Parameter Estimations of Geometric Extreme Exponential Distribution Based on Dual Generalized Order Statistics

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Abstract
In this study, we consider the maximum likelihood and Bayes estimation of the parameters of geometric extreme exponential distribution based on dual generalized order statistics. However, the Bayes estimator does not exist in an explicit form for the parameters. We used an approximation based on Lindley method for obtaining Bayes estimates under squared error loss function. We also discuss the asymptotic variance-covariance matrix of maximum likelihood estimators of two parameters. Through Monte Carlo simulation, we compare the maximum likelihood and Bayes estimates of the parameters. And we include one real data analysis.
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1. Introduction

Order statistics are widely used in statistical modelling and inference. As a unified approach to various models of ordered random variables such as ordinary order statistics, upper record values and sequential order statistics, the concept of generalized order statistics (GOS) was introduced by Kamps (1995). Based on GOS, Burkschat et al. (2003) introduced the concept of dual generalized order statistics (DGOS) as a dual model of GOS and a unification of several models of decreasingly ordered random variables such as reversed order statistics, lower record values, and lower Pfeifer records.

Let $F(x)$ denote an absolutely continuous distribution function with the corresponding density function $f(x)$ and $X(r, n, \tilde{m}, k)$, $r = 1, 2, \cdots, n$ be the corresponding DGOS. Then, the joint probability density function of the first $n$ DGOS is

$$f_{X(1, n, \tilde{m}, k), \ldots, X(n, n, \tilde{m}, k)}(x_1, \ldots, x_n) = k \left( \prod_{j=1}^{n-1} \gamma_j \right) \left[ \prod_{i=1}^{n-1} F^{m_i}(x_i) f(x_i) \right]$$

$$\times F^{k-1}(x_n) f(x_n) \quad (1.1)$$

for $F^{-1}(1) > x_1 \geq \cdots \geq x_n > F^{-1}(0)$, $\tilde{m} = (m_1, m_2, \cdots, m_{n-1}) \in \mathbb{R}^{n-1}$, with parameters $n \in \mathbb{N}$, $n \geq 2$, $k > 0$ and $M_r = \sum_{j=r}^{n-1} m_j$ such that $\gamma_r = k + n - r + M_r > 0$ for all $r \in \{1, 2, \ldots, n\}$. In our paper, we assume $m_1 = m_2 = \cdots = m_{n-1} = m$. Depending on $m$ and $k$, there are a few special cases. If $m = 0$ and $k = 1$, (1.1) produces the joint probability density function of $n$ reversed order statistics from the independent and identically distributed random sample coming from $F(x)$. If $m = -1$, then $X(r, n, m, k)$ becomes to the $r$th lower $k$-record value of the iid random variables. A lot of distributional properties and applications of DGOS are studied by Ahsanullah (2004), Barakat and El-Adll (2009), Burkschat et al. (2003), Jaheen (2005), Mbah and Ahsanullah (2007), Kim and Kim (2013), Kim and Kim (2014), and Kim et al. (2016).

Marshall and Olkin (1997) introduced Geometric Extreme Exponential (GEE) distribution, which is right-skewed and suitable for lifetime data in Pakyari (2012). GEE distribution was further investigated and applied by Pakyari (2012), Adamidis et al (2005), and Adamidis and Loukas (1998).
The probability density function and the cumulative distribution function of GEE distribution are

\[
    f(x) = \frac{\alpha \lambda \exp(-\lambda x)}{(1 - \alpha \exp(-\lambda x))^2}, \quad \overline{\alpha} = 1 - \alpha,
\]

\[
    F(x) = \frac{1 - \exp(-\lambda x)}{1 - \overline{\alpha} \exp(-\lambda x)}, \quad x > 0, \quad \alpha > 0, \quad \lambda > 0.
\]

(1.2)

Note that, if \( \alpha = 1 \), then GEE distribution becomes the exponential distribution. If \( 0 < \lambda < 1 \), then GEE distribution becomes the exponential geometric distribution, which was discussed by Adamidis and Loukas (1998). One well-known fact about GEE distribution is that maximum likelihood estimators of two parameters can be numerically solved.

We discuss maximum likelihood estimation in Section 2 and Lindley approximation to obtain Bayes estimators for the parameters \( \alpha \) and \( \lambda \) in Section 3. In Section 4, to compare maximum likelihood estimators with Bayes estimators, Monte Carlo simulation is performed. And we analyze a real data of survival times.

