On an Inverse Boundary Value Problem for the Boussinesq-Love Equation with an Integral Condition

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Abstract

In the paper an inverse boundary value problem for the Boussinesq - Love equation with the integral condition of the first kind is investigated. First given problem is reduced to the equivalent problem in a known sense. Then, using the method of Fourier equivalent problem is reduced to the solution of the system of integral equations. Further, the existence and uniqueness of the integral equation is proved by means of the contraction mappings principle, which is also the unique solution of the equivalent problem. Finally, using the equivalence, the theorem on the existence and uniqueness of a classical solution of the given problem is proved.

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1 Introduction

There are many cases where the needs of the practice leads to problems in determining the coefficients or the right-hand side of the differential equations
according to some known data of its solutions. Such problems are called inverse value problems of mathematical physics. Inverse value problems arise in various areas of human activity such as seismology, mineral exploration, biology, medicine, quality control of industrial products, etc., that states them in a number of actual problems of modern mathematics.

The inverse problems are favorably developing section of up-to-date mathematics. Recently, the inverse problems are widely applied in various fields of science.

Different inverse problems for various types of partial differential equations have been studied in many papers. First of all we note the papers of A.N.Tikhonov [8], M.M.Lavrentyev [4,5], A.M.Denisov [1], M.I.Ivanchov [2] and their followers.

In this paper, due to the [6,7], we proved the existence and uniqueness of the solution of the inverse boundary value problem for the Boussinesq-Love equation, modeling the longitudinal waves in an elastic bar with the transverse inertia.

2 Problem statement and its reduction to equivalent problem

Consider for the Boussinesq-Love equation [9]

$$u_{tt}(x, t) - u_{tttx}(x, t) - \alpha u_{txx}(x, t) - \beta u_{xx}(x, t) = a(t)u(x, t) + f(x, t)$$  \hspace{1cm} (1)

in the domain $D_T = \{(x, t) : 0 \leq x \leq 1, \ 0 \leq t \leq T\}$ an inverse boundary problem with the initial conditions

$$u(x, 0) = \varphi(x), \quad u_t(x, 0) = \psi(x) \quad (0 \leq x \leq 1),$$  \hspace{1cm} (2)

the boundary condition Neyman

$$u_x(0, t) = 0 \quad (0 \leq t \leq T),$$  \hspace{1cm} (3)

the non-local integral condition

$$\int_0^1 u(x, t)dx = 0 \quad (0 \leq t \leq T)$$  \hspace{1cm} (4)

and with the additional condition

$$u(0, t) = h(t) \quad (0 \leq t \leq T),$$  \hspace{1cm} (5)

where $\alpha > 0$, $\beta > 0$ are the given numbers, $f(x, t), \varphi(x), \psi(x), h(t)$ are the given functions, and $u(x, t)$, $a(t)$ are the required functions.
**Definition.** The classic solution of problem (1) – (5) is the pair \( \{ u(x,t), a(t) \} \) of the functions \( u(x,t) \) and \( a(t) \) possessing the following properties:
1) the function \( u(x,t) \) is continuous in \( D_T \) together with all its derivatives contained in equation (1);
2) the function \( a(t) \) is continuous on \([0,T]\);
3) all the conditions of (1) – (5) are satisfied in the ordinary sense.

**Lemma 1.** Let \( f(x,t) \in C(D_T), \int_0^1 f(x,t)dx = 0 \) \( (0 \leq t \leq T) \) \( \varphi(x), \psi(x) \in C^1[0,1], h(t) \in C^2[0,T], h(t) \neq 0 \) \( (0 \leq t \leq T) \) and
\[ \varphi'(1) = 0, \quad \psi'(1) = 0, \]
\[ \int_0^1 \varphi(x)dx = 0, \quad \int_0^1 \psi(x)dx = 0, \quad \varphi(0) = h(0), \quad \psi(0) = h'(0). \]

Then the problem on finding the classic solution of problem (1) – (5) is equivalent to the problem on defining of the function \( u(x,t) \) and \( a(t) \), possessing the properties 1) and 2) of definition of the classic solution of problem (1) – (5), from relations (1) – (3), and
\[ u_x(1,t) = 0 \quad (0 \leq t \leq T), \]
\[ u_{txx}(0,t) - \alpha u_{txx}(0,t) - \beta u_{xx}(0,t) = a(t)h(t) + f(0,t) \quad (0 \leq t \leq T). \]

