GQVIs for Studying Competitive Equilibrium Problem when Utilities are Locally Lipschitz and Quasi-Concave

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Abstract

We establish an existence result on competitive equilibrium problem for an exchange economy when the consumers’ utilities are represented by a locally Lipschitz continuous and quasi-concave functions. The consumer’s demand is found to be actually a multi-valued map. Furthermore, any competitive equilibrium satisfies Walras’ law, too. To achieve this goal, the theory of Nonsmooth Analysis combined with the Generalized Quasi-Variational Inequalities (GQVIs) is used.

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1 Introduction

Starting from the papers of Benedetti et al. [6] and of Rockafellar et al. [10], our purpose here is to generalize the existence result on Arrow-Debreu equilibrium problem for the pure exchange economies (see for details [4, 9]). At this aim, we weak the assumptions on how the consumer expresses the preferences
when has to choose a commodity bundle among the feasible ones. Specifically, the utility functions are assumed locally Lipschitz continuous and quasi-concave. These facts allow to treat with utility functions not differentiable (but however, existing in real contexts) and to obtain a multi-valued demand function, because quasi-concavity is weaker than concavity and than strictly concavity, too. Non-differentiability implies the use of the Clarke subdifferential (see [7]).

Our main results are Theorems 7 and 10, which derive by variational approach, i.e. by using generalized quasi-variational inequalities (briefly GQVI). In Theorem 7, we show that any solution to a suitable GQVI is a solution to competitive economic equilibrium problem. In Theorem 10, recalling a result of Cubiotti in [8], we show that GQVI problem admits at least one solution. Furthermore, under the assumptions of non-local satiation and global desirability defined in a suitable compact set, in Proposition 5 we show that any competitive equilibrium is a Walras competitive equilibrium.

As said above our existence result on competitive equilibrium is general for the pure exchange economies because it includes the results in [1, 2] and, probability, also the ones in [3, 12] related to the dynamic case introduced in [11, 13] if we replace the concavity condition with quasi-concavity condition.

This paper is organized as follows: in Section 2, we shall describe the model, formulate the competitive economic equilibrium problem and list the assumptions; in Section 3, we shall establish that a solution of the competitive economic equilibrium problem is a solution to a generalized quasi-variational inequality defined on a suitable compact; in Section 4, we shall prove the existence of a solution to GQVI and thus, as consequence, the existence of a competitive equilibrium for the pure exchange economy.

2 Model and assumptions

We consider a pure exchange economy $E$ with $\ell$ finite types of commodities indexed in $H = \{1, ..., \ell\}$ and $n$ consumers indexed in $I = \{1, ..., n\}$ (where $\ell, n \in \mathbb{N}$).\footnote{We denote by $\mathbb{N}$ and $\mathbb{R}$, the set of natural number and the field of real numbers, respectively. We write $\mathbb{R}^n$ for the $n$-dimensional Euclidean space of the real $n$-vectors $x = (x_1, ..., x_n)$; while, we write $\mathbb{R}_+^n$, $\mathbb{R}_+^n$ and $\mathbb{R}_{0+}^n$ for the cone of non-negative, positive and strongly positive vectors, respectively. Furthermore, the set $\Delta_{n-1} = \{x \in \mathbb{R}_+^n : \sum_{i=1}^{n} x_i = 1\}$ indicates the unit simplex of $\mathbb{R}_+^n$. We adopt the usual notation for vector inequalities, that is: for any $x, y \in \mathbb{R}^n$ one has $x \geq y$ if $x - y \in \mathbb{R}_+^n$; $x > y$ if $x - y \in \mathbb{R}_{0+}^n$ and $x \gg y$ if $x - y \in \mathbb{R}_{0+}^n$. We assume $\mathbb{R}^n$ equipped with the standard inner product $\langle x, y \rangle := \sum_{i=1}^{n} x_i y_i$ and with the norm $|x| := \langle (x, x) \rangle^{\frac{1}{2}}$ for any $x, y \in \mathbb{R}^n$. Open and closed balls of centre $x \in \mathbb{R}^n$ with radius $\varepsilon > 0$ are denoted by $B\varepsilon(x)$ and $\overline{B}\varepsilon(x)$, respectively. Given $X \subseteq \mathbb{R}^n$, its interior is $\text{int}(X)$, its boundary is $\partial X$ and its closure is $\overline{X}$.} We regard $\mathbb{R}_+^\ell$ as the commodity space. Each consumer $i \in I$ is
characterized by his consumption set $X^{(i)} \subseteq \mathbb{R}^\ell_+$, his preferences described by an utility function $u_i : X^{(i)} \to \mathbb{R}$ and of his initial endowment $\omega^{(i)} \in X^{(i)}$. The aggregate endowment is $\omega := \sum_{i \in I} \omega^{(i)}$. A pure exchange economy is the tuple $\mathcal{E} = \{X^{(i)}, u_i, \omega^{(i)}\}_{i \in I, \omega}$. An allocation is denoted by $x = (x_1, ..., x_n) \in \mathbb{R}^{n \times \ell}$ where $x^{(i)} = (x^{(i)}_1, ..., x^{(i)}_\ell) \in X^{(i)}$ for any $i \in I$. An allocation $x$ is feasible if $\sum_{i \in I} x^{(i)} \leq \omega$. By assuming the vector $p = (p_1, ..., p_\ell) \in P := \Delta_{\ell-1}$ as a price system, for any initial endowment $\omega^{(i)}$ the consumer $i$’s budget set for $\omega^{(i)}$ is given by the multifunction $B^{(i)} : P \to 2^{\mathbb{R}^\ell_+}$ defined by

$$B^{(i)}(p) = \{ x \in \mathbb{R}^\ell_+ : \langle p, x \rangle \leq \langle p, \omega^{(i)} \rangle \}.$$

A state of $\mathcal{E}$ is a pair $(p, x) \in \mathbb{R}^{\ell \times \ell}_+$ where $p$ is a price system and $x$ is a feasible allocation.

