On the Structure of Paper-Floor Sequences

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Abstract

Paperfolding sequences are represented by 0’s and 1’s. In this paper, we introduce paper-floor sequences which are represented by non-negative integers. We investigate the structure of paper-floor sequences which represent the floors obtained by paperfoldings.

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1 Introduction

Since Davis and Knuth [3] introduced paperfolding sequences, they have been investigated by many researchers because of its application such as dynamical systems and binary reflected gray codes. Paperfolding sequences are represented by 0’s and 1’s along with upward creases and downward creases, respectively. Dekking, France and Poorten [4] improved the relation between the paperfolding sequences and dragon curves which were introduced in [3]. France
and Poorten [5] explored the algebraic properties of paperfolding sequences and suggested the associated Fourier series. Bates, Bunder and Tognetti [2] showed the relation between paperfolding sequences and binary reflected gray codes. France also studied the relation between paperfolding sequences and dynamical systems in [6]. Yun and Hahm [7, 8] investigated the counting problems of generalized paperfolding sequences $X_p^n$ and $(X_pY_q)^n$. In addition, Yun, Lim and Hahm classified generalized paperfolding sequences in [9].

In this paper, we assume that the left end of a sheet of paper is taped to the ground. Before we introduce paper-floor sequences, we define a positive paperfolding and a negative paperfolding as follows.

**Definition 1.1** (1) If we fold a sheet of paper right over left in half, this paperfolding is called a positive paperfolding.
(2) If we fold a sheet of paper right under left in half, this paperfolding is called a negative paperfolding.

Figure 1 shows a positive paperfolding and a negative paperfolding in Definition 1.1. We denote a positive paperfolding by $P$ and a negative paperfolding by $N$.

![Positive paperfolding and negative paperfolding](image)

**2 Preliminaries**

Now, we introduce paper-floor sequences using some examples. If we perform 1 positive paperfolding, then the paper has 2 floors and we give floor numbers from the bottom. In addition, if we unfold it, the paper is divided into 2 parts by a crease and the floor numbers correspond to 2 parts from the left as in Figure 2. Similarly, if we perform 2 positive paperfolding, then the paper has 4 floors and we give floor numbers from the bottom. In addition, if we unfold it, the paper is divided into 4 parts by 3 creases and the floor numbers correspond to 4 parts from the left as in Figure 2. Figure 2 also shows the relation between the floor numbers and 8 parts by 7 creases when we perform 2 positive paperfolding and then perform 1 negative paperfolding.

**Definition 2.1** A sequence of numbers of divided parts of a paper corresponding to floor numbers by a paperfolding is called a paper-floor sequence.
In Figure 2, \((1, 2), (1, 4, 3, 2)\) and \((5, 4, 1, 8, 7, 2, 3, 6)\) are examples of paper-floor sequences.

We use the notations in [1, 7]. Let \(X = X_1X_2\cdots X_n\) be a \(n\) paperfolding, where \(X_i \in \{P, N\}\) for \(i = 1, 2, \ldots, n\). Then \(S_X = ((S_X)_i)_{1 \leq i \leq 2^n - 1}\) and \(F_X = ((F_X)_i)_{1 \leq i \leq 2^n}\) denote the paperfolding sequence and the paper-floor sequence with respect to \(X\), respectively.

**Example 2.2**

1. Let \(X = PNP\). Then
   
   \[S_X = (1, 0, 0, 1, 1, 0)\quad \text{and} \quad F_X = (3, 6, 7, 2, 1, 8, 5, 4)\]

2. Let \(X = PPPNN\). Then
   
   \[S_X = (0, 1, 1, 1, 0, 0, 1, 1, 0, 1, 0, 0, 0, 1, 1, 0, 1, 0, 0, 1, 1, 1, 0, 0, 0, 1)\]
   
   and
   
   \[F_X = (17, 16, 1, 32, 25, 8, 9, 24, 21, 12, 5, 28, 29, 4, 13, 20, 19, 14, 3, 30, 27, 6, 11, 22, 23, 10, 7, 26, 31, 2, 15, 18)\]

   When \(X = PPPNN = P^3N^2\), we have \((S_X)_1 = 0, (F_X)_1 = 17\) and \((S_X)_4 = 1, (F_X)_4 = 32\) from Example 2.2.

