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On Computing Supply Chain Scheduling Using Max-Plus Algebra

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Abstract

In this paper we discuss the problems of supply chain scheduling for the distribution of oil feedstock from a fuel terminal to another one using tankers transportation. We derive some algorithms of supply chain scheduling computational problems using max-plus algebra. The resulting algorithms are used in some cases of supply chain scheduling in order to obtain an optimal scheduling i.e., when the tankers must leave so that the arrival time of the tankers is appropriate to meet the customer demand.

Keywords: Supply chain; max-plus algebra; optimal scheduling

1 Introduction

Many problems in operation research, performance analysis, manufacturing, communication network, etc. can be modeled as discrete event systems with maximum timing constraints (see [7, 8]). An algebra underlying such systems is based on two operations maximization and addition and called Max-Plus Algebra (see [1, 2]). Some works of the transportation systems by using max-plus algebra, especially scheduling problem of city bus routes, busway, analysis of aircraft transit timetable in airport, and monorail and train using max-plus algebra can be found in [3, 4, 5] and [6]. Elmahi et al ([9]) discussed modeling and control of a supply chain by using max plus algebra. Ema et al ([10]) discussed scheduling problem of the supply chain using max-plus algebra. In the [10], they discussed the distribution of oil feedstocks from a fuel terminal to another one in order for tankers to arrive on time which means the date of tanker arrivals is equal to the date of demand. In the discussion it is assumed that the number of tankers is two and their capacity exceeds all requests so that all of tankers arrive on time. In this paper, we consider the capacity of each fuel tanker is not always exceed request, and focus on at most two tankers cases . We discuss some possible cases and analyze it; and derive some algorithms for supply chain scheduling problem from computational point of view.

2 Max-plus algebra and Supply chain

In this section we briefly introduce the notion of max-plus algebra and some related notation used in the discussion, and supply chain . Detailed discussion about the max-plus algebra can be found in [1, 2] while the definition of the supply chain can be seen in [10].

Supply chain is a system process involving the production, delivery, storage, distribution and sale of products in order to meet demand for such products. We can also consider that supply chain is delivery of goods or services from supplier to customer. The objective of supply chain is to ensure a product is delivered to the right locations and at appropriate time to meet consumer demand without creating excess or shortage of stock.

Max-plus algebra is defined as $\mathbb{R}_{\max} = (\mathbb{R}_\varepsilon, \oplus, \otimes)$, where $\mathbb{R}_\varepsilon = \mathbb{R} \cup \{\varepsilon\}$ with $\mathbb{R}$ is the set of real numbers, $\varepsilon \stackrel{\text{def}}{=} -\infty$[1], $x \oplus y \stackrel{\text{def}}{=} \max\{x, y\}$ and $x \otimes y \stackrel{\text{def}}{=} x + y$ for every $x, y \in \mathbb{R}_\varepsilon$. Furthermore, for brevity, the idempotent commutative semiring $(\mathbb{R}_\varepsilon, \oplus, \otimes)$ is written as $\mathbb{R}_{\max}$. Power in the max-plus algebra is commonly introduced by using associative properties. The set of natural numbers combined with the usual zero number is denoted by $\mathbb{N}$. We define $x^{\otimes n} \stackrel{\text{def}}{=} \underbrace{x \otimes x \otimes \ldots \otimes x}_{n}$ for $x \in \mathbb{R}_\varepsilon$ and for all $n \in \mathbb{N}$ with $n \neq 0$, while for $n = 0$, $x^{\otimes n} \stackrel{\text{def}}{=} e \ \ (= 0)$.