In a pure exchange economy $\mathcal{E}$ where the consumers are considered as price takers, for a suitable price $\bar{p}$, the problem of any consumer $i$ is to obtain the best commodity bundle $\bar{x}^{(i)}$ into his budget set $B^{(i)}(\bar{p})$ in according to own utility function $u_i$ and, contextually, all together the consumers have to respect the equality between demand and supply for any commodity $h \in H$. Formally, set $B(p) := \prod_{i \in I} B^{(i)}(p)$ one has

**Problem 1.** Find $(\bar{p}, \bar{x}) \in P \times \mathbb{R}^{\ell \times \ell}_+$ with $x \in B(\bar{p})$ such that:

$$u_i(\bar{x}^{(i)}) = \max_{x^{(i)} \in B^{(i)}(\bar{p})} u_i(x^{(i)}) \quad \forall i \in I \quad (1a)$$

$$\sum_{i \in I} (\bar{x}^{(i)}_h - \omega^{(i)}_h) \leq 0 \quad \forall h \in H \quad (1b)$$

**Definition 1.** A pure exchange economy $\mathcal{E}$’s state $(\bar{p}, \bar{x})$ solving Problem 1 is said to be a competitive equilibrium for the $\mathcal{E}$.

**Definition 2.** A competitive equilibrium $(\bar{p}, \bar{x})$ is said to be a Walras competitive equilibrium if in addition to (1a) and (1b) satisfies also the following condition

$$\langle \bar{p}, \bar{x}^{(i)} - \omega^{(i)} \rangle = 0 \quad \forall i \in I \quad (1c)$$

We list below the assumptions that will use to solve Problem 1.

For any $h \in H$:

**Assumption 1.** There exists a price $q_h$ such that $q_h \geq \frac{1}{\ell}$ and $q_h \leq p_h$ for $p_h \neq 0$, where $p_h$ is the $h$th entry of a price system $p \in P$.

For any $i \in I$:

**Assumption 2.** $\omega^{(i)}$ is such that $\sum_{i \in I} \omega^{(i)} \gg 0$ and contains at least one good $h \in H$ with price $q_h > 0$ or greater $p_h > q_h$;
Assumption 3. \( X^{(i)} \) is a closed, convex subset of \( \mathbb{R}^\ell_+ \).

Assumption 4. Consider an open convex \( A \supseteq X^{(i)} \)

a. \( u_i(x^{(i)}) \) is a local Lipschitz continuous function on \( A \);

b. \( u_i(x^{(i)}) \) is a quasi-concave function on \( A \);

\[
\text{Set } K^{(i)} = \prod_{i \in I} \left( \bigcup_{h \in \mathcal{H}} \left\{ x^{(i)} \in X^{(i)} : x_h^{(i)} = \omega_h^{(i)} \right\} \right) \cap \prod_{h \in \mathcal{H}} \left[ 0, \sum_{a \in I} \omega_b^{(a)} \right] 
\]

c. \( 0 \notin \partial^0 (-u_i)(x^{(i)}) \) for all \( x^{(i)} \in K^{(i)} \),

d. \( (-u_i)^0(x^{(i)}, i_h) < 0 \) for all \( x^{(i)} \in K^{(i)} \) and \( h \in \mathcal{H} \) such that \( x_h^{(i)} = 0 \) where \( i_h \)

is the unit vector of the \( h \)th axis.

**Remark 3.** The Assumption 1 guarantees the existence of a positive minimum price \( q_k \) for any commodity \( h \in \mathcal{H} \). This fact, in addition with the Assumption 2, where the initial endowment \( \omega^{(i)} \in \mathbb{R}^\ell_+ \), permits to any consumer \( i \) to be always active in the trades. The Assumption 3 is standard and from \( \omega^{(i)} \in X^{(i)} \) and above considerations one has in particular \( X^{(i)} \subseteq \mathbb{R}^\ell_+ \). The Assumption 4 regards the alternative commodity bundles which each consumer \( i \) choices on the basis of own utility function. In details: 4.a extends the condition of continuity of the preference relation considering, now, not differentiable utilities, too; 4.b assures that the indifference surfaces are convex; 4.c represents the non-satiation condition rewritten in terms of Clark subdifferential; 4.d indicates how the global desirability of a commodity bundle \( x^{(i)} \) changes through the increasing of the Clarke derivatives along each direction.

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2 A function \( f : X \to \mathbb{R} \) is said to be *locally Lipschitz continuous* on \( X \subseteq \mathbb{R}^n \) if, for each \( x \in X \), there exist constants \( L > 0 \) and \( \varepsilon > 0 \) such that \( y \in B_\varepsilon(x) \) implies \( |f(y) - f(x)| \leq L|y - x|_n \).

3 A function \( f : X \to \mathbb{R} \) is said to be *quasi-concave* on a convex \( X \subseteq \mathbb{R}^n \) iff for every \( r < \sup_X f \), the set \( \{ x \in X : f(x) \geq r \} \) is convex.

4 Given an open \( A \subseteq \mathbb{R}^n \), a function \( f : A \to \mathbb{R} \), \( x_0 \in A \) and \( z \in \mathbb{R}^n \), the Clarke derivative of \( f \) at \( x_0 \) along the direction \( z \) is defined by \( f^\sigma(x_0, z) := \limsup_{x \to x_0, \sigma \to 0^+} \frac{f(x + \sigma z) - f(x)}{\sigma} \) and the Clarke subdifferential of \( f \) at \( x_0 \) is defined by \( \partial^0 f(x_0) := \{ T \in \mathbb{R}^n : f^\sigma(x_0, z) \geq \langle T, z \rangle, \text{ for all } z \in \mathbb{R}^n \} \).

