On the Differential Geometric Elements of Mannheim Darboux Ruled Surface in $\mathbb{E}^3$

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Abstract

In this paper we consider two special ruled surfaces associated to Mannheim pair $\{\alpha, \alpha^*\}$. First, Mannheim Darboux Ruled surface (MDRS) of the curve $\alpha$ be defined and examined in terms of the Frenet-Serret apparatus of the Mannheim curve $\alpha$, in $\mathbb{E}^3$. Further we have examined the differential geometric elements such as, Weingarten map $S$, Gauss curvature $K$ and mean curvature $H$ of Darboux ruled surface (DRS) and Mannheim Darboux ruled surface (MDRS) relative to each other. Also the first, second and third fundamental forms of Mannheim Darboux ruled surface (MDRS) have been examined in terms of the Mannheim curve $\alpha$ too.

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1 Introduction and Preliminaries

Involute-evolute curves, Bertrand curves, and Mannheim partner curves are the curves derived based on the other curves in geometry. Mannheim curve

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was firstly defined by A. Mannheim in 1878. A curve is called a Mannheim curve if and only if \( \frac{k_1}{k_1+k_2} \) is a nonzero constant, with curvatures \( k_1 \) and \( k_2 \).

Recently, a new definition of the associated curves was given by Liu and Wang [4]. According to this new definition, if the principal normal vector of first curve and binormal vector of second curve are linearly dependent, then first curve is called Mannheim curve, and the second curve is called Mannheim partner curve. As a result they called these new curves as Mannheim pair curves. For more detail see in [3] and [4]. The quantities \( \{V_1, V_2, V_3, D, k_1, k_2\} \) are collectively Frenet-Serret apparatus of the curve \( \alpha : I \to E^3 \). Also

\[
\begin{bmatrix}
\dot{V}_1 \\
\dot{V}_2 \\
\dot{V}_3
\end{bmatrix} =
\begin{bmatrix}
0 & k_1 & 0 \\
-k_1 & 0 & k_2 \\
0 & -k_2 & 0
\end{bmatrix}
\begin{bmatrix}
V_1 \\
V_2 \\
V_3
\end{bmatrix}.
\]

are well known the Frenet formulae. Darboux vector \( D \) is the areal velocity vector of the Frenet frame of a space curve. For any unit speed curve \( \alpha \), in terms of the Frenet-Serret apparatus, the Darboux vector can be expressed as

\[
D(s) = k_2(s)V_1(s) + k_1(s)V_3(s)
\]

where curvature functions are \( k_1 \) and \( k_2 \) [1]. Along curve \( \alpha \) under the condition that \( k_1 \neq 0 \), vector field

\[
\tilde{D}(s) = \frac{k_2}{k_1}(s)V_1(s) + V_3(s)
\]

is called the modified Darboux vector field of curve \( \alpha \) in [2].

Let \( \alpha : I \to E^3 \) and \( \alpha^* : I \to E^3 \) be the \( C^2 \)-class differentiable unit speed and \( \alpha^* : I \to E^3 \) be two curves and let \( V_1(s), V_2(s), V_3(s) \) and \( V_1^*(s^*), V_2^*(s^*), V_3^*(s^*) \) be the Frenet frames of the curves \( \alpha \) and \( \alpha^* \), respectively. If the principal normal vector \( V_2 \) of the curve \( \alpha \) is linearly dependent on the binormal vector \( V_3^* \) of the curve \( \alpha^* \), then the pair \( \{\alpha, \alpha^*\} \) is said to be Mannheim pair, then \( \alpha \) is called a Mannheim curve and \( \alpha^* \) is called Mannheim partner curve of \( \alpha \) where \( <(V_1, V_1^*)> = \cos \theta \) and besides the equality \( \frac{k_1}{k_1+k_2} = \) nonzero constant is known the offset property. In [5] and [6] Mannheim partner curves and Mannheim offsets of ruled surfaces are defined and characterized. Mannheim pair \( \{\alpha, \alpha^*\} \) can be represented by

\[
\alpha(s^*) = \alpha^*(s^*) + \lambda(s^*)V_3^*(s^*)
\]

for some function \( \lambda \), since \( V_2 \) and \( V_3 \) are linearly dependent. This equation can be rewritten as

\[
\alpha^*(s) = \alpha(s) - \lambda V_2(s)
\]
where $\lambda = \frac{-k_1}{k_1^2 + k_2^2}$. Frenet-Serret apparatus of Mannheim partner curve $\alpha^*$, based in Frenet-Serret vectors of Mannheim curve $\alpha$ are

\begin{align*}
V_1^* &= \cos \theta \ V_1 - \sin \theta \ V_3 \\
V_2^* &= \sin \theta \ V_1 + \cos \theta \ V_3 \\
V_3^* &= V_2.
\end{align*}

