Laplace Type Problems for a Lattice with Cell Composed by an Isosceles Triangle and $n - 1$ Isosceles Trapezoids

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Abstract

In this paper we consider a lattice with a cell composed by an isosceles triangle and two isosceles trapezoids and we compute the probability that a random segment and of constant length intersects a side of the lattice. Moreover, we show the case of a lattice with a cell composed by an isosceles triangle and $n-1$ isosceles trapezoids.

Mathematics Subject Classification: 60D05, 52A22

Keywords: Geometric Probability, stochastic geometry, random sets, random convex sets and integral geometry

1 Introduction

Starting from the fundamental results obtained by Poincaré [4] and Stoka [5] for the geometric probability problems some authors have extended their results [1], [2] and [3]. In particular Caristi and Stoka [3] have proved some
extension of the Laplace problem. In this paper a classical Buffon-Laplace problem for two lattices was considered, the first lattice is composed by an isosceles triangle and an isosceles trapezium, and the second is composed by an isosceles triangle and (n-1) isosceles trapezoids. We study the probabilities that a random segment of constant length intersects the fundamental cells represented in fig. 1 and fig. 3.

2 Cell composed by an isosceles triangle and two isosceles trapezoids

Let $\mathcal{R}_1(a, \alpha)$ be the lattice with the fundamental cell $C_0^{(1)}$ represented in fig.1 where $\frac{\pi}{4} \leq \alpha \leq \frac{\pi}{3}$

![Diagram](image)

fig.1

From this figure we obtain the following:

$$\text{area}C_0 = \frac{9a^2}{2} \sin 2\alpha. \quad (1)$$

We want to compute the probability that a random segment $s$ of constant length $l < \frac{a}{2}$ intersects a side of lattice $\mathcal{R}_1$, i.e. the probability $P_{int}^{(1)}$ that the segment $s$ intersects a side of the fundamental cell $C_0^{(1)}$.

The position of the segment $s$ is determined by the centre and by the angle $\varphi$ that $s$ forms with line $BC$ (or $DE$ or $FG$).
To compute the probability $P_{\text{int}}^{(1)}$ we consider the limiting positions of segment $s$, for a fixed value of $\varphi$, in the cells $C_{0i}^{(1)}$, $(i = 1, 2, 3)$.

So, we have fig. 2

\[ \text{area} \hat{C}_{01}^{(1)}(\varphi) = \text{area}C_{01}^{(1)} - \sum_{i=1}^{5} \text{area}a_i(\varphi), \quad (2) \]

\[ \text{area} \hat{C}_{02}^{(1)}(\varphi) = \text{area}C_{02}^{(1)} - \sum_{i=1}^{6} \text{area}b_i(\varphi), \quad (3) \]

\[ \text{area} \hat{C}_{03}^{(1)}(\varphi) = \text{area}C_{03}^{(1)} - \sum_{i=1}^{6} \text{area}c_i(\varphi). \quad (4) \]

To compute $\text{area} \hat{C}_{01}^{(1)}(\varphi)$ we have:

\[ \text{area}a_1(\varphi) = \frac{l^2 \sin(\alpha - \varphi) \sin(\alpha + \varphi)}{2 \sin 2\alpha}, \]
\[
\text{areaa}_2 (\varphi) = \frac{a l}{2} \sin (\alpha - \varphi) - \frac{l^2 \sin (\alpha - \varphi) \sin (\alpha + \varphi)}{2 \sin 2\alpha},
\]
\[
\text{areaa}_4 (\varphi) = \frac{l^2 \sin \varphi \sin (\varphi + \alpha)}{2 \sin \alpha},
\]
\[
\text{areaa}_3 (\varphi) = a l \cos \alpha \sin \varphi - \frac{l^2 \sin \varphi \sin (\varphi + \alpha)}{2 \sin \alpha},
\]
\[
\text{areaa}_5 (\varphi) = \frac{a l}{2} \sin (\alpha + \varphi) - \frac{l^2}{4} \left( \sin 2\varphi - \cot 2\alpha \cos 2\varphi + \frac{1}{\sin 2\alpha} \right),
\]

Then, we can write
\[
A_1^{(1)} (\varphi) = \sum_{i=1}^{5} \text{areaa}_i (\varphi) = a l \sin (\alpha + \varphi) - \frac{l^2}{4} \left( \sin 2\varphi - \cot 2\alpha \cos 2\varphi + \frac{1}{\sin 2\alpha} \right). \tag{5}
\]

Replacing this formula in (2), we obtain:
\[
\text{area}_C^{(1)} (\varphi) = \text{area}_C^{(1)} (\varphi) - A_1^{(1)} (\varphi).
\]