Note that, from the definition, the Clarke subdifferential is a convex set. Furthermore, if \( f \) is a locally Lipschitz continuous function the Clarke derivative is finite and the Clarke subdifferential is compact in \( \mathbb{R}^n \) expressed by \( f^\sigma(x_0, z) = \max\langle T, z \rangle : T \in \partial^0 f(x_0) \), for all \( z \in \mathbb{R}^n \).
3 Variational approach

Now, the competitive economic equilibrium problem for a pure exchange economy $\mathcal{E}$ is revisited in key of generalized quasi-variational inequality problem.\footnote{Given a set $X$, we write $2^X$ for the family of all non-empty subsets of $X$. A correspondence or a multifunction between two sets $X$ and $Y$ is a function $F: X \to 2^Y$. The graph of a multifunction $F: X \to 2^Y$ is the subset of $X \times X$ defined by $\text{gr}(F) = \{(x, y) \in X \times Y : x \in X \land y \in F(x)\}$. Given $X \subseteq \mathbb{R}^n$ and two multifunctions $\Gamma: X \to 2^X$, $\Phi: X \to 2^{\mathbb{R}^n}$, the generalized quasi-variational inequality problem associated to $X$, $\Gamma$, $\Phi$ and denoted briefly $\text{GQVI}(X, \Gamma, \Phi)$, is to find $(\bar{x}, \bar{z}) \in X \times \mathbb{R}^n$ such that $\bar{x} \in \Gamma(\bar{x})$, $\bar{z} \in \Phi(\bar{x})$ and $(\bar{z}, \bar{x} - y) \leq 0$ for all $y \in \Gamma(\bar{x})$.}

**Problem 2.** Find $(\bar{p}, \bar{x}) \in P \times \mathbb{R}_{n+}^{nxT}$, with $x \in \mathcal{B}(\bar{p})$, such that there exists $T = (T^{(1)}, ..., T^{(n)}) \in \prod_{i \in I} \partial^p(\partial u_i(x^{(i)}))$ satisfying:

$$-\sum_{i \in I} \langle T^{(i)}, x^{(i)} - \bar{x}^{(i)} \rangle + \sum_{i \in I} \langle \bar{x}^{(i)} - \omega^{(i)}, p - \bar{p} \rangle \leq 0, \quad \forall (p, x) \in P \times \mathcal{B}(\bar{p}).$$

Let $K = \prod_{i \in I} K_i$, where $K^{(i)}$ is as in Assumptions 4.c and 4.d.

**Proposition 4.** Let $(\bar{p}, \bar{x}) \in P \times \mathbb{R}_{n+}^{nxT}$, with $\bar{x} \in \mathcal{B}(\bar{p})$, be satisfying condition (1b) of Problem 1. Then, $\bar{x} \in K$.

*Proof.* See proof of Proposition 1 of [1].

**Proposition 5.** Let Assumptions 4.c and 4.d be satisfied. Then, any competitive equilibrium is a Walras competitive equilibrium.

*Proof.* Let $(\bar{p}, \bar{x}) \in P \times \mathcal{B}(\bar{p})$ be a competitive equilibrium and fix $i \in I$. From Proposition 4, we have $\bar{x}^{(i)} \in K^{(i)}$. So, by Assumption 4.c and condition (1a) of Problem 1 it cannot be $\bar{x}^{(i)} \in \text{int}(\mathcal{B}(\bar{p}))$.

Moreover, Assumption 4.d and again condition (1a) of Problem 1 imply that $x^{(i)}_h > 0$ for all $h \in \mathcal{H}$. Therefore, the equation (1c) is verified, or equivalently, $(\bar{p}, \bar{x})$ is a Walras competitive equilibrium.

For the next result we need the following Propositions.

**Proposition 6.** Let Assumptions 4.a and 4.b be satisfied. Let $i \in I$ and let $x^{(i)}$, $z^{(i)} \in \mathbb{R}_{+}^n$ be such that $u_i(x^{(i)}) < u_i(z^{(i)})$. Then, $(-u_i)^\circ(x^{(i)}, z^{(i)} - x^{(i)}) \leq 0$.

*Proof.* Let $\{(y^{(n)}, t^{(n)})\}$ be a sequence in $A \times (0, 1)$ (the set $A$ is as in Assumptions 4.a and 4.b) such that $(y^{(n)}, t^{(n)}) \to (x^{(i)}, 0)$. From Assumption 4.a (and hence from continuity of $u_i$) and $u_i(x^{(i)}) < u_i(z^{(i)})$, we can suppose $u_i(y^{(n)}) < u_i(z^{(i)})$ for all $n \in \mathbb{N}$. From Assumption 4.b, one has $u_i(y^{(n)}) \leq u_i(y^{(n)} + t^{(n)}(z^{(i)} - y^{(n)}))$ for all $n \in \mathbb{N}$. Consequently,

$$\limsup_{n \to \infty} \frac{-u_i(y^{(n)} + t^{(n)}(z^{(i)} - y^{(n)})) + u_i(y^{(n)})}{t^{(n)}} \leq 0.$$
Taking into account the arbitrariness of the sequence \(\{(y^{(n)}, t^{(n)})\}\), conclusion follows. \(\square\)

**Theorem 7.** Let Assumption 1 and 4 entirely be satisfied. Moreover, let 
\((\bar{p}, \bar{x}) \in P \times \mathbb{R}^n_+\), with \(\bar{x} \in B(\bar{p})\), be a solution of Problem 2. Then, \((\bar{p}, \bar{x})\) is a solution of Problem 1.