**Proof.** Let \( \{ u(x,t), a(t) \} \) be a classical solution to the problem (1) – (5). Integrating equation (1) with respect to \( x \) from 0 to 1, we have:
\[ \frac{d^2}{dt^2} \int_0^1 u(x,t)dx - \alpha \frac{d}{dt} (u_x(1,t) - u_x(0,t)) \]
\[ - \beta (u_x(1,t) - u_x(0,t)) = a(t) \int_0^1 u(x,t)dx + \int_0^1 f(x,t)dx \quad (0 \leq t \leq T). \]

Taking into account that \( \int_0^1 f(x,t)dx = 0 \) \( (0 \leq t \leq T) \) and (3), we find that :
\[ \frac{d^2}{dt^2} u_x(1,t) - \alpha \frac{d}{dt} u_x(1,t) - \beta u_x(1,t) = 0 \quad (0 \leq t \leq T). \]

By (2) and \( \varphi'(1) = 0, \quad \psi'(1) = 0 \) we obtain:
\[ u_x(1,0) = \varphi'(1) = 0, \quad u_{tx}(1,0) = \psi'(1) = 0. \]
Since the problem (9), (10) has only a trivial solution, we have \( u_x(1,t) = 0 \), i.e. the condition (6) is fulfilled.

Assume now that \( h(t) \in C^2[0,T] \). Differentiating (5) twice, we get:

\[
\begin{align*}
  u_t(0,t) &= h'(t), \\
  u_{tt}(0,t) &= h''(t) \quad (0 \leq t \leq T).
\end{align*}
\]

\( (11) \)

It follows from (1) that:

\[
\begin{align*}
  &u_{ttt}(0,t) - u_{txx}(x_0,t) - \alpha u_{txx}(0,t) - \beta u_{xx}(0,t) = \\
  &= a(t)u(0,t) + f(0,t) \quad (0 \leq t \leq T).
\end{align*}
\]

\( (12) \)

Now suppose that \( \{u(x,t), a(t)\} \) is a solution to the problem (1) - (3), (6), (7), then from (8) and (3), (6) we find that:

\[
\begin{align*}
  &\frac{d^2}{dt^2} \int_0^1 u(x,t)dx - a(t) \int_0^1 u(x,t)dx = 0 \quad (0 \leq t \leq T). \\
  \end{align*}
\]

\( (13) \)

By (2) and \( \int_0^1 \varphi(x)dx = 0 \), \( \int_0^1 \psi(x)dx = 0 \), it is obvious that

\[
\begin{align*}
  &\int_0^1 u(x,0)dx = \int_0^1 \varphi(x)dx = 0, \\
  &\int_0^1 u_t(x,0)dx = \int_0^1 \psi(x)dx = 0.
\end{align*}
\]

\( (14) \)

Since the problem (13), (14) has only a trivial solution, \( \int_0^1 u(x,t)dx = 0 \) \( (0 \leq t \leq T) \), i.e. the condition (4) is fulfilled.

From (7) and (12) we obtain:

\[
\begin{align*}
  &\frac{d^2}{dt^2}(u(0,t) - h(t)) = a(t)(u(0,t) - h(t)) \quad (0 \leq t \leq T).
\end{align*}
\]

\( (15) \)

By (2) and \( \varphi(0) = h(0) \), \( \psi(0) = h'(0) \), we have:

\[
\begin{align*}
  \begin{cases}
    u(0,0) - h(0) = \varphi(0) - h(0) = 0, \\
    u_t(0,0) - h'(0) = \psi(0) - h'(0) = 0.
  \end{cases}
\end{align*}
\]

\( (16) \)

From (15) and (16) we conclude that condition (5) is fulfilled. The lemma is proved.
3 Investigation of the existence and uniqueness of the classic solution of the inverse boundary value problem

We’ll look for the first component \( u(x,t) \) of the solution \( \{u(x,t), a(t)\} \) of problem (1) – (3), (6), (7) in the form:

\[
u(x,t) = \sum_{k=0}^{\infty} u_k(t) \cos \lambda_kx \quad (\lambda_k = k\pi), \tag{17}\]

where

\[
u_k(t) = l_k \int_0^1 u(x,t) \cos \lambda_kxdx \quad (k = 0, 1, \ldots), \tag{18}\]

moreover,

\[
m_k = \begin{cases} 1, & k = 0 \\ 2, & k = 1, 2, \ldots \end{cases} \]