To compute \(\text{area}_\hat{C}_0^{(1)} (\varphi)\) we have:
\[
\text{areab}_1 (\varphi) = \frac{l^2 \sin \varphi \sin (\varphi - \alpha)}{2 \sin \alpha},
\]
\[
\text{areab}_2 (\varphi) = \frac{a l}{2} \sin (\alpha - \varphi) - \frac{l^2 \sin \varphi \sin (\alpha - \varphi)}{2 \sin \alpha},
\]
\[
\text{areab}_4 (\varphi) = \text{areaa}_4 (\varphi) = \frac{l^2 \sin \varphi \sin (\varphi + \alpha)}{2 \sin \alpha},
\]
\[
\text{areab}_3 (\varphi) = 2a l \cos \alpha \sin \varphi - \frac{l^2 \sin \varphi \sin (\varphi + \alpha)}{2 \sin \alpha},
\]
\[
\text{areab}_5 (\varphi) = \frac{a l}{2} \sin (\varphi + \alpha) - \frac{l^2 \sin \varphi \sin (\varphi + \alpha)}{2 \sin \alpha},
\]
\[
\text{areab}_6 (\varphi) = a l \cos \alpha \sin \varphi - \frac{l^2 \sin \varphi \sin (\alpha - \varphi)}{2 \sin \alpha}.
\]

Then we can write
An isosceles triangle and \( n - 1 \) isosceles trapezium

\[
A_2^{(1)} (\varphi) = \sum_{i=1}^{6} \text{area} b_i (\varphi) = a l \left( \sin \alpha \cos \varphi + 3 \cos \alpha \sin \varphi \right) - \frac{l^2}{2} \sin 2\varphi. \tag{6}
\]

Replacing this formula in (3), we have:

\[
\text{area} \hat{C}_{02}^{(1)} (\varphi) = \text{area} C_{01}^{(1)} - A_2^{(1)} (\varphi). \tag{7}
\]

To compute \( \text{area} \hat{C}_{03}^{(1)} (\varphi) \) we have:

\[
\text{area} c_1 (\varphi) + \text{area} c_2 (\varphi) = \frac{a l}{2} \sin (\varphi - \alpha), \\
\text{area} c_4 (\varphi) = \frac{l^2 \sin \varphi \sin (\varphi + \alpha)}{2 \sin \alpha}, \\
\text{area} c_3 (\varphi) = 3 a l \cos \alpha \sin \varphi - \frac{l^2 \sin \varphi \sin (\varphi + \alpha)}{2 \sin \alpha}, \\
\text{area} c_5 (\varphi) = \frac{a l}{2} \sin (\varphi + \alpha) - \frac{l^2 \sin \varphi \sin (\varphi + \alpha)}{2 \sin \alpha}, \\
\text{area} c_6 (\varphi) = 2 a l \cos \alpha \sin \varphi - \frac{l^2 \sin \varphi \sin (\alpha - \varphi)}{2 \sin \alpha},
\]

then

\[
A_3^{(1)} (\varphi) = \sum_{i=1}^{6} \text{area} c_i (\varphi) = a l \left( \sin \alpha \cos \varphi + 5 \cos \alpha \sin \varphi \right) - \frac{l^2}{2} \sin 2\varphi.
\]

Considering this relationship, the formula (4) becomes

\[
\text{area} \hat{C}_{03}^{(1)} (\varphi) = \text{area} C_{03}^{(1)} - A_3^{(1)} (\varphi).
\]

Denoting by \( M_i^{(1)} \), the set of all segment \( s \) which have their centre in the cell \( C_{0i}^{(1)} \), and with \( N_i \) the set of all the segment entirely contained in the cell \( C_{oi}^{(1)} \). We have (cf. [5]):

\[
P^{(1)}_{\text{int}} = 1 - \frac{\sum_{i=1}^{3} \mu \left( N_i^{(1)} \right)}{\sum_{i=1}^{3} \mu \left( M_i^{(1)} \right)}, \tag{8}
\]

where \( \mu \) is the Lebesgue measure in the Euclidean plane.
To compute the measure $\mu \left( M_i^{(1)} \right)$ and $\mu \left( N_i^{(1)} \right)$ we use the kinematic measure of Poincaré [4]:

$$ dk = dx \wedge dy \wedge d\varphi, $$

where $x$, $y$ are the coordinates of centre of $s$ and $\varphi$ is a fixed angle.

For $\varphi \in [0, \alpha]$ we can write

$$ \mu \left( M_i^{(1)} \right) = \int_0^\alpha \frac{d\varphi}{2} \int \frac{\text{area} C_0^{(1)} (\varphi)}{\left( \left( x, y \right) \in C_0^{(1)} (\varphi) \right)} dxdy = \int_0^\alpha \left( \text{area} C_0^{(1)} \right) d\varphi = a\text{area} C_0^{(1)}, \quad (i = 1, 2, 3), $$

then

$$ \sum_{i=1}^3 \mu \left( M_i^{(1)} \right) = a\text{area} C_0^{(1)}, \quad (9) $$

and

$$ \mu \left( N_i^{(1)} \right) = \int_0^\alpha \frac{d\varphi}{2} \int \frac{\text{area} \tilde{C}_0^{(1)} (\varphi)}{\left( \left( x, y \right) \in \tilde{C}_0^{(1)} (\varphi) \right)} dxdy = \int_0^\alpha \left[ \text{area} \tilde{C}_0^{(1)} (\varphi) - A_i^{(1)} (\varphi) \right] d\varphi = a\text{area} C_0^{(1)} - \int_0^\alpha \left[ A_i^{(1)} (\varphi) \right] d\varphi, \quad (i = 1, 2, 3). \quad (10) $$