**Proof.** Assume that \((\bar{p}, \bar{x})\) is a solution to Problem 2.

Let \(T \in \prod_{i \in I} \partial^p(-u_i)(\bar{x}^{(i)})\) satisfying inequality (2). Testing (2) with \((p, \bar{x})\), \(p \in P\), one has \(\langle \sum_{i \in I}(\bar{x}^{(i)} - \omega^{(i)}), \bar{p} - \bar{p}\rangle \leq 0\) for all \(p \in P\). Moreover, from \(\bar{x} \in B(\bar{p})\), we promptly obtain \(\langle \sum_{i \in I}(\bar{x}^{(i)} - \omega^{(i)}), \bar{p}\rangle \leq 0\) for all \(p \in P\). Hence,

\[
\langle \sum_{i \in I}(\bar{x}^{(i)} - \omega^{(i)}), p \rangle = \langle \sum_{i \in I}(\bar{x}^{(i)} - \omega^{(i)}), p - \bar{p}\rangle + \langle \sum_{i \in I}(\bar{x}^{(i)} - \omega^{(i)}), \bar{p}\rangle \leq 0
\]

for all \(p \in P\). Now, from Assumption 1 choosing \(p = (0, 0, 1, 0, ..., 0) \in P\), with 1 at the \(h\)th-position, we obtain condition (1b) of Problem 1.

At this point, we prove condition (1a) of Problem 1. Fix \(i \in I\). Testing (2) with \((\bar{p}, (\bar{x}^{(1)}, ..., \bar{x}^{(i-1)}, x^{(i)}, \bar{x}^{(i+1)}, ..., \bar{x}^{(n)})), x_i \in B^{(i)}(\bar{p})\), we obtain

\[
\langle T^{(i)}, x^{(i)} - \bar{x}^{(i)} \rangle \geq 0.
\] (3)

From Assumption 4.c, there exists \(h \in \mathbb{R}^n\) such that \(\langle T^{(i)}, h \rangle \neq 0\). Without loss of generality, we can suppose \(\langle T^{(i)}, h \rangle < 0\). Now, let \(z^{(i)} \in B^{(i)}(\bar{p})\) and put \(y^{(i)}_{\theta,1} = (1 - \theta)(\bar{x}^{(i)} + h) + \theta z^{(i)}\) and \(y^{(i)}_{\theta,2} = (1 - \theta)(\bar{x}^{(i)} + h) + \theta x^{(i)}\) for all \(\theta \in (0, 1)\). Then, we have

\[
\langle T^{(i)}, x^{(i)} - y^{(i)}_{\theta,1} \rangle = -(1 - \theta)\langle T^{(i)}, h \rangle > 0
\]

and, taking (3) into account,

\[
\langle T^{(i)}, y^{(i)}_{\theta,1} - \bar{x}^{(i)} \rangle \geq 0
\]

for all \(\theta \in (0, 1)\). Adding side to side the above inequality, we obtain:

\[
0 < \langle T^{(i)}, y^{(i)}_{\theta,1} - y^{(i)}_{\theta,2} \rangle = \langle T^{(i)}, z^{(i)} - \bar{x}^{(i)} \rangle \leq (-u_i)\varphi(\bar{x}^{(i)}, z^{(i)} - \bar{x}^{(i)}).
\]

From the above inequality and Proposition 6, it follows \(u_i(\bar{x}^{(i)}) \geq u_i(z^{(i)})\). From the arbitrariness of \(z^{(i)} \in B^{(i)}(\bar{p})\), condition (1a) of Problem 1 follows. \(\square\)

**Remark 8.** Under the assumptions of Theorem 7, condition (1b) actually holds as equality. Indeed, fix \(i \in I\) and define

\[
g_i(x^{(i)}) := -\sum_{h \in H} p_h(x^{(i)}_h - \omega_h^{(i)}) \quad \text{for all } x^{(i)} \in B^{(i)}(\bar{p}).
\] (4)
We claim that \( g_i(\bar{x}^{(i)}) = 0 \). Indeed, if not, taking in mind that \( \bar{x}_i \in \mathcal{B}^{(i)}(\bar{p}, \bar{y}) \), it should be \( g_i(\bar{x}^{(i)}) > 0 \). Then, for each \( h = 1, \ldots, l \), there exists \( \rho_h > 0 \) such that

\[
\bar{x}(\rho) := (\bar{x}^{(1)}, \ldots, \bar{x}^{(i-1)}, \bar{x}^{(i)} + \rho \bar{y}_h, \bar{x}^{(i+1)}, \ldots, \bar{x}^{(n)}) \in \mathcal{B}(\bar{p}) \text{ for all } \rho \in [0, \rho_h].
\]

(5)

Testing (2) with \((\bar{p}, \bar{x}(\rho))\), we obtain

\[
\rho \langle \bar{T}^{(i)}, \bar{i}_h \rangle \leq 0, \text{ for all } \rho \in [0, \rho_h].
\]

(6)

Thus, in view of Assumption 4.d, it must be \( \bar{x}^{(i)}_h > 0 \) for all \( h \in \mathcal{H} \). This fact, together with \( g_i(\bar{x}^{(i)}) > 0 \), implies \( \bar{x}^{(i)} \in \text{int}(\mathcal{B}^{(i)}(\bar{p})) \). Testing (2) with \((\bar{p}, x)\), where \( x = (\bar{x}^{(1)}, \ldots, \bar{x}^{(i-1)}, \bar{x}^{(i)}, \bar{x}^{(i+1)}, \ldots, \bar{x}^{(n)}) \), being \( x^{(i)} \) arbitrarily chosen in \( \mathcal{B}^{(i)}(\bar{p}) \), we obtain \( \langle \bar{T}^{(i)}, x^{(i)} - \bar{x}^{(i)} \rangle \leq 0 \) for all \( x^{(i)} \in \mathcal{B}^{(i)}(\bar{p}) \). Since \( \bar{x}^{(i)} \in \text{int}(\mathcal{B}^{(i)}(\bar{p})) \), from this inequality it follows \( T^{(i)} = \theta_{\mathbb{R}^l} \) in contradiction with Assumption 4.d.