We introduce a matrix over $\mathbb{R}_{\max}$. The set of matrices of size $m \times n$ in max-plus algebra is denoted by $\mathbb{R}_{\max}^{m \times n}$. For $n \in \mathbb{N}$ with $n \neq 0$, $\underline{n} \stackrel{\text{def}}{=} \{1, 2, \ldots, n\}$. An element $A \in \mathbb{R}_{\max}^{m \times n}$ in i^{th} row and j^{th} column is denoted by $a_{i,j}$ for $i \in \underline{m}$

[1] In max-plus algebra ε is defined as $-\infty$ where is in conventional algebra usually ε is an arbitrary positive number close to zero.

and $j \in \underline{n}$. Sometimes the elements $a_{i,j}$ is also denoted as $[A]_{i,j},\ i \in \underline{m},\ j \in \underline{n}$. An identity matrix of size $n \times n$ in $\mathbb{R}_{\max}$ is denoted by E, which is the main diagonal elements of the matrix are equal to $0 = (e)$ and the other elements are equal to ε.

The addition of matrices $A, B \in \mathbb{R}_{\max}^{m\times n}$ is denoted by $A \oplus B$ and defined by $[A \oplus B]_{i,j} = a_{i,j} \oplus b_{i,j} = \max\{a_{i,j}, b_{i,j}\}$. For $A \in \mathbb{R}_{\max}^{m\times n}$ and scalar $\alpha \in \mathbb{R}_{\max}$ multiplication $\alpha \otimes A$ is defined as $[\alpha \otimes A]_{i,j} \stackrel{\text{def}}{=} \alpha \otimes a_{i,j}$ for $i \in \underline{n}$ and $j \in \underline{m}$. For $A \in \mathbb{R}_{\max}^{m\times p}$ and $B \in \mathbb{R}_{\max}^{p\times n}$, the matrix multiplication $A \otimes B$ is defined as $[A \otimes B]_{i,j} = \bigoplus_{k=1}^{p} a_{i,k} \otimes b_{k,j} = \max_{k\in\underline{p}}\{a_{i,k} + b_{k,j}\}$, for $i \in \underline{m}$ and $j \in \underline{n}$. Matrix multiplication is similar to ordinary algebra matrix multiplication where $+$ and $\times$ are replaced by $\oplus$ and $\otimes$, respectively.

Next we discuss the form of the equation $A \otimes x = b$, with $A \in \mathbb{R}_{\max}^{m\times n},\ x \in \mathbb{R}_{\max}^{n}$ and $b \in \mathbb{R}_{\max}^{m}$. This equation is related to the supply chain scheduling, i.e. determining a date of departure of a tanker carrying oil feedstock over a consumer demand. As in the regular algebra, the equation $A \otimes x = b$ does not always have solution. Moreover, the solution is not necessarily unique. But this equation always has a Largest sub-solution that satisfies $A \otimes x \leq b$. This solution is denoted by $x^*(A, b)$, where

$$[x^*(A,b)]_j = \min\{b_i - a_{i,j} \mid i \in \underline{m},\ \text{and } a_{i,j} > \varepsilon\} \text{ and } j \in \underline{n}. \tag{1}$$

3 Main Results

In this section we derive the max-plus algebra models of a supply chain by considering a number of requests, fuel demand, and a capacity of each fuel tanker. In this study we consider at most two tankers based on data. The date represents date of request, capacity of tankers, fuel demand, the time it takes the ship to deliver fuel from the supplier to the consumer and to return to the supplier. Time is needed by the ship to deliver fuel from the supplier to consumer and to return to supplier are five and two days respectively. Date of requests are 20-09-2013, 28-09-2013, 06-10-2013, 14-10-2013, 17-10-2013, 23-10-2013, 03-11-2013, 10-11-2013, 18-11-2013, 26-11-2013, and 04-12-2013. In this study the capacity of each tanker and the fuel demand are based on appropriate cases. We will study this in more detail in sub-section 3.1 and 3.2.

Before we derive a model of the supply chain, to provide an overview of the system we give Petri nets of the system. The Petri nets in Figure 1 represents a transport system which consists of two tankers, each of which has a certain carrying capacity. The tanker carries the product from the supplier P_1 to the consumer P_4. Figure 1 shows the entry time of premium fuel products $u(k)$ to the supplier, which is then ready to be dispatched to the consumer. The entry time $u(k)$ depends on data of the request time. The premium is ready for deployment (P_1) and will leave ($x_1(k)$) if there is a ship that is ready for a new departure (P_2). Furthermore, there is a delivery process of premium carried by tankers (P_3) from suppliers (P_1) to the consumer (P_4) through the sea route. After the tankers have arrived in the consumer ($x_2(k)$), then the tankers return to the supplier location. This means the premium in the consumer is ready to be distributed to the surrounding area ($y(k)$).