Then, applying the formal scheme of the Fourier method, from (1) and (2) we have:

\[
\left(1 + \lambda_k^2\right) u''_k(t) + \alpha \lambda_k^2 u'_k(t) + \beta \lambda_k^2 u_k(t) = \\
F_k(t; u, a) \quad (0 \leq t \leq T; \quad k = 0, 1, 2, \ldots), \tag{19}\]

\[
u_k(0) = \varphi_k, \quad u'_k(0) = \psi_k \quad (k = 0, 1, 2, \ldots), \tag{20}\]

where

\[
F_k(t; u, a) = f_k(t) + a(t)u_k(t), \quad f_k(t) = m_k \int_0^1 f(x, t) \cos \lambda_kxdx
\]

\[
\varphi_k = m_k \int_0^1 \varphi(x) \cos \lambda_kxdx, \quad \psi_k = m_k \int_0^1 \psi(x) \cos \lambda_kxdx \quad (k = 0, 1, 2, \ldots)
\]

It is obvious that

\[
\lambda_k^2 < 1 + \lambda_k^2 < 2\lambda_k^2.
\]
Therefore
\[ \frac{\alpha^2}{8} - \beta < \frac{\alpha^2\lambda_k^2}{4(1+\lambda_k^2)} - \beta < \frac{\alpha^2}{4} - \beta. \]

(21)

Now suppose that
\[ \frac{\alpha^2}{8} - \beta > 0. \]

(22)

Solving the problem (19), (20), we find:
\[ u_0(t) = \varphi_0 + t\psi_0 + \int_0^t (t-\tau)F_0(\tau; u, a)d\tau \quad (0 \leq t \leq T), \]

(23)

\[ u_k(t) = \frac{1}{\gamma_k} \left[ \left( \mu_{2k}e^{\mu_{1k}t} - \mu_{1k}e^{\mu_{2k}t} \right) \varphi_k + \left( e^{\mu_{2k}t} - e^{\mu_{1k}t} \right) \psi_k + \int_0^t F_k(\tau; u, a) \left( e^{\mu_{2k}(t-\tau)} - e^{\mu_{1k}(t-\tau)} \right) d\tau \right] \quad (0 \leq t \leq T; \quad k = 1, 2, \ldots), \]

(24)

where
\[ \mu_{ik} = -\frac{\alpha\lambda_k^2}{2(1+\lambda_k^2)} + (-1)^i \lambda_k \sqrt{\frac{\alpha^2\lambda_k^2}{4(1+\lambda_k^2)^2} - \frac{\beta}{1+\lambda_k^2}} \quad (i = 1, 2), \]

(25)

\[ \gamma_k = \mu_{2k} - \mu_{1k} = 2\lambda_k \sqrt{\frac{\alpha^2\lambda_k^2}{4(1+\lambda_k^2)^2} - \frac{\beta}{1+\lambda_k^2}}. \]

After substituting the expressions \( u_k(t) \) \( (k = 0, 1, \ldots) \) into (17), for the component \( u(x,t) \) of the solution \( \{u(x,t),a(t)\} \) to the problem (1) – (3), (6), (7) we get:
\[ u(x,t) = \varphi_0 + t\psi_0 + \int_0^t (t-\tau)F_0(\tau; u, a)d\tau + \]
\[ + \sum_{k=1}^{\infty} \left\{ \frac{1}{\gamma_k} \left[ \left( \mu_{2k}e^{\mu_{1k}t} - \mu_{1k}e^{\mu_{2k}t} \right) \varphi_k + \left( e^{\mu_{2k}t} - e^{\mu_{1k}t} \right) \psi_k + \right. \right. \]
\[ \left. \left. \int_0^t F_k(\tau; u, a) \left( e^{\mu_{2k}(t-\tau)} - e^{\mu_{1k}(t-\tau)} \right) d\tau \right] \cos \lambda_k x \right\}. \]

(26)
Now, from (7) and (17) we have:

\[
a(t) = [h(t)]^{-1} \{h''(t) - f(0, t) + \\
+ \sum_{k=1}^{\infty} \left[ \lambda_k^2 u_k''(t) + \alpha \lambda_k^2 u_k'(t) + \beta \lambda_k^2 u_k(t) \right] \}.
\]  

(27)