4 Existence Results

Theorem 7 in Section 3 states that any solution to Problem 2 is a solution to Problem 1. Thus, to find a solution of Problem 1, we will prove, in this Section, that Problem 2 admits at least a solution.

First, we need the following Proposition:

Proposition 9. For any \( i \in \mathcal{I} \), the map \( \partial^o(-u_i) : \mathbb{R}_+^\ell \to 2^{\mathbb{R}^\ell} \) has closed graph.

Proof. Fix \( i \in \mathcal{I} \). Let \( (x^{(i)}, T^{(i)}) \in \mathbb{R}_+^\ell \times \mathbb{R}^\ell \) and let \( \{(x^{(m)}, T^{(m)})\} \) be a sequence in \( \text{gr}(\partial^o(-u_i)) \subset \mathbb{R}_+^\ell \times \mathbb{R}^\ell \) such that \( (x^{(m)}, T^{(m)}) \to (x^{(i)}, T^{(i)}) \in \mathbb{R}_+^\ell \times \mathbb{R}^\ell \), as \( m \to +\infty \). Clearly, one has \( (x^{(i)}, T^{(i)}) \in \mathbb{R}_+^\ell \times \mathbb{R}^\ell \). It remains to show that

\[
(x^{(i)}, T^{(i)}) \in \text{gr}(\partial^o(-u_i)).
\]

(7)

Fix \( z^{(i)} \in \mathbb{R}^\ell \). From \( (x^{(m)}, T^{(m)}) \in \text{gr}(\partial^o(-u_i)(x^{(m)}, z^{(i)})) \), for all \( m \in \mathbb{N} \), one has \( \langle T^{(m)}, z^{(i)} \rangle \leq \langle -u_i \rangle^o(x^{(m)}, z^{(i)}) \), for all \( m \in \mathbb{N} \). Then,

\[
\langle T^{(i)}, z^{(i)} \rangle = \lim_{m \to +\infty} \langle T^{(m)}, z^{(i)} \rangle = \lim_{m \to +\infty} \langle T^{(m)}, z^{(i)} \rangle \leq \lim_{m \to +\infty} \langle -u_i \rangle^o(x^{(m)}, z^{(i)}).
\]

(8)

We claim that

\[
\lim_{m \to +\infty} \langle -u_i \rangle^o(x^{(m)}, z^{(i)}) \leq \langle -u_i \rangle^o(x^{(i)}, z^{(i)}).
\]

(9)
Indeed, assume that (9) does not hold. Then, it should exist \( m \in \mathbb{N} \) and \( M \in \mathbb{R} \) such that

\[
(-u_i) \circ (x^{(m)}, z^{(i)}) > M > (-u_i) \circ (x^{(i)}, z^{(i)}),
\]

(10)

for all \( m \in \mathbb{N} \), with \( m \geq m \). From (10), for each fixed \( m \in \mathbb{N} \), with \( m \geq m \), we can find a sequence \((y_k^{(m)}, t_k^{(m)}) \in A \times \mathbb{R}_+\), where \( A \) is as in Assumptions 4.a and 4.b, such that \((y_k^{(m)}, t_k^{(m)}) \to (x^{(m)}, 0)\) as \( k \to +\infty \), and

\[
\frac{(-u_i)(y_k^{(m)} + t_k^{(m)} z^{(i)}) - (-u_i)(y_k^{(m)})}{t_k^{(m)}} > M,
\]

for all \( k \in \mathbb{N} \). Moreover, again from (10), we can find \( \delta > 0 \) such that

\[
\frac{(-u_i)(y^{(i)} + tz^{(i)}) - (-u_i)(y^{(i)})}{t} < M,
\]

for all \( y^{(i)} \in A \), with \(|y^{(i)} - x^{(i)}| < \delta\), and all \( t \in ]0, \delta[\). Thus, if we fix \( m \in \mathbb{N} \), with \( m \geq m \), such that \(|x^{(m)} - x^{(i)}| < \delta\), we can find \( k' \in \mathbb{N} \) such that \(|y_k^{(m)} - x^{(i)}| < \delta\) and \( t_k^{(m)} \in ]0, \delta[\). Then, the couple \((y_k^{(m)}, t_k^{(m)})\) should satisfy

\[
\frac{(-u_i)(y_k^{(m)} + t_k^{(m)} z^{(i)}) - (-u_i)(y_k^{(m)})}{t_k^{(m)}} < M < \frac{(-u_i)(y_k^{(m)} + t_k^{(m)} z^{(i)}) - (-u_i)(y_k^{(m)})}{t_k^{(m)}},
\]

a contradiction. Therefore, inequality (9) holds. At this point, observe that from inequalities (4) and (9) it easily follows that \((x^{(i)}, T^{(i)}) \in \text{gr}(\partial^o(-u_i))\). \( \square \)

**Theorem 10.** Let Assumptions 2, 4.a, 4.c, 4.d be satisfied. Then, Problem 2 admits at least a solution in \( P \times C \times Y \), where

\[
C := \left\{ x = \left( x_h^{(i)} \right)_{i \in I, h \in H} \in \mathbb{R}_+^{n \times \ell} : \sum_{i \in I} \sum_{h \in H} x_h^{(i)} \leq \sum_{i \in I} \sum_{h \in H} \omega_h^{(i)} \right\}.
\]