(1.6)

The curvature and the torsion have the following equalities,

\begin{align*}
k_1^* &= -\frac{d\theta}{ds^*} = \frac{\dot{\theta}}{\cos \theta}, \\
k_2^* &= k_1 \sin \theta \cos \theta - k_2 \cos \theta \cos \theta
\end{align*}

or

\begin{equation}
k_2^* = \frac{k_1}{\lambda k_2}.
\end{equation}

(1.8)

We use dot to denote the derivative with respect to the arc length parameter of the curve $\alpha$. Also

\begin{equation}
\frac{ds}{ds^*} = \frac{1}{\cos \theta} = \frac{-\lambda k_2^*}{\sin \theta},
\end{equation}

(1.9)

where $|\lambda|$ is the distance between the curves $\alpha$ and $\alpha^*$, since $d(\alpha(s), \alpha^*(s)) = |\lambda|$. For more detail see in [5]. Also we can write

\begin{equation}
\frac{ds}{ds^*} = \frac{1}{\sqrt{1 + \lambda k_2}}.
\end{equation}

(1.10)

The product of Frenet vector fields of the Mannheim pair $\{\alpha, \alpha^*\}$ has the following matrix form;

\begin{equation}
\begin{bmatrix}
V_1^* \\
V_2^* \\
V_3^*
\end{bmatrix}
\begin{bmatrix}
V_1 & V_2 & V_3
\end{bmatrix}
= \begin{bmatrix}
\cos \theta & 0 & -\sin \theta \\
\sin \theta & 0 & \cos \theta \\
0 & 1 & 0
\end{bmatrix}.
\end{equation}

(1.11)

**Definition 1.1** Ruled surface is said to be “Darboux Ruled surface” if it is generated by moving Darboux vector fields, with the parametrization

\begin{equation}
\varphi(s,u) = \alpha(s) + u\tilde{D}(s)v = \alpha(s) + u\frac{k_2}{k_1}(s)V_1(s) + uV_3(s).
\end{equation}

(1.12)

Also it has been called rectifying developable surface in [2].

## 2 Mannheim Darboux Ruled surface

In this section we will define and work on MDRS, which is known as rectifying developable ruled surface, or $\tilde{D}$ scroll as in [7] where the differential geometric elements of the involute $\tilde{D}$ scroll are examined too. Here first we will give Darboux vector field of the Mannheim partner $\alpha^*$ as in the following theorem.
Theorem 2.1 The modified Darboux vector of Mannheim partner curve \( \alpha^* \) of a Mannheim curve \( \alpha \), based on the Frenet apparatus of Mannheim curve \( \alpha \) is

\[
\tilde{D}^* = -\frac{k_1^2 + k_2^2}{\theta k_2} \cos^2 \theta V_1 + V_2 + \frac{k_1^2 + k_2^2}{\theta k_2} \cos \theta \sin \theta V_3. \tag{2.1}
\]

Proof. The desired result is obtained from equations (1.3), (1.6) and (1.8).

Definition 2.2 The parametrization of MDRS, in terms of the Frenet-Serret apparatus of the Mannheim partner curve \( \alpha \) is

\[
\varphi^*(s, v) = \alpha - v\frac{k_1^2 + k_2^2}{\theta k_2} \cos^2 \theta V_1 + (v - \lambda) V_2 + v\frac{k_1^2 + k_2^2}{\theta k_2} \cos \theta \sin \theta V_3. \tag{2.2}
\]

since

\[
\varphi^*(s, v) = \alpha^*(s) + v\tilde{D}^*(s),
\]

where

\[
\tilde{D}^* = -\frac{k_1^2 + k_2^2}{\theta k_2} \cos^2 \theta V_1 + V_2 + \frac{k_1^2 + k_2^2}{\theta k_2} \cos \theta \sin \theta V_3.
\]

Theorem 2.3 DRS and MDRS are intersect each other along the curve

\[
\varphi^*(s) = \alpha - \frac{\sin \theta \cos \theta}{\theta} V_1 + \frac{\sin^2 \theta}{\theta} V_3. \tag{2.3}
\]