Differentiating (24) twice, we get:

\[
u_k'(t) = \frac{1}{\gamma_k} \left[ \mu_{1k} \mu_{2k}(e^{\mu_{1k} t} - e^{\mu_{2k} t}) \varphi_k + (\mu_{2k} e^{\mu_{2k} t} - \mu_{1k} e^{\mu_{1k} t}) \psi_k + \\
+ \int_0^t F_k(\tau; u, a)(\mu_{2k} e^{\mu_{2k}(t-\tau)} - \mu_{1k} e^{\mu_{1k}(t-\tau)}) d\tau \right]
\]

\[
(0 \leq t \leq T; \quad k = 0, 1, 2, \ldots),
\]  

(28)

\[
u_k''(t) = \frac{1}{\gamma_k} \left[ \mu_{1k} \mu_{2k}(\mu_{1k} e^{\mu_{1k} t} - \mu_{2k} e^{\mu_{2k} t}) \varphi_k + (\mu_{2k}^2 e^{\mu_{2k} t} - \mu_{1k}^2 e^{\mu_{1k} t}) \psi_k + \\
+ \int_0^t F_k(\tau; u, a)(\mu_{2k}^2 e^{\mu_{2k}(t-\tau)} - \mu_{1k}^2 e^{\mu_{1k}(t-\tau)}) d\tau \right] + \\
+ F_k(t; u, a) \quad (0 \leq t \leq T; \quad k = 1, 2, \ldots).
\]  

(29)

By (19) and (29) we have:

\[
\lambda_k^2 u_k''(t) + \alpha \lambda_k^2 u_k'(t) + \beta \lambda_k^2 u_k(t) = F_k(t; u, a) - u_k''(t) = \\
= -\frac{1}{\gamma_k} \left[ \mu_{1k} \mu_{2k}(\mu_{1k} e^{\mu_{1k} t} - \mu_{2k} e^{\mu_{2k} t}) \varphi_k + (\mu_{2k}^2 e^{\mu_{2k} t} - \mu_{1k}^2 e^{\mu_{1k} t}) \psi_k + \\
+ \int_0^t F_k(\tau; u, a)(\mu_{2k} e^{\mu_{2k}(t-\tau)} - \mu_{1k} e^{\mu_{1k}(t-\tau)}) d\tau \right] (0 \leq t \leq T; \quad k = 1, 2, \ldots).
\]  

(30)

To obtain the equation for the second component \( a(t) \) of the solution \{u(x, t), a(t)\} to the problem (1) – (3), (6), (7), we substitute expression (30) into (27) and have:

\[
a(t) = [h(t)]^{-1} \{h''(t) - f(0, t) -
\]
\[-\sum_{k=1}^{\infty} \frac{1}{\gamma_k} \left[ \mu_{1k} \mu_{2k} (\mu_{1k} e^{\mu_{1k} t} - \mu_{2k} e^{\mu_{2k} t}) \varphi_k + (\mu_{2k}^2 e^{\mu_{2k} t} - \mu_{1k}^2 e^{\mu_{1k} t}) \psi_k + \int_{0}^{t} F_k(\tau; u, a)(\mu_{2k}^2 e^{\mu_{2k}(t-\tau)} - \mu_{1k}^2 e^{\mu_{1k}(t-\tau)}) d\tau \right] \right\}. \quad (31)\]

Thus, the problem (1) – (3), (6), (7) is reduced to solving the system (26), (31) with respect to the unknown functions \( u(x, t) \) and \( a(t) \).

The following lemma is important for studying the uniqueness of the solution of problem (1) – (3), (6), (7).

Similarly [7], it is possible to prove the following lemma.

**Lemma 2.** If \( \{u(x, t), a(t)\} \) is any solution of problem (1) – (3), (6), (7), then the functions

\[ u_k(t) = m_k \frac{1}{\gamma_k} \int_{0}^{1} u(x, t) \cos \lambda_k x \, dx \quad (k = 1, 2, \ldots), \]

satisfy system (23), (24) in \([0, T]\).

**Remark.** It follows from lemma 2 that to prove the uniqueness of the solution to the problem (1) – (3), (6), (7), it suffices to prove the uniqueness of the solution to the system (26), (31).