**Proof.** Let us divide the proof in several steps.

**Step 1.** Put \( X = P \times \mathbb{R}_+^{n \times \ell} \) and define
- \( u(x) = (u_1(x^{(1)}),..., u_n(x^{(n)})) \), for all \( x = (x^{(1)},..., x^{(n)}) \in \mathbb{R}_+^{n \times \ell} \),
- \( \Gamma(p, x) = P \times B(p) \), for all \((p, x) \in X\),
- \( \Phi(p, x) = (\sum_{i \in I} (x^{(i)} - \omega^{(i)}), \partial^o(-u)(x)) \), for all \((p, x) \in X\), where \( \partial^o(-u)(x) = (\partial^o(-u_1)(x^{(1)}),..., \partial^o(-u_n)(x^{(n)})) \), for all \( x \in \mathbb{R}_+^{n \times \ell} \).
By means of these notations, we can rewrite Problem 2 and the variational inequality (2) as follows:

**Problem 2 bis.** find \((\bar{p}, \bar{x})\) s.t. \((\bar{p}, \bar{x}) \in \Gamma(\bar{p}, \bar{x})\), and \((\hat{z}, T) \in \Phi(\bar{p}, \bar{x})\) such that

\[
\langle (\hat{z}, T), (\bar{p}, \bar{x}) - (p, x) \rangle \leq 0, \quad \text{for all } (p, x) \in \Gamma(\bar{p}, \bar{x}).
\]

\((11)\)

**Step 2.** Note that:

- the set \(X\) is nonempty closed and convex in \(\mathbb{R}^\ell \times \mathbb{R}^{n+\ell}\);
- the set \(K := P \times C \subset X\) is nonempty and compact in \(\mathbb{R}^\ell \times \mathbb{R}^{n+\ell}\);
- \(\Gamma(p, x)\) is a nonempty convex subset of \(X\), for all \((p, x) \in X\).

Moreover, recalling that \(\partial^i(-u_i)(x^{(i)})\) is (nonempty) convex and compact in \(\mathbb{R}^\ell\), for all \(i \in I\) and for all \(x^{(i)} \in \mathbb{R}^\ell_+\), we also have that

- \(\Phi(p, x)\) is a nonempty convex and compact subset of \(\mathbb{R}^\ell \times \mathbb{R}^{n+\ell}\), for all \((p, x) \in X\).

In the next steps we check the conditions of Theorem 3.2 of [8]6.

**Step 3.** Prove that the below condition holds true:

\[
(a_1) \quad \text{the set } \Lambda(\rho, \tau) := \left\{ (p, x) \in X : \inf_{(z, T) \in \Phi(p, x)} \langle (z, T), (\rho, \tau) \rangle \leq 0 \right\} \quad \text{is closed,}
\]

for each \((\rho, \tau) \in X - X\).

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6 For convenience of reader we report below the following

**Theorem 3.2 (of [8]).** Let \(X\) be a closed convex subset of \(\mathbb{R}^n\), \(K \subseteq X\) a nonempty compact set, \(\Phi : X \to 2^{\mathbb{R}^n}\) and \(\Gamma : X \to 2^X\) two multifunctions. Assume that:

(i) the set \(\Phi(x)\) is convex for each \(x \in K\), with \(x \in \Gamma(x)\);

(ii) the set \(\Phi(x)\) is nonempty and compact for each \(x \in X\);

(iii) for each \(y \in X - X\), the set \( \{ x \in X : \inf_{z \in \Phi(x)} \langle z, y \rangle \leq 0 \} \) is closed;

(iv) \(\Gamma\) is a lower semicontinuous multifunction (i.e. \( \{ x \in X : \Gamma(x) \cap A \neq \emptyset \} \) is open in \(X\), for each open set \(A \) in \(X\)) with closed graph and convex values.

Moreover, assume that there exists an increasing sequence \(\epsilon_k\) of positive real numbers, with \(X \cap B(0, \epsilon_1) \neq \emptyset\) and \(\lim_{k \to \infty} \epsilon_k = +\infty\) such that, if one puts \(D_k = B(0, \epsilon_k)\), for each \(k \in \mathbb{N}\) one has:

(v) \(\Gamma(x) \cap D_k \neq \emptyset\), for all \(x \in X \cap D_k\);

(vi) \(\sup_{y \in \Gamma(x) \cap D_k} \inf_{z \in \Phi(x)} \langle z, x - y \rangle > 0\), for each \(x \in (X \cap D_k) \setminus K\), with \(x \in \Gamma(x)\).

Then, there exists at least one solution to GQVI\((X, \Gamma, \Phi)\) belonging to \(K \times \mathbb{R}^n\).
Fix \((\rho, \tau) \in X - X\) and let \(\{(p^k, x_k)\}\) be a sequence in \(\Lambda(\rho, \tau)\) such that \((p^k, x_k) \to (p^*, x_*)\) as \(k \to \infty\). Let us to show that \((p^*, x_*) \in \Lambda(\rho, \tau)\). At first observe that, since \(X\) is closed, one has \((p^*, x_*) \in X\). Moreover, since \(\Phi(p^k, x_k)\) is compact for each \(k \in \mathbb{N}\), and the function \((z, T) \in \mathbb{R}^\ell \times \mathbb{R}^{n \times \ell} \to \langle (z, T), (\rho, \tau) \rangle\) is continuous in \(\mathbb{R}^\ell \times \mathbb{R}^{n \times \ell}\), then, for each \(k \in \mathbb{N}\), we can find \((z_k^k, T_k) \in \Phi(p^k, x_k)\) such that
\[
\langle (z_k^k, T_k), (\rho, \tau) \rangle \leq 0.
\] (12)

Note that, from the definition of \(\Phi\), for each \(k \in \mathbb{N}\), one has:
\[
z^k = -\sum_{i \in I} (x_k^{(i)} - \omega^{(i)}); \quad T_k \in \partial(-u)(x_k).
\] (13)

(14)