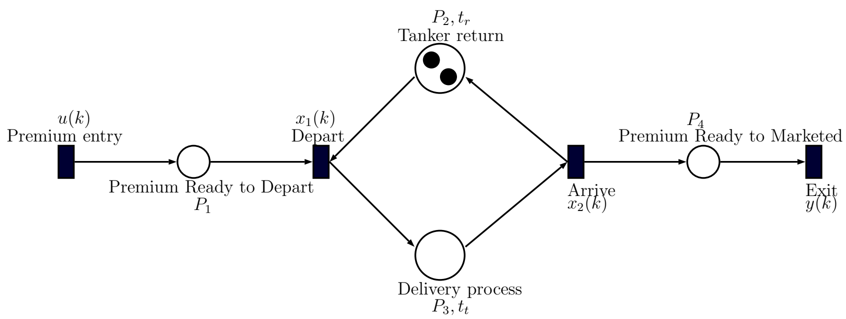

Figure 1: Petri net model for transport system of supply chain management from the supplier to the consumer

Based on Figure 1, if the number of tankers is n, then we obtain a system of equations in max-plus algebra as follows

$$\left.\begin{array}{rcl} x_1(k) &=& t_r \otimes x_2(k-n) \oplus u(k) \\ x_2(k) &=& t_t \otimes x_1(k) \\ y(k) &=& x_2(k), \end{array}\right\} \quad (2)$$

with t_t and t_r are time spent by the tanker to bring premium fuel products from suppliers to consumers and time spent by the tanker to return from the consumer to the supplier respectively, $u(k)$ is the entry time of the premium which occurs at the k^{th} in supplier location ready to be dispatched to the consumer location, $x_1(k)$ is departure time of premium which occurs at the k^{th} from the supplier, $x_2(k)$ is the arrival time of premium in the consumer which occurs at the k^{th} and $y(k)$ is time premium marketed in the consumer area which occurs at the k^{th}. Then in the form of a matrix equation, the model of Equations 2 can be written as follows

$$\boldsymbol{x}(k) = A_0 \otimes \boldsymbol{x}(k) \oplus A_1 \otimes \boldsymbol{x}(k-n) \oplus B_0 \otimes u(k) \quad (3)$$

$$y(k) = C \otimes \boldsymbol{x}(k), \quad (4)$$

where

$$\boldsymbol{x}(k) = \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}, \ A_0 = \begin{bmatrix} \varepsilon & \varepsilon \\ t_t & \varepsilon \end{bmatrix}, \ A_1 = \begin{bmatrix} \varepsilon & t_r \\ \varepsilon & \varepsilon \end{bmatrix}, \ B_0 = \begin{bmatrix} 0 \\ \varepsilon \end{bmatrix}, \ \text{and } C = [\varepsilon \ \ 0].$$

Furthermore, we substitute $\boldsymbol{x}(k)$ in the right hand side of Equation 3 and repeat it β times. Then for β approaches infinity and suppose

$$A_0^* = \bigoplus_{i=0}^{\infty} A_0^{\otimes i} = \begin{bmatrix} 0 & \varepsilon \\ \varepsilon & 0 \end{bmatrix} \oplus \begin{bmatrix} \varepsilon & \varepsilon \\ t_t & \varepsilon \end{bmatrix} \oplus \begin{bmatrix} \varepsilon & \varepsilon \\ \varepsilon & \varepsilon \end{bmatrix} \oplus \begin{bmatrix} \varepsilon & \varepsilon \\ \varepsilon & \varepsilon \end{bmatrix} \oplus \cdots = \begin{bmatrix} 0 & \varepsilon \\ t_t & 0 \end{bmatrix}.$$

It follows that

$$\boldsymbol{x}(k) = A \otimes \boldsymbol{x}(k-n) \oplus B \otimes u(k), \tag{5}$$

where $A = A_0^* \otimes A_1$ and $B = A_0^* \otimes B_0$.