In order to investigate problem (1) – (3), (6), (7), consider the following spaces:

1. Denote by \( B_{3, T}^2 \) [3] the totality of all the functions \( u(x, t) \) of the

\[ u(x, t) = \sum_{k=0}^{\infty} u_k(t) \cos \lambda_k x \quad (\lambda_k = k\pi), \]

considered in \( D_T \), where each of the functions \( u_k(t) (k = 0, 1, 2, \ldots) \) is continuous on \([0, T]\) and

\[ J(u) = \|u_0(t)\|_{C[0, T]} + \left\{ \sum_{k=1}^{\infty} \left( \lambda_k^3 \|u_k(t)\|_{C[0, T]} \right)^2 \right\}^{1/2} \]

< +\infty

The norm of this set is determined as follows:

\[ \|u(x, t)\|_{B_{3, T}^2} = J_T(u). \]

2. Denote by \( E_{3, T}^3 \) the space \( B_{3, T}^2 \times C[0, T] \) of the vector-functions \( z(x, t) = \{u(x, t), a(t)\} \) with the norm

\[ \|z\|_{E_{3, T}^3} = \|u(x, t)\|_{B_{3, T}^2} + \|a(t)\|_{C[0, T]}.\]
It is known that $B_{2,T}^3$ and $E_{T}^3$ are Banach spaces. Now, in the space $E_{T}^3$ consider the operator

$$\Phi(u,a) = \{\Phi_1(u,a), \Phi_2(u,a)\},$$

where

$$\Phi_1(u,a) = \bar{u}(x,t) \equiv \sum_{k=0}^{\infty} \bar{u}_k(t) \cos \lambda_k x,$$

$$\Phi_2(u,a) = \bar{a}(t),$$

$\bar{u}_0(t), \bar{u}_k(t), k = 1, 2, \ldots$ and $\bar{a}(t)$ equal the right sides of (23), (24) and (31), respectively.

It is easy to see that

$$|\mu_{ik}| < 0, \quad e^{\mu_{ik} t} \leq 1, \quad e^{\mu_{ik} (t-\tau)} \leq 1 \quad (i = 1, 2; k = 1, 2, \ldots; 0 \leq t \leq T, 0 \leq \tau \leq t),$$

$$\mu_{1k}\mu_{2k} = \frac{\beta \lambda_k^2}{1 + \lambda_k^2} \leq \beta,$$

$$\frac{1}{\gamma_k} = \frac{1}{\frac{\lambda_k^2}{2(1+\lambda_k^2)} - \beta} \leq \frac{1}{2\sqrt{\frac{1}{2} \left( \frac{\alpha^2}{\beta} - \beta \right)}} \equiv \gamma_0 \quad (k = 1, 2, \ldots).$$

Taking into account these relations, by means of simple transformations we find:

$$\|\bar{u}_{10}(t)\|_{C[0,T]} \leq |\varphi_{10}| + T |\psi_{10}| +$$

$$+ T \sqrt{T} \left( \int_0^T |f_{10}(\tau)|^2 \, d\tau \right)^{\frac{1}{2}} + T^2 \|a(t)\|_{C[0,T]} \|u_{10}(t)\|_{C[0,T]}, \quad (32)$$

$$\left( \sum_{k=1}^{\infty} \lambda_k^3 \|\bar{u}_k(t)\|_{C[0,T]} \right)^{\frac{1}{2}} \leq 4\alpha \gamma_0 \left( \sum_{k=1}^{\infty} \lambda_k^3 |\varphi_k| \right)^{\frac{1}{2}} + 4\gamma_0 \left( \sum_{k=1}^{\infty} \lambda_k^3 |\psi_k| \right)^{\frac{1}{2}} +$$

$$+ 4\gamma_0 \sqrt{T} \left( \int_0^T \sum_{k=1}^{\infty} \lambda_k^3 |f_k(\tau)|^2 \, d\tau \right)^{\frac{1}{2}} + 4\gamma_0 T \|a(t)\|_{C[0,T]} \left( \sum_{k=1}^{\infty} \lambda_k^3 \|u_k(t)\|_{C[0,T]} \right)^{\frac{1}{2}}, \quad (33)$$
\[
\|\tilde{a}(t)\|_{C[0,T]} \leq \left\| [h(t)]^{-1} \right\|_{C[0,T]} \left\{ \|h''(t) - f(x_0, t)\|_{C[0,T]} +
\right.
\]
\[
+ \left( \sum_{k=1}^{\infty} \lambda_k^{-2} \right)^{\frac{1}{2}} \left[ 2\alpha \beta \gamma_0 \left( \sum_{k=1}^{\infty} (\lambda_k^3 |\varphi_k|)^2 \right)^{\frac{1}{2}} + 2\alpha^2 \gamma_0 \left( \sum_{k=1}^{\infty} (\lambda_k^3 |\psi_k|)^2 \right)^{\frac{1}{2}} +
\right.
\]
\[
+ 2\alpha^2 \gamma_0 \sqrt{T} \left( \int_0^T \sum_{k=1}^{\infty} (\lambda_k^3 |f_k(\tau)|)^2 d\tau \right)^{\frac{1}{2}} +
\right.
\]
\[
+ 2\alpha^2 \gamma_0 T \|a(t)\|_{C[0,T]} \left( \sum_{k=1}^{\infty} (\lambda_k^3 \|u_k(t)\|_{C[0,T]})^2 \right)^{\frac{1}{2}} \right\}.
\] (34)