Moreover, in force of Assumption 4.a, then, for each \(k \in \mathbb{N}\), there exist an open neighborhood \(A_k\) of \(x_k\) in \(\mathbb{R}^{n \times \ell}\), and a constant \(L_k \geq 0\) such that:
\[
\sup_{T \in \partial(-u)(x_k)} |T| \leq L_k, \quad \text{for each } x \in A_k \cap \mathbb{R}^{n \times \ell}.
\] (15)

Furthermore, there exist an open neighborhood \(A_0\) of \(x_*\) in \(\mathbb{R}^{n \times \ell}\), and a constant \(L_0 \geq 0\) such that
\[
\sup_{T \in \partial(-u)(x_*)} |T| \leq L_0, \quad \text{for each } x \in A_0 \cap \mathbb{R}^{n \times \ell}.
\] (16)

At this point, observe that the family of open sets \(\{A_k\}_{k \in \mathbb{N} \cup \{0\}}\) is a covering of the compact set \(\{x_k\}_{k \in \mathbb{N}} \cup \{x_*\}\). Therefore, from (15) and (16), we infer that there exists a constant \(L \geq 0\) such that
\[
\sup_{T \in \partial(-u)(x_k)} |T| \leq L, \quad \text{for each } k \in \mathbb{N}.
\]

Consequently, from (14), up to a subsequence, we can suppose that the sequence \(\{T_k\}_{k \in \mathbb{N}}\) converges to some \(T_* \in \mathbb{R}^{n \times \ell}\). Now, from (13), we infer that
\[
z^k \to z_* := -\sum_{i \in I} (x_*^{(i)} - \omega^{(i)}), \quad \text{as } k \to \infty;
\] (17)

and, from Proposition 9 and (14), we also infer that
\[
T_* \in \partial(-u)(x_*).
\] (18)

Furthermore, from (12), passing to the limit as \(k \to \infty\), one has
\[
\langle (z^*, T_*), (\rho, \tau) \rangle \leq 0.
\] (19)

At this point, observe that conditions (17) and (18) mean that \((z^*, T_*) \in \Phi(p^*, x_*)\) and this latter condition, together with (19), gives \((p^*, x_*) \in \Lambda(\rho, \tau)\). Therefore, condition \((a_1)\) is proved.

**Step 4.** Now, let us to show that the below condition holds true
(a2) the map $\Gamma : X \to 2^X$ is lower semicontinuous with closed graph;

To this end, it is sufficient to prove that for every $(p^0, x_0) \in X$, every $(p, x) \in \Gamma(p^0, x_0)$, and every sequence $\{(p^k, \hat{x}_k)\}$ in $X$ such that $(\hat{p}^k, \hat{x}_k) \to (p^0, x_0)$ as $k \to +\infty$, there exists a sequence $\{(p^k, x_k)\}$ in $X$ such that $(p^k, x_k) \in \Gamma(\hat{p}^k, \hat{x}_k)$ for all $k \in \mathbb{N}$, and $(p^k, x_k) \to (p, x)$ as $k \to +\infty$ (see [5] at page 39, for instance).

So, let $(p^0, x_0), (p, x)$ and $\{(\hat{p}^k, \hat{x}_k)\}$ be as above. For each $i \in I$, using the fact that $(p, x) \in \Gamma(p^0, x_0)$, we have the following two situations:

either

$$\langle p^0, x^{(i)} \rangle < \langle p^0, \omega^{(i)} \rangle,$$

(20)

or

$$\langle p^0, x^{(i)} \rangle = \langle p^0, \omega^{(i)} \rangle.$$

(21)

Suppose that (20) holds. Then, since $\hat{p}^k \to p^0$, there exists $k_0 \in \mathbb{N}$ such that $\langle \hat{p}^k, x^{(i)} \rangle < \langle \hat{p}^k, \omega^{(i)} \rangle$, for all $k \in \mathbb{N}$, with $k \geq k_0$. So, in this case, if we put $x_k^{(i)} = x^{(i)}$ for $k \geq k_0$, and $x_k^{(i)} = 0$ for $k = 1, \ldots, k_0 - 1$, it is easy to check that $x_k^{(i)} \in \mathcal{B}^{(i)}(\hat{p}^k)$, for all $k \in \mathbb{N}$. Moreover, it is clear that $x_k^{(i)} \to x^{(i)}$ as $k \to +\infty$.

Suppose that (21) holds. Then, from the Assumption 2, we have $\langle p^0, x^{(i)} \rangle = \langle p^0, \omega^{(i)} \rangle > 0$. Consequently, since $\hat{p}^k \to p^0$, we have

$$\lim_{k \to \infty} \frac{\langle \hat{p}^k, \omega^{(i)} \rangle}{\langle \hat{p}^k, x^{(i)} \rangle} = \frac{\langle p^0, \omega^{(i)} \rangle}{\langle p^0, x^{(i)} \rangle} = 1.$$

Therefore, if we put $a_k = \max \left\{0, 1 - \frac{\langle \hat{p}^k, \omega^{(i)} \rangle}{\langle \hat{p}^k, x^{(i)} \rangle}\right\}$, for all $k \in \mathbb{N}$, and $x_k^{(i)} = (1 - a_k)x^{(i)}$ for all $k \in \mathbb{N}$, it is easy to check that $x_k^{(i)} \in \mathcal{B}^{(i)}(\hat{p}^k)$, for all $k \in \mathbb{N}$, and that $x_k^{(i)} \to x^{(i)}$ as $k \to +\infty$.