**Definition 2.3**

Let \(X = X_1X_2\cdots X_n\) be a \(n\) paperfolding with \(X_i \in \{P, N\}\) for \(i = 1, 2, \ldots, n\). Then \(|X|\), \(|X|_P\) and \(|X|_N\) denote the number of paperfolding, positive paperfolding and negative paperfolding in \(X\), respectively.

By Definition 2.3, we get \(|X| = 8, |X|_P = 5\) and \(|X|_N = 3\) if \(X = PNP^3N^2P\). The following theorem can be easily obtained from Definition 4.

**Theorem 2.4**

Let \(X = X_1X_2\cdots X_n\) be a \(n\) paperfolding with \(X_i \in \{P, N\}\) for \(i = 1, 2, \ldots, n\). Then

1. \(|X| = |X|_P + |X|_N\).
2. \(2^n\) floors are built if we fold a strip of paper by \(X\).
If we rotate a strip of paper 180 degree, then the left side and the right side of a strip of paper are exchanged. When we fold a strip of paper once and then unfold it, the left side and the right side of the left half of a strip are unchanged but the left side and the right side of the right half of a strip are exchanged.

Note that the left side and the right side of some divided parts of paper are unchanged if we fold a strip of paper several times and then unfold it. In this cases, we give $+$ sign to those divided parts. Similarly, the left side and the right side of some divided parts of paper are exchanged if we fold a strip of paper several times and then unfold it. In this cases, we give $-$ sign to those divided parts.

![Figure 3: Rotation by paperfolding](image)

When $X = P^2$ in Figure 3, the signs of $(F_X)_1$, $(F_X)_2$, $(F_X)_3$ and $(F_X)_4$ are $+$, $-$, $+$ and $-$, respectively. We denote them by $(F_X)_1 = +$, $(F_X)_2 = -$,

$(F_X)_3 = +$ and $(F_X)_4 = +$.

**Definition 2.5** Let $X = X_1 X_2 \cdots X_n$ be a $n$ paperfolding with $X_i \in \{P, N\}$ for $i = 1, 2, \ldots, n$. Then $F_X = ((F_X)_i)_{1 \leq i \leq 2^n}$ is called the sign sequence of the paper-floor sequence $F_X$.

By Definition 2.5, we get, for example,

$F_{NP} = (+, -, +, -)$ and $F_{PN^2} = (+, -, +, -, +, -, +, -)$.

### 3 Main Results

First of all, we prove that any sign sequence $F_X$ in Definition 2.5 is always alternating.

**Lemma 3.1** Let $X = X_1 X_2 \cdots X_n$ be a $n$ paperfolding with $X_i \in \{P, N\}$ for $i = 1, 2, \ldots, n$. Then we have

$$(F_X)_i = \begin{cases} +, & i \text{ is odd} \\ -, & i \text{ is even} \end{cases}$$

for all $i$ with $1 \leq i \leq 2^n$. 

Proof. We prove the lemma using the mathematical induction.
When $X = P$ or $X = N$, the result is trivial.
We assume that the statement is true for $Y = X_1X_2 \cdots X_k$, where $X_i \in \{P, N\}$
for $1 \leq i \leq k$. By hypothesis, we have

\[
\overline{(F_Y)_i} = \begin{cases} 
  +, & i \text{ is odd} \\
  -, & i \text{ is even}
\end{cases} \tag{2}
\]

Let $X = X_1X_2 \cdots X_kX_{k+1}$, where $X_{k+1} \in \{P, N\}$. Then $X = YX_{k+1}$. Note
that $(F_Y)_i$th floor is divided into $(F_X)_{2i-1}$th floor and $(F_X)_{2i}$th floor by $X_{k+1}$.
If $i$ is odd, then $(F_Y)_i = +$ by (2). Note that the left side and the right side of
$(F_X)_{2i-1}$th floor are unchanged by $X_{k+1}$, and the left side and the right side
of $(F_X)_{2i}$th floor are exchanged by $X_{k+1}$. Hence

\[
(F_X)_{2i-1} = \overline{(F_Y)_i} = + \quad \text{and} \quad (F_X)_{2i} = -. \tag{3}
\]

If $i$ is even, then $(F_Y)_i = -$ by (2). Note that the left side and the right side
of $(F_X)_{2i-1}$th floor are exchanged by $X_{k+1}$, and the left side and the right side
of $(F_X)_{2i}$th floor are unchanged by $X_{k+1}$. Hence

\[
(F_X)_{2i-1} = + \quad \text{and} \quad (F_X)_{2i} = \overline{(F_Y)_i} = -. \tag{4}
\]

By the mathematical induction, we complete the proof.