In order to produce a form of equation $Y = H \otimes U$, then by substituting $\boldsymbol{x}(k)$ of Equation 5 in the right hand side of Equation 4 and Repeating the argument for α times with $\alpha = [k/n]$ which is the the Euclidean division of k by n, it follows that

$$y(k) \quad = \quad C \otimes A^{\alpha} \otimes \boldsymbol{x}(k - \alpha \cdot n) \oplus (\bigoplus_{i=0}^{\alpha-1} C \otimes A^{\otimes i} \otimes B \otimes u(k - i \cdot n)).$$

Due to the inequality $k - \alpha \cdot n \leq n$ then we obtain $y(k) = \bigoplus_{i=0}^{\alpha} C \otimes A^{\otimes i} \otimes B \otimes u(k - i \cdot n)$. So, if the number of tankers departure is n_d and $l = 1, 2, ..., n$, then we obtain

$$Y(l) = H \otimes U(l), \tag{6}$$

where

$$Y = \begin{bmatrix} y(l) \\ y(l+n) \\ y(l+2n) \\ \vdots \\ y(l+\gamma n) \end{bmatrix}, \ H = \begin{bmatrix} CB & \boldsymbol{\mathcal{E}} & \cdots & \boldsymbol{\mathcal{E}} \\ CAB & CB & \cdots & \boldsymbol{\mathcal{E}} \\ \vdots & \vdots & \ddots & \vdots \\ CA^{\gamma}B & CA^{\gamma-1}B & \cdots & CB \end{bmatrix}, \ U = \begin{bmatrix} u(l) \\ u(l+n) \\ u(l+2n) \\ \vdots \\ u(l+\gamma n), \end{bmatrix}.$$

We consider γ as an integer such that $l + \gamma n$ is the greatest integer less or equal to n_d and $\boldsymbol{\mathcal{E}}$ as the zero-matrix of the appropriate size with all elements equal to ε.

The problem in this study is to determine the value of $u(k)$ which is obtained by solving Equation 6, while Equation 6 can be solved using Equation 1. To determine the value $u(k)$,we first determine the value of $y(k)$ by means of processing the date of demand, tanker capacity and the number of demands. Algorithm to determine $y(k)$ and n_d is given by Algorithm 1.

Algorithm 1 Determine y and n_d where Date Of Demand(Dd), Number of Demand(Nd), Capacity of Tanker(Ct)

Input: n, array Dd, array Nd and array Ct
Output: n_d and array y

```
procedure
    convert Dd to integer
    Initialization n_d = 0 and hit=0
    while hit < length of Dd do
        hit ← hit+1
        temp ← Nd[hit]
        while temp>0 do
            temp ← temp - Ct[mod(n_d,n)+1]
            n_d ← n_d+1
            y[n_d]← Dd[hit]
        end while
    end while
    return n_d and array y
end procedure
```

Algorithm 2 Determine y and n_d where Date Of Demand(Dd), Number of Demand(Nd), Capacity of Tanker(Ct)

Input: n, t_r, t_t array Dd, array Nd and array Ct
Output: array u

```
procedure
    Determine y and n_d                              ▷ using Algorithm 1
    for l in {1,2,...,n} do
        Determine matrix Y and H                     ▷ using Equation 6
        Determine U                                  ▷ using Equation 1
    end for
    return u
end procedure
```

Note that if the order of the array arrangement Ct changes, then y also changes. In other words, the order of the tankers departure has an influence on scheduling. Thus, in this study we make scheduling for all possibilities and choose the best schedule according to certain criteria. An algorithm to determine schedule with the input array Ct can be seen in Algorithm 2.

In order to obtain a more optimal scheduling, the value of $u(k)$ obtained by Algorithm 2 still needs to be reprocessed using Algorithm 3. This algorithm reduces the number of departures that is "not on-time". From the latter process we obtain the schedule of departure tankers.