So, for each $i \in I$, in both cases (20) and (21), we can find a sequence $x_k^{(i)}$ which converges to $x^{(i)}$ and such that $x_k^{(i)} \in \mathcal{B}^{(i)}(\hat{p}^k)$, for all $k \in \mathbb{N}$. Consequently, if we put $x_k = (x_k^{(1)}, \ldots, x_k^{(n)})$, for all $k \in \mathbb{N}$, the sequence $\{(p^k, x_k)\}$, where $p^k = p$, for all $k \in \mathbb{N}$, satisfies $(p^k, x_k) \in \Gamma(\hat{p}^k, \hat{x}_k)$ for all $k \in \mathbb{N}$, and $(p^k, x_k) \to (p, x)$ as $k \to +\infty$, as desired. Therefore, $\Gamma$ is lower semicontinuous in $X$.

To show condition (a2) holds true, it remains to prove that $\Gamma$ has closed graph. To this end, let $\{(\hat{p}^k, \hat{x}_k)\}$ and $\{(p^k, x_k)\}$ be two sequences in $X$ such that $(p^k, x_k) \in \Gamma(\hat{p}^k, \hat{x}_k)$, for all $k \in \mathbb{N}$, and suppose that $(\hat{p}^k, \hat{x}_k) \to (p^0, x_0)$, $(p^k, x_k) \to (p, x)$, as $k \to \infty$. Let us to show that $(p, x) \in \Gamma(p^0, x_0)$.
Since $P$ is closed, one has $p \in P$. Moreover, being $(p^k, x_k) \in \Gamma(\hat{p}^k, \hat{x}_k)$ for all $k \in \mathbb{N}$, one has $\langle p^k, x_k \rangle \leq \langle \hat{p}^k, \omega^i \rangle$, for all $k \in \mathbb{N}$, and $i \in I$. Passing to the limit as $k \to \infty$, we obtain $\langle p^0, x^{(i)} \rangle \leq \langle p^0, \omega^i \rangle$, for all $i \in I$. Thus, $x \in B(p^0)$ which, together $p \in P$, gives $(p, x) \in \Gamma(p^0, x^0)$, as desired.

**Step 5.** It remains to show that condition

(a3) there exists $R_0$ such that, if for each $R \in [R_0, \infty[$ one has $\overline{B}_R(0) \cap X \neq \emptyset$, and:

(i) $\Gamma(p, x) \cap \overline{B}_R(0) \neq \emptyset$, for all $(p, x) \in X \cap \overline{B}_R(0);

(ii) $\sup_{(p', x') \in \Gamma(p, x) \cap \overline{B}_R(0)} \inf_{(z, T) \in \Phi(p, x)} \langle (z, T), (p, x) - (p', x') \rangle > 0$,

for all $(p, x) \in X \cap \overline{B}_R(0) \setminus K$, with $(p, x) \in \Gamma(p, x)$.

holds true as well.

Let $R_0 > 0$ be such that $\overline{B}_{R_0}(0) \subset \mathbb{R}^n \times \mathbb{R}^{n \times \ell}$ contains the compact set $K$. Then, for each $R \in [R_0, \infty[$, one $K \subset X \cap \overline{B}_R(0)$ and $P \times \{0\} \subset \Gamma(p, x) \cap \overline{B}_R(0)$, for all $(p, x) \in X$. Therefore, condition (i) of (a3) holds. Suppose that condition (ii) of (a3) does not hold. Then, it should exist $(\bar{p}, \bar{x}) \in X \cap \overline{B}_{R_0}(0) \setminus K$, with $(\bar{p}, \bar{x}) \in \Gamma(\bar{p}, \bar{x})$, such that:

$$\inf_{(z, T) \in \Phi(\bar{p}, \bar{x})} \langle (z, T), (\bar{p}, \bar{x}) - (p', x') \rangle \leq 0, \text{ for all } (p', x') \in \Gamma(\bar{p}, \bar{x}) \cap \overline{B}_{R_0}(0). \quad (22)$$

Now, let us put $p^* := (1/l, ..., 1/l) \in P$. Then, $(p^*, \bar{x}) \in \overline{B}_{R_0}(0) \cap X$. Moreover, from $(\bar{p}, \bar{x}) \in \Gamma(\bar{p}, \bar{x})$, it trivially follows $(p^*, \bar{x}) \in \Gamma(\bar{p}, \bar{x})$. Thus, we can test (22) with $(p', x') = (p^*, \bar{x})$. Doing so, we get $\inf_{(z, T) \in \Phi(\bar{p}, \bar{x})} \langle (z, T), (\bar{p} - p^*, 0) \rangle \leq 0$. Therefore, being $\Phi(\bar{p}, \bar{x})$ a compact set, it should exist $(\bar{z}, \bar{T}) \in \Phi(\bar{p}, \bar{x})$ such that $\langle (\bar{z}, \bar{T}), (\bar{p} - p^*, 0) \rangle \leq 0$. From the definition of $\Phi$, the previous inequality is equivalent to

$$\left\langle \sum_{i \in I} \left( \bar{x}^{(i)} - \omega^{(i)} \right), p^* - \bar{p} \right\rangle \leq 0 \implies \left\langle \sum_{i \in I} \left( \bar{x}^{(i)} - \omega^{(i)} \right), p^* \right\rangle \leq 0$$

taking in mind that $\bar{x} \in B(\bar{p})$. Consequently,

$$\sum_{i \in I} \sum_{h \in H} \left( \bar{x}_h^{(i)} - \omega_h^{(i)} \right) = \ell \left\langle \sum_{i \in I} \left( \bar{x}^{(i)} - \omega^{(i)} \right), p^* \right\rangle \leq 0.$$

Therefore, $\bar{x} \in C$. But this contradicts the fact that $(\bar{p}, \bar{x}) \in X \cap \overline{B}_{R_0}(0) \setminus K = (P \times \mathbb{R}^{n \times \ell}_+) \cap \overline{B}_{R_0}(0) \setminus (P \times C)$.

$\square$
GQVIs for competitive equilibrium problem

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