Using Lemma 3.1, we show the relation between $F_X$ and $F_{XY}$, where $X$ is
a $n$ paperfolding and $Y \in \{P, N\}$.

**Theorem 3.2** Let $X = X_1X_2 \cdots X_n$ be a $n$ paperfolding with $X_j \in \{P, N\}$
for $j = 1, 2, \ldots, n$ and let $1 \leq i \leq 2^n$.
1. If $Y = P$, then $(F_{XY})_{2i-1} = (F_X)_i$ for odd $i$ and $(F_{XY})_{2i} = (F_X)_i$ for even $i$.
2. If $Y = N$, then $(F_{XY})_{2i-1} = 2^n + (F_X)_i$ for odd $i$ and $(F_{XY})_{2i} = 2^n + (F_X)_i$
for even $i$.

**Proof.** For $i$ with $1 \leq i \leq 2^n$, $(F_X)_i$th floor is divided into $(F_{XY})_{2i-1}$th floor
and $(F_{XY})_{2i}$th floor by $Y$.

1. Assume that $Y = P$. If $i$ is odd, then $(F_X)_i = +$ by Lemma 3.1. Since
$2i - 1$ is also odd, we get $(F_{XY})_{2i-1} = +$ by Lemma 3.1. Thus the left half of
$(F_X)_i$th floor is unchanged by $Y$ and hence $(F_{XY})_{2i-1} = (F_X)_i$. If $i$ is even,
then $(F_X)_i = -$ by Lemma 3.1. Since $2i$ is also even, we get $(F_{XY})_{2i} = -$ by
Lemma 3.1. Thus the right half of $(F_X)_i$th floor is unchanged by $Y$ and so
$(F_{XY})_{2i} = (F_X)_i$.

2. Assume that $Y = N$. If $i$ is odd, then $(F_X)_i = + = (F_{XY})_{2i-1}$ by Lemma
3.1. Note that the right half of $(F_X)_i$th floor goes down and so $2^n$ floors are
built at the bottom by $Y$. Hence $(F_{XY})_{2i-1} = (F_X)_i + 2^n$. If $i$ is even, then $(F_X)_i = - = (F_{XY})_{2i}$ by Lemma 3.1. Note that the left half of $(F_X)_i$th floor goes down and so $2^n$ floors are built at the bottom by $Y$. Hence $(F_{XY})_{2i} = (F_X)_i + 2^n$.

Thus we complete the proof.

In Example 2.2, we have

$$(F_X)_i + (F_X)_{i+1} = 9 = 2^3 + 1$$

for $X = PNP$ and odd $i$ with $1 \leq i \leq 2^3$. In addition, we also get

$$(F_X)_i + (F_X)_{i+1} = 33 = 2^5 + 1$$

for $X = P^3N^2$ and odd $i$ with $1 \leq i \leq 2^5$.

Now we generalize these results.

**Theorem 3.3** Let $X = X_1X_2\cdots X_n$ be a $n$ paperfolding with $X_j \in \{P, N\}$ for $j = 1, 2, \ldots, n$. For any odd $i$ with $1 \leq i \leq 2^n$, we have

$$(F_X)_i + (F_X)_{i+1} = 2^n + 1. \quad (5)$$

**Proof.** We prove the theorem by mathematical induction.

If $n = 1$, then $X = P$ or $X = N$. When $X = P$, we have

$$(F_X)_1 + (F_X)_2 = (F_P)_1 + (F_P)_2 = 1 + 2 = 2^1 + 1. \quad (6)$$

Similarly, we have, for $X = N$,

$$(F_X)_1 + (F_X)_2 = (F_N)_1 + (F_N)_2 = 2 + 1 = 2^1 + 1. \quad (7)$$

Thus $(F_X)_i + (F_X)_{i+1} = 2^n + 1$ when $n = 1$.