Suppose that the data of problem (1) – (3), (6), (7) satisfy the following conditions:

1. \( \alpha > 0, \beta > 0, \frac{\alpha^2}{8} - \beta > 0. \)

2. \( \varphi(x) \in C^2[0,1], \ \varphi''(x) \in L_2(0,1), \ \varphi'(0) = \varphi'(1) = 0. \)

3. \( \psi(x) \in C^2[0,1], \ \psi''(x) \in L_2(0,1), \ \psi'(0) = \psi'(1) = 0. \)

4. \( f(x,t), f_x(x,t), f_{xx}(x,t) \in C(D_T), \ f_{xxx}(x,t) \in L_2(D_T), \)

\[
f_x(0,t) = f_x(1,t) = 0 \quad (0 \leq t \leq T).
\]

5. \( h(t) \in C^2[0,T], \ h(t) \neq 0 \quad (0 \leq t \leq T). \)

Then, from (32)–(34), we get:

\[
\|\tilde{u}(x,t)\|_{B^3_{2,\infty}} \leq A_1(T) + B_1(T) \|a(t)\|_{C[0,T]} \|u(x,t)\|_{B^3_{2,\infty}},
\] (35)

\[
\|\tilde{a}(t)\|_{C[0,T]} \leq A_2(T) + B_2(T) \|a(t)\|_{C[0,T]} \|u(x,t)\|_{B^3_{2,\infty}},
\] (36)

where

\[
A_1(T) = \|\varphi(x)\|_{L_2(0,1)} + T \|\psi(x)\|_{L_2(0,1)} + T \sqrt{T} \|f(x,t)\|_{L_2(D_T)} +
\]
\[
+ 4\alpha \gamma_0 \|\varphi''(x)\|_{L_2(0,1)} + 4\gamma_0 \|\psi''(x)\|_{L_2(0,1)} + 4\gamma_0 \sqrt{T} \|f_{xxx}(x,t)\|_{L_2(D_T)},
\]
\[
B_1(T) = T^2 + 4\gamma_0 T,
\]
\[
A_2(T) = \left\| [h(t)]^{-1} \right\|_{C[0,T]} \left\{ \|h''(t) - f(0,t)\|_{C[0,T]} +
\right.
\]
Inverse BVP for the Boussinesq-Love equation

\[ \lambda_k \sum_{k=1}^{\infty} \left( \lambda_k - \frac{2}{k} \right)^{-\frac{1}{2}} \left[ 4\alpha \gamma_0 \| \varphi^{(3)}(x) \|_{L^2(0,1)} + 4\alpha^2 \gamma_0 \| \psi^{(3)}(x) \|_{L^2(0,1)} + 4\alpha^2 \gamma_0 \sqrt{T} \| f_{xxx}(x,t) \|_{L^2(D_T)} \right] \}

\[ B_2(T) = 4\alpha^2 \gamma_0 \| h(t) \|_{C[0,T]} - \| a(t) \|_{C[0,T]} \| u(x,t) \|_{B^3_{2,T}}. \]

It follows from the inequalities (35), (36) that:

\[ \| \tilde{u}(x,t) \|_{B^3_{2,T}} + \| \tilde{a}(t) \|_{C[0,T]} \leq A(T) + B(T) \| a(t) \|_{C[0,T]} \| u(x,t) \|_{B^3_{2,T}}, \quad (37) \]

where

\[ A(T) = A_1(T) + A_2(T), \quad B(T) = B_1(T) + B_2(T). \]

Now we can prove the following theorem.

**Theorem 1.** Let conditions 1-5 be fulfilled and

\[ (A(T) + 2)^2 B(T) < 1. \quad (38) \]

Then problem (1) – (3), (6), (7) has a unique solution in the ball \( K = K_R(\| z \|_{E^3_T} \leq R = A(T) + 2) \) of the space \( E^3_T \).