Proof. With the equations

\[
\varphi^*(s, v) = \alpha + v\frac{k_1}{\lambda k_2} \cos^2 \frac{\theta}{\theta} V_1 + (v - \lambda) V_2 - v\frac{k_1}{\lambda k_2} \cos \frac{\theta}{\theta} \sin \frac{\theta}{\theta} V_3
\]

and

\[
\varphi(s, u) = \alpha + u\frac{k_2}{k_1} V_1 + u V_3
\]

under the conditions,

\[
\begin{cases}
\frac{v}{\lambda k_2} \frac{k_1 \cos^2 \theta}{\theta} = u\frac{k_2}{k_1} \\
v - \lambda = 0 \\
-\frac{v}{\lambda k_2} \frac{k_1 \cos \theta \sin \theta}{\theta} = u
\end{cases}
\]

\[
\begin{cases}
\frac{k_1}{k_2} \frac{\cos^2 \theta}{\theta} = u\frac{k_2}{k_1} \\
v = \lambda \\
-\frac{k_1}{k_2} \frac{\cos \theta \sin \theta}{\theta} = u
\end{cases}
\]

we have

\[
\frac{k_1}{k_2} = -\tan \theta.
\]

This complete the proof. ■
**Theorem 2.4** Normal vector field $N$ of DRS is perpendicular to the normal vector field $N^*$ of MDRS.

**Proof.** Since the normal vector field $N$ of $DR$ is

$$N = \frac{\varphi_s \Lambda \varphi_u}{\| \varphi_s \Lambda \varphi_u \|} = V_2$$

[7], and the normal vector field $N^*$ of MDRS of the curve $\alpha$ is

$$N^* = \frac{\varphi_s^* \Lambda \varphi_u^*}{\| \varphi_s^* \Lambda \varphi_u^* \|} = V_2^* = \sin \theta \, V_1 + \cos \theta \, V_3.$$

it is trivial that $\langle V_2, V_2^* \rangle = 0$. ■

**Theorem 2.5** The matrix corresponding to the Weingarten map (Shape Operator) $S^*$ of MDRS is

$$S^* = \begin{bmatrix} -\frac{\hat{d}}{\cos \theta} & 0 \\ 1 + \frac{v}{\sin \theta} \left( \frac{k_1^2 + k_2^2}{\kappa_2^2} \right) \hat{\theta} \cos \theta + \frac{k_1 + k_2}{\kappa_2^2} (\hat{\theta} \sin \theta + \hat{\theta} \cos \theta) \right) & 0 \\ 0 & 0 \end{bmatrix}. \quad (2.4)$$

**Proof.** In the Euclidean $3$-space, the matrix corresponding to the Weingarten map (Shape Operator) $S$ of DRS of curve $\alpha$ is

$$S = \begin{bmatrix} -\frac{k_1}{1 + \frac{v}{\sin \theta} \left( \frac{k_1^2 + k_2^2}{\kappa_2^2} \right) \hat{\theta} \cos \theta + \frac{k_1 + k_2}{\kappa_2^2} (\hat{\theta} \sin \theta + \hat{\theta} \cos \theta) \right) & 0 \\ 0 & 0 \end{bmatrix}.$$ [7]. Hence the matrix corresponding to the Weingarten map (Shape Operator) $S^*$ of MDRS is

$$S^* = \begin{bmatrix} -\frac{k_1^*}{1 + \frac{v}{\sin \theta} \left( \frac{k_1^* + k_2^*}{\kappa_2^*} \right) \hat{s} \cos \theta} & 0 \\ 0 & 0 \end{bmatrix}.$$ by substituing $k_2^*$ and $k_1^*$ in matrix $S^*$ we get

$$S^* = \begin{bmatrix} -\frac{\hat{d}}{\cos \theta} & 0 \\ 1 + \frac{v}{\sin \theta} \left( \frac{k_1^* + k_2^*}{\kappa_2^*} \right) \hat{s} \cos \theta & 0 \end{bmatrix}.$$

Since the derivative with respect to parameter $s^*$ is

$$\left( \frac{k_1 \cos \theta}{\lambda k_2} \right) = \frac{d}{ds} \left( \frac{k_1 \cos \theta}{\lambda k_2} \right) \frac{ds}{ds^*}$$

$$= \frac{1}{\theta^2 \cos \theta} \left[ \left( \frac{k_1^2 + k_2^2}{-k_2} \right) \hat{\theta} \cos \theta + \frac{k_1^2 + k_2^2}{-k_2} (\hat{\theta} \sin \theta + \hat{\theta} \cos \theta) \right].$$

we have the proof. Where $\frac{k_1}{\lambda k_2} = \frac{k_1^2 + k_2^2}{-k_2}$. ■
Corollary 2.1 The Gaussian curvature of MDRS is

\[ K = \det S^* = 0. \]  \hfill (2.5)

Corollary 2.2 The mean curvature of MDRS is

\[ H = \text{trace} S^* = \frac{-k_1^*}{1 + u \left( \frac{k_2^*}{k_1^*} \right)}, \]  \hfill (2.6)

\[ H = \frac{-\dot{\theta}}{\cos \theta} \left[ \left( \frac{k_1 + k_2}{-k_2} \right) \dot{\theta} \cos \theta + \frac{k_1}{k_2} \left( -\dot{\theta} \sin \theta + \ddot{\theta} \cos \theta \right) \right]. \]

where \(<(V_1, V^*_1) = \cos \theta \) and \( \theta \) is not constant.