Assume that the statement is true for $n = k$. Let $Y = X_1X_2\cdots X_k$. Then we have

$$(F_Y)_i + (F_Y)_{i+1} = 2^k + 1$$

for any odd $i$ with $1 \leq i \leq 2^k$ by hypothesis.
Let $X = YX_{k+1}$, where $X_{k+1} \in \{P, N\}$. For $i$ with $1 \leq i \leq 2^k$, $(F_Y)_i$th floor is divided into $(F_X)_{2i-1}$th floor and $(F_X)_{2i}$th floor by $X_{k+1}$. For $i$ with $1 \leq i \leq 2^k$, let $(F_Y)_i = m_i$.

Suppose that $X_{k+1} = P$. Since $2^k$ floors are built by $Y$, one of two parts of $m_i$th floor remains and the other part of $m_i$th floor goes to $(2^k - m_i + 1)$th floor by $X_{k+1}$.

If $i$ is odd, then $(F_X)_{2i-1} = (F_YX_{k+1})_{2i-1} = (F_Y)_i = m_i$ by Theorem 3.2. Hence $(F_X)_{2i} = (F_YX_{k+1})_{2i} = 2^{k+1} - m_i + 1$. Thus

$$(F_X)_{2i-1} + (F_X)_{2i} = m_i + 2^{k+1} - m_i + 1 = 2^{k+1} + 1.$$  (9)

If $i$ is even, then $(F_X)_{2i} = (F_YX_{k+1})_{2i} = (F_Y)_i = m_i$ by Theorem 3.2. Hence $(F_X)_{2i-1} = (F_YX_{k+1})_{2i-1} = 2^{k+1} - m_i + 1$. Thus

$$(F_X)_{2i-1} + (F_X)_{2i} = m_i + 2^{k+1} - m_i + 1 = 2^{k+1} + 1.$$  (10)

Suppose that $X_{k+1} = N$. Since $2^k$ floors are built by $Y$, one of two parts of $m_i$th floor goes to $(2^k + m_i)$th floor and the other part of $m_i$th floor goes to $(2^k - m_i + 1)$th floor by $X_{k+1}$.

If $i$ is odd, then $(F_X)_{2i-1} = 2^k + (F_Y)_i = 2^k + m_i$ by Theorem 3.2. Hence $(F_X)_{2i} = 2^k - m_i + 1$. Thus

$$(F_X)_{2i-1} + (F_X)_{2i} = 2^k + m_i + 2^k - m_i + 1 = 2^{k+1} + 1.$$  (11)

If $i$ is even, then $(F_X)_{2i} = (F_YX_{k+1})_{2i} = (F_Y)_i = 2^k + m_i$ by Theorem 3.2. Hence $(F_X)_{2i-1} = 2^k - m_i + 1$. Thus

$$(F_X)_{2i-1} + (F_X)_{2i} = 2^k - m_i + 1 + 2^k + m_i = 2^{k+1} + 1.$$  (12)

If we set $l = 2i - 1$ in (10) and (12), then we get

$$(F_X)_l + (F_X)_{l+1} = 2^{k+1} + 1$$  (13)

for odd $l$ with $1 \leq l \leq 2^{k+1}$.

By the mathematical induction, we complete the proof.

Next theorem shows the relation between the first term of a paper-floor sequence and the first floor number of a $n$ paperfolding.
Theorem 3.4  Let $X = X_1X_2\cdots X_n$ be a $n$ paperfolding, where $X_i \in \{P, N\}$ for $i = 1, 2, \ldots, n$. Then

$$ (F_X)_1 = 1 + \sum_{i=1}^{n} d(X_i)2^{i-1}, \quad (14) $$

where $d(X_i) = 0$ for $X_i = P$ and $d(X_i) = 1$ for $X_i = N$.

Proof. We use the mathematical induction. Assume that $n = 1$.
If $X_1 = P$, then $(F_{X_1})_1 = 1$ and $l = 1 = 1 + \sum_{i=1}^{1} d(P)2^{i-1}$ since $d(P) = 0$.
If $X_1 = N$, then $(F_{X_1})_1 = 2$ and $l = 2 = 1 + \sum_{i=1}^{1} d(N)2^{i-1}$ since $d(N) = 1$.
Assume that the statement is true for $n = k$. Let $Y = X_1X_2\cdots X_k$. Then

$$ (F_Y)_1 = 1 + \sum_{i=1}^{k} d(X_i)2^{i-1} \quad (15) $$

by hypothesis. Let $X = X_1X_2\cdots X_kX_{k+1}$, where $X_{k+1} \in \{P, N\}$.
If $X_{k+1} = P$, then $(F_X)_1 = (F_{YX_{k+1}})_1 = (F_Y)_1$ by Theorem 3.2. Hence