**Proof.** In the space \( E^3_T \) consider the equation

\[ z = \Phi z, \quad (39) \]

where \( z = \{ u, a \} \), the components \( \Phi_i(u,a) \) \( (i = 1, 2) \) of the operator \( \Phi(u,a) \) are given by the right hand sides of the equations (26), (31).

Consider the operator \( \Phi(u,a) \) in the ball \( K = K_R \) from \( E^3_T \). Similarly to (37), we see that for any \( z, z_1, z_2 \in K_R \) the following estimates are valid:

\[ \| \Phi z \|_{E^3_T} \leq A(T) + B(T) \| a(t) \|_{C[0,T]} \| u(x,t) \|_{B^3_{2,T}}, \quad (40) \]

\[ \| \Phi z_1 - \Phi z_2 \|_{E^3_T} \leq B(T) R(\| a_1(t) - a_2(t) \|_{C[0,T]} + \| u_1(x,t) - u_2(x,t) \|_{B^3_{2,T}}). \quad (41) \]

Then, it follows from (38) together with the estimates (40) and (41) that the operator \( \Phi \) acts in the ball \( K = K_R \) and is contractive. Therefore, in the ball \( K = K_R \) the operator \( \Phi \) has a unique fixed point \( \{ u, a \} \), that is a unique solution of equation (39) in the ball \( K = K_R \), i.e. a unique solution to the system (26), (31) in the ball \( K = K_R \).

The function \( u(x,t) \), as an element of the space \( B^3_{2,T} \), is continuous and has continuous derivatives \( u_x(x,t) \) and \( u_{xx}(x,t) \) in \( D_T \).
Now from (28) it is obvious that $u'_k(t)(k = 1, 2, ...)$ is continuous in $[0, T]$ and from the same relation we get:

$$\left( \sum_{k=1}^{\infty} (\lambda_k^3 \|u'_k(t)\|_{C[0,T]})^2 \right)^{\frac{1}{2}} \leq 2\sqrt{2} \beta \gamma_0 \|\varphi''(x)\|_{L_2(0,1)} + 2\sqrt{2} \alpha \|\psi''(x)\|_{L_2(0,1)} + 2\alpha \sqrt{2} \|f(x,t) + a (t) u_x(x,t)\|_{C[0,T]} \right)\bigg) \bigg),$$

Hence, it follows that $u_t(x,t)$, $u_{tx}(x,t)$, $u_{txx}(x,t)$ are continuous in $D_T$.

Next, from (19) it follows that $u''_k(t)(k = 1, 2, ...)$ is continuous in $[0, T]$ and consequently we have:

$$\left( \sum_{k=1}^{\infty} \lambda_k^3 \|u''_k(t)\|_{C[0,T]} \right)^{\frac{1}{2}} \leq 2\alpha \left( \sum_{k=1}^{\infty} (\lambda_k^3 \|u'_k(t)\|_{C[0,T]})^2 \right)^{\frac{1}{2}} +$$

$$+ \beta \left( \sum_{k=1}^{\infty} \lambda_k^3 \|u_k(t)\|_{C[0,T]} \right)^{\frac{1}{2}} + 2 \left\| f_x(x,t) + a (t) u_x(x,t) \right\|_{C[0,T]} \right),$$

From the last relation it is obvious that $u_{tt}(x,t)$, $u_{tx}(x,t)$, $u_{txx}(x,t)$ are continuous in $D_T$.

It is easy to verify that equation (1) and conditions (2), (3), (6), (7) are satisfied in the ordinary sense. Consequently, $\{u(x,t), a(t)\}$ is a solution of problem (1)–(3), (6), (7), and by Lemma 2 it is unique in the ball $K = K_R$. The theorem is proved.

By Lemma 1 the unique solvability of the initial problem (1)-(5) follows from the theorem.

**Theorem 2.** Let all the conditions of Theorem 1 be fulfilled and

$$\int_0^1 \varphi(x)dx = 0, \quad \int_0^1 \psi(x)dx = 0, \quad \varphi(0) = h(0), \quad \psi(0) = h'(0).$$

Then problem (1)–(5) has a unique classic solution in the ball $K = K_R(\|z\|_{E^3_T} \leq R = A(T) + 2)$ of the space $E^3_T$.

**References**


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