Corollary 2.3 MDRS is not minimal surface since \( \frac{\dot{\theta}}{\cos \theta} = 0 \) and

\[ v \neq \frac{\dot{\theta}^2 \cos \theta}{\left( \frac{k_1^2 + k_2^2}{-k_2} \right) \dot{\theta} \cos \theta - \frac{k_1^2 + k_2^2}{k_2} \left( -\dot{\theta} \sin \theta + \ddot{\theta} \cos \theta \right)} \]  \hfill (2.7)

with curvatures \( k_1 \) and \( k_2 \) of the Mannheim curve \( \alpha \).

Proof. Minimal surfaces are classically defined as surfaces of zero mean curvature in the Euclidean 3-space. Since \( H \neq 0 \) we have the proof. \( \blacksquare \)

We know that a surface can be characterized by the basic intrinsic properties such as the fundamental forms of a surface; usually called the first, second and third fundamental forms. They are extremely important and useful in determining the metric properties of a surface, such as line element, area element, normal curvature, Gaussian curvature, and mean curvature. The third fundamental form is given in terms of the first and second forms by \( III - 2HII + KI = 0 \) where \( H \) is the mean curvature and \( K \) is the Gaussian curvature. The fundamental forms of the involute \( D \) scroll are examined in [7].

The first fundamental form characterizes the interior geometry of the surface in a neighbourhood of a given point \( M \). Suppose that the surface is given by the equation \( \varphi(s, u) \); where \( s \) and \( u \) are parameters of the surface; and \( d\bar{\varphi} = \bar{\varphi}_s ds + \bar{\varphi}_u du \) is the differential of the radius vector of \( \varphi \) along a chosen direction from a point \( M \) to an infinitesimally close point \( M' \); [8].

Theorem 2.6 The first fundamental form of MDRS is

\[ I^* = \cos^2 \theta ds ds + dv dv \]  \hfill (2.8)
Proof. The first fundamental form $I$ of DRS can be calculated as
\[ I = \langle d\bar{\varphi}, d\bar{\varphi} \rangle = dsds + dudu \]
Hence the first fundamental form $I^*$ of MDRS is
\[ I^* = ds^*ds^* + dvdv \]
Since $ds^* = \cos \theta ds$, it is trivial. Now we will examine the second fundamental form of MDRS, already defined.

**Theorem 2.7** The second fundamental form of MDRS is
\[ II^* = -\dot{\theta} \cos \theta dsds - \frac{k_1^2 + k_2^2}{k_2} \cos \theta dsv. \]  

**Proof.** The second fundamental form $II$ of DRS is given by
\[ II = \langle d\bar{\varphi}, dN \rangle = -k_1dsds + k_2dudu, \]
where $N$ is the unit normal vector of the surface at the point $M$. Hence the second fundamental form $II^*$ of MDRS is
\[ II^* = -k_1^*ds^*ds^* + k_2^*ds^*dv = -\frac{\dot{\theta}}{\cos \theta} \cos \theta dsds + \left( \frac{k_1^2 + k_2^2}{-k_2} \right) \cos \theta dsv. \]
Since $ds^* = \cos \theta ds$, it is trivial.

**Theorem 2.8** The third fundamental form of MDRS is
\[ III^* = \frac{k_2^2\dot{\theta}^2 + (k_1^2 + k_2^2)^2 \cos^2 \theta}{k_2} dsds. \]  

**Proof.** The third fundamental form of MDRS is the square of the differential of the unit normal vector $N$ of the surface at the point $M$ which is denoted by $III$ and given by
\[ III = \langle dN, dN \rangle = (k_1^2 + k_2^2) dsds, \]
Hence third fundamental form of MDRS is
\[ III^* = (k_1^2 + k_2^2) ds^*ds^* = \left( \frac{\dot{\theta}}{\cos \theta} \right)^2 + \left( \frac{k_1}{\lambda k_2} \right)^2 \cos^2 \theta dsds \]
\[ = \left( \frac{\dot{\theta}}{\cos \theta} \right)^2 \cos^2 \theta dsds \]
\[ = \left( \frac{k_2^2\dot{\theta}^2 + (k_1^2 + k_2^2)^2 \cos^2 \theta}{k_2^2 \cos^2 \theta} \right) \cos^2 \theta dsds = \frac{k_2^2\dot{\theta}^2 + (k_1^2 + k_2^2)^2 \cos^2 \theta}{k_2^2} dsds. \]
Since $ds^* = \cos \theta ds$, it is trivial.
References


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