$$ (F_X)_1 = (F_Y)_1 = 1 + \sum_{i=1}^{k} d(X_i)2^{i-1} = 1 + \sum_{i=1}^{k+1} d(X_i)2^{i-1}, \quad (16) $$

since $d(X_{k+1}) = 0$.
If $X_{k+1} = N$, then $(F_X)_1 = (F_{YX_{k+1}})_1 = 2^k + (F_Y)_1$ by Theorem 3.2. Then

$$ (F_X)_1 = 2^k + (F_Y)_1 = 2^k + 1 + \sum_{i=1}^{k} d(X_i)2^{i-1} = 1 + \sum_{i=1}^{k+1} d(X_i)2^{i-1}, \quad (17) $$

since $d(X_{k+1}) = 1$.
By the mathematical induction, we complete the proof.

Example 3.5  Let $X = PNP^2N^2P$. Then

$$ (F_X)_1 = 1 + \sum_{i=1}^{7} d(X_i)2^{i-1} = 1 + 2^1 + 2^4 + 2^5 = 51. \quad (18) $$

Example 3.5 shows that the first term of paper-floor sequence $F_X$ with $X = PNP^2N^2P$ is 51. Corollary 3.6 can be directly obtained from Theorem 3.4.
Corollary 3.6 Let $X = X_1 X_2 \cdots X_n$ be a $n$ paperfolding, where $X_i \in \{P, N\}$ for $i = 1, 2, \ldots, n$. Assume that $(F_X)_1 = s$ and $d(X_i) = 0$ for $X_i = P$ and $d(X_i) = 1$ for $X_i = N$. Then
\[
s - 1 = a_1 + a_2 2^1 + a_3 2^2 + \cdots + a_n 2^{n-1},
\] (19)
where $a_i = d(X_i)$ for $1 \leq i \leq n$.

When $n$ is given, Corollary 3.6 tells us that we can find $X$ if we know $|X|$ and the number of first term of paper-floor sequence $F_X$.

Example 3.7 If $|X| = 6$ and $(F_X)_1 = 34$, then $X = X_1 X_2 \cdots X_6$, where $X_i \in \{P, N\}$ for $i = 1, 2, \ldots, 6$. Note that
\[
34 - 1 = 1 + 2^5 = 1 + 0 \cdot 2^1 + 0 \cdot 2^2 + 0 \cdot 2^3 + 0 \cdot 2^4 + 1 \cdot 2^5.
\]
Hence $X_1 = N$, $X_2 = P$, $X_3 = P$, $X_4 = P$, $X_5 = P$ and $X_6 = N$. Therefore we conclude that $X = NPPPPN = NP^4N$.

Using Corollary 3.6, we obtain the following.

Theorem 3.8 Let $X$ and $Y$ be two paperfoldings. Then $X = Y$ if and only if $F_X = F_Y$.

Proof. If $X = Y$, then it is trivial that $F_X = F_Y$. Assume that $F_X = F_Y$. Since the numbers of terms in $F_X$ and $F_Y$ are same, we get $|X| = |Y|$. Let $X = X_1 X_2 \cdots X_n$ and $Y = Y_1 Y_2 \cdots Y_n$. Since $(F_X)_1 = (F_Y)_1$, we have, by Corollary 3.6,
\[
s - 1 = a_1 + a_2 2^1 + \cdots + a_n 2^{n-1} = b_1 + b_2 2^1 + \cdots + b_n 2^{n-1},
\] (20)
where $(F_X)_1 = s$, $a_i = d(X_i)$ and $b_i = d(Y_i)$ for $1 \leq i \leq n$. Note that $a_i, b_i \in \{0, 1\}$ for $1 \leq i \leq n$. Since $(a_1 - b_1) + (a_2 - b_2) 2^1 + \cdots + (a_n - b_n) 2^{n-1} = 0$, we have $a_i = b_i$ for $1 \leq i \leq n$. So, $X_i = Y_i$ for all $i = 1, 2, \ldots, n$. Thus $X = Y$. Therefore we complete the proof.

Theorem 3.8 tells us that a paper-floor sequence is uniquely determined by a paperfolding.

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