The formulas (1), (8), (9) and (10) give

$$ P_{nt}^{(1)} = \frac{2}{9a^2 \alpha \sin 2\alpha} \int_0^\alpha \left[ \sum_{i=1}^3 A_i^{(1)} (\varphi) \right] d\varphi, \quad (11) $$

and

$$ \int_0^\alpha \left[ \sum_{i=1}^3 A_i^{(1)} (\varphi) \right] d\varphi = 3a \left( 1 + 3 \cos \alpha - 4 \cos^2 \alpha \right) - \frac{l^2}{8} \left( 11 - 12 \cos^2 \alpha + \frac{2\alpha}{\sin 2\alpha} \right). $$

Replacing this relationship in (11) we have

$$ P_{nt}^{(1)} = \frac{2l}{9a^2 \alpha \sin 2\alpha} \left[ 3a \left( 1 + 3 \cos \alpha - 4 \cos^2 \alpha \right) - \frac{l}{8} \left( 11 - 12 \cos^2 \alpha + \frac{2\alpha}{\sin 2\alpha} \right) \right]. $$
3 Cell composed by an isosceles triangle and (n-1) isosceles trapezoids.

Let $\mathcal{R}_2 (a, \alpha)$ be the lattice with the fundamental cell $C_0^{(2)}$ represented in fig.3 where $\frac{\pi}{4} \leq \alpha \leq \frac{\pi}{3}$.

From fig. 3 we have

$$\text{area} C_0^{(2)} = \frac{n^2 a^2 \sin 2\alpha}{2}. \quad (12)$$

With the notions of paragraph 1 the probability $P_{\text{int}}^{(2)}$ that the random segment $s$ of constant length $l < a/2$ intersects a side of the fundamental cell $C_0^{(2)}$ is

$$P_{\text{int}}^{(2)} = \frac{1}{\text{area} C_0^{(2)}} \int_0^\alpha \left[ \sum_{i=1}^n A_i (\varphi) \right] d\varphi. \quad (13)$$

Since $\varphi$ is the angle between the segment $s$ and the line $M_{n+2} M_{n+3}$, we have:

$$A_1 (\varphi) = al (\sin \alpha \cos \varphi + \cos \alpha \sin \varphi) - \frac{l^2}{4} \left( \sin 2\varphi - \ctg 2\alpha \cos 2\varphi + \frac{1}{\sin 2\alpha} \right).$$
\[ A_2 (\varphi) = al (\sin \alpha \cos \varphi + 3 \cos \alpha \sin \varphi) - \frac{l^2}{2} \sin 2\varphi, \]

\[ A_3 (\varphi) = al (\sin \alpha \cos \varphi + 5 \cos \alpha \sin \varphi) - \frac{l^2}{2} \sin 2\varphi, \]

\[ A_n (\varphi) = al [\sin \alpha \cos \varphi + (2n - 1) \cos \alpha \sin \varphi] - \frac{l^2}{2} \sin 2\varphi. \]

Then

\[ \sum_{i=1}^{n} A_i (\varphi) = nal (\sin \alpha \cos \varphi + n \cos \alpha \sin \varphi) - \]

\[ \frac{l^2}{4} \left[ (2n - 1) \sin 2\varphi - \cot g2\alpha \cos 2\varphi + \frac{1}{\sin 2\alpha} \right], \]

and

\[ \int_0^{\alpha} \left[ \sum_{i=1}^{n} A_i (\varphi) \right] d\varphi = nal \left[ 1 + n \cos \alpha - (n + 1) \cos^2 \alpha \right] - \]

\[ \frac{l^2}{8} \left( 2n - 1 - 2n \cos 2\alpha + \frac{2\alpha}{\sin 2\alpha} \right). \tag{14} \]

The formulas (12), (13) and (14) give us

\[ P_{int}^{(2)} = \frac{2l}{n^2 a^2 \alpha \sin 2\alpha} \left\{ na \left[ 1 + n \cos \alpha - (n + 1) \cos^2 \alpha \right] - \right. \]

\[ \left. \frac{l}{8} \left( 2n - 1 - 2n \cos 2\alpha + \frac{2\alpha}{\sin 2\alpha} \right) \right\}. \]

For \( n = 3 \) we find the probability \( P_{int}^{(1)} \).

\section*{References}


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Received: May 5, 2016; Published: November 15, 2016