beta^* - i\xi} \hat{u}(\xi) d\xi, \\
\varepsilon^-_q u(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\xi x} \frac{-\beta^-}{-\beta^- + i\xi} \hat{u}(\xi) d\xi, \\
\beta^+_{(n,k)} &= -\frac{\gamma_{(n,k)} - \sqrt{\gamma_{(n,k)}^2 + 2\sigma_{(n,k)}^2 q}}{\sigma_{(n,k)}^2}, \\
\beta^-_{(n,k)} &= -\frac{\gamma_{(n,k)} + \sqrt{\gamma_{(n,k)}^2 + 2\sigma_{(n,k)}^2 q}}{\sigma_{(n,k)}^2},
\end{aligned}
\]

(11)

where \( \hat{u}(\xi) = \int_{-\infty}^{\infty} e^{-i\xi x} u(x) dx \) is the Fourier transform of \( u(x) \). Using a technique from [8] one can efficiently implement formulae (11) by means of a Fast Fourier Transform procedure. The convergence of the method can be accelerated by means of the technique described in [9].
6 Numerical results

In this section we provide the results of the numerical experiments, that demonstrates the accuracy and computational performance of the method proposed. We will refer to our method as M&FWH, meaning Markov Chain & Fast Wiener-Hopf approach. In this example we use the European down-and-out put in Heston model with the following parameters: \( \nu_0 = 0.01, \kappa = 2, \theta = 0.01, \sigma = 0.2, \rho = 0.5, \xi = 0.095, T = 1, K = 100, H = 90. \) To perform the experiment we used a PC with a configuration: Intel CPU, 2.4 GHz, RAM 8 Gb, Windows 10. We checked the performance of our method against prices obtained by the advanced M-C method [1], and a standard M-C. The advanced Alfonsi M-C results were obtained using the code implemented into the program platform Premia [10].

The algorithm parameters are as follows: \( N \) is the number of time steps, \( L \) is a scale coefficient, which controls the localization error of the FFT, and \( m \) is a number of points, meant to control its the discretization error. \( S \) is a price of the underlying asset.

<table>
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<tr>
<th>( S ) – spot</th>
<th>M-C Alfonsi, 600 000 iterations, Generator: Knuth, conf, int.: 0.05</th>
<th>M&amp;FWH ( N=50, M=2^{13}, L=3 )</th>
<th>M&amp;FWH ( N=100, M=2^{13}, L=3 )</th>
<th>M&amp;FWH ( N=300, M=2^{13}, L=3 )</th>
<th>standard M-C (traject: 15000, steps: 1000)</th>
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7 Conclusions

The numerical experiments show that the method has a nice computational performance in comparison to the advanced Monte-Carlo method [1], and reflects the behavior of the function in the proximity of the barrier quite well. Another advantage that should be mentioned is that to evaluate an option for all the prices in quite a vast range you only need to use the method once. This can massively improve the speed of the computations where large arrays of underlying asset prices are involved, given all other parameters constant. The method allows a generalization to Lévy process with jumps, and, with slight modification, can be used other types of barrier options as well. Numerical experiments also demonstrate the accuracy (results obtained by Monte-Carlo methods are in good agreement with the method results) and fast convergence of the proposed method.

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