Algorithm 3 Determine component of departure u will be removed in every demand where Number of Demand(Nd), Capacity of Tanker(Ct) and time difference of arrival and demand (diff)

Input: Nd, Ct and diff
Output: Array del consist of component departure will be removed

```
procedure
    nc ← length of Ct
    Initialization del ← number 1 to nc and small ← infinity
    perpossible ← permutation of del
    lperposs ← row size of perpossible
    for i in {1,2,...,nc} do
        possible ← perpossible(1:factorial(nc-i):lperposs,:)
        lpossible ← row size of possible
        for j in {1,2,...,lpossible} do
            if sum(Ct(possible(j,:))) ≥ Nd and sum(diff(possible(j,:))) < small then
                temp=possible(j,:);
            end if
        end for
    end for
    remove temp from del
    return del
end procedure
```

3.1 Scheduling By Using One Tanker

In this sub-section we discuss tankers scheduling of several possibility related to the capacity of the tanker.The number of tankers that are discussed in this sub-section is one. The data obtained consisted of eleven requests (recall date of request in early part of this section) which we have twelve possibility cases. they are exceeds the capacity of the tankers then scheduling of previous requests will be changed, while scheduling of next requests will not be changed. The total number of days that is "not-on-time" of these case can be seen in Table 1.

Table 1: The total number of days is "not-on time" when the number of tanker is one

Tanker capacities less than k^{th} request	n_d	The total number of days is "not-on time"	The number of requests on time
otherwise	11	15	6
1	12	24	6
2	12	32	6
3	12	40	6
4	12	48	6
5	12	51	6
6	12	57	6
7	12	40	5
8	12	47	4
9	12	46	3
10	12	44	2
11	12	41	2

The first row of Table 1 explains scheduling of tankers where its capacity exceeds each request. This scheduling happen 15 days "not on time" i.e., on the first until the fifth request. Moreover, in the second until the seventh row of this table have 6 requests on time. While in the next row, on time demand decreases. This happens because when k^{th} request exceeds the capacity of the tankers then scheduling of previous requests will be changed, while scheduling of next requests will not be changed.

3.2 Scheduling By Using Two Tankers

In this sub-section we discuss tankers scheduling of several possibilities related to the capacity of the tanker. The number of tankers that are discussed in this sub-section is two. By considering the results of the discussion in the previous sub-section, scheduling of tanker where its capacity exceeds each request. Thus in this sub-section we assume the first tanker capacity exceeds each request. With this assumption, we make scheduling for 6 cases. They are the capacity of the second tanker is less than all requests and capacity of the second tanker is less than all requests except the first until the fifth request. The total number of days is "not-on time" of these cases can be seen in Table 2.

In the first to the sixth row of the Table 2, it appears that the number of requests that are "not-on time" decreases. Finally, the fifth case can be scheduled for all requests on time. This happens because when the second tanker capacity exceeds k^{th} request then scheduling of previous requests will be changed, while scheduling of next requests will not be changed.

Table 2: The total number of days is "not-on time" when the number of tanker is two

Second tanker capacities less than k^{th} request	n_d	The total number of days is "not-on time"	The number of requests is "not-on time"
all	11	15	5
1	11	13	4
2	11	10	3
3	11	6	2
4	11	1	1
5	11	0	0

4 Conclusions and Further Research

Based on the results of the discussion it can be concluded that the problems of supply chain scheduling can be solved by applying Algorithm 1, 2 and 3. In addition, from the results of scheduling that have been done for some cases ones obtained a conclusion that adds the number of tankers with a capacity of less than all the requests had no effect on the reduction schedule that is "not-on time". This can be seen in the first row of Table 1 and Table 2. Furthermore, we obtained the number of requests are "not-on time" is five for the case number of tanker is one with a capacity meet all consumer demand. By providing additional tankers with a capacity greater than or equal to the fifth request, we can eliminate the "not-on time" request. We have discussed the problem of supply chain scheduling only for one customer. For further research, we are planning to consider arbitrary number of customers.

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