### 2. Maximum Likelihood Estimation

In this paper, we assume \( m_1 = m_2 = \cdots = m_{n-1} = m \) for simplicity. Suppose that \( X(1, n, m, k), X(2, n, m, k), \ldots, X(n, n, m, k) \) are \( n \) DGOS drawn from GEE distribution with parameters \( \alpha \) and \( \lambda \). With (1.1) and (1.2), we get the likelihood function, which is

\[
    L(\alpha, \lambda | x) = k \alpha^n \lambda^n \prod_{j=1}^{n-1} \gamma_j \prod_{i=1}^{n-1} \frac{\exp(-\lambda x_i) \{1 - \exp(-\lambda x_i)\}^m}{\{1 - \alpha \exp(-\lambda x_i)\}^{m+2}} \\
    \times \frac{\exp(-\lambda x_n) \{1 - \exp(-\lambda x_n)\}^{k-1}}{\{1 - \overline{\alpha} \exp(-\lambda x_n)\}^{k+1}}.
\]

(2.1)

The log-likelihood function is

\[
    \ell = \ln L(\alpha, \lambda | x)
    = \ln(k) \prod_{j=1}^{n-1} \gamma_j + n \ln \alpha + n \ln \lambda - \lambda \sum_{i=1}^{n} x_i
    + \sum_{i=1}^{n-1} \{m \ln \{1 - \exp(-\lambda x_i)\} - (m + 2) \ln \{1 - \overline{\alpha} \exp(-\lambda x_i)\}\}
    + (k - 1) \ln \{1 - \exp(-\lambda x_n)\} - (k + 1) \ln \{1 - \overline{\alpha} \exp(-\lambda x_n)\}.
\]

(2.2)
Now, we want to find maximum likelihood estimators $\hat{\alpha}_M$ and $\hat{\lambda}_M$ of $\alpha$ and $\lambda$, respectively.

$$\frac{\partial l^*}{\partial \alpha} = n - \sum_{i=1}^{n-1} \frac{(m + 2) \exp(-\lambda x_i)}{1 - \alpha \exp(-\lambda x_i)} - \frac{(k + 1) \exp(-\lambda x_n)}{1 - \alpha \exp(-\lambda x_n)}, \quad (2.3)$$

$$\frac{\partial l^*}{\partial \lambda} = n \frac{\lambda}{\bar{x}} - \sum_{i=1}^{n} \left( m x_i \exp(-\lambda x_i) - \frac{(m + 2) \bar{x} x_i \exp(-\lambda x_i)}{1 - \alpha \exp(-\lambda x_i)} \right)$$

$$+ \frac{(k - 1) x_n \exp(-\lambda x_n)}{1 - \exp(-\lambda x_n)} - \frac{(k + 1) \bar{x} x_n \exp(-\lambda x_n)}{1 - \bar{x} \exp(-\lambda x_n)}. \quad (2.4)$$

Unfortunately, we cannot find the exact forms of $\hat{\alpha}_M$ and $\hat{\lambda}_M$. In Section 4, we use Newton’s method for nonlinear systems to find them under the lower record value case (which means $m = -1$ and $k = 1$). To apply Newton-Raphson method, with $m = -1$ and $k = 1$, we define

$$f_1(\alpha, \lambda) = n \frac{\alpha}{\bar{x}} - \sum_{i=1}^{n-1} \frac{\exp(-\lambda x_i)}{1 - \alpha \exp(-\lambda x_i)} - \frac{2 \exp(-\lambda x_n)}{1 - \alpha \exp(-\lambda x_n)}, \quad (2.5)$$

$$f_2(\alpha, \lambda) = n \frac{\lambda}{\bar{x}} - \sum_{i=1}^{n} x_i \left( \exp(-\lambda x_i) - \frac{\bar{x} x_i \exp(-\lambda x_i)}{1 - \alpha \exp(-\lambda x_i)} \right)$$

$$- \frac{2 \bar{x} x_n \exp(-\lambda x_n)}{1 - \bar{x} \exp(-\lambda x_n)}. \quad (2.6)$$

From these two, we should have

$$\frac{\partial f_1(\alpha, \lambda)}{\partial \alpha} = -n \frac{\alpha^2}{\bar{x}} + \sum_{i=1}^{n-1} \frac{\exp(-2\lambda x_i)}{\{1 - \alpha \exp(-\lambda x_i)\}^2} + \frac{2 \exp(-2\lambda x_n)}{\{1 - \alpha \exp(-\lambda x_n)\}^2},$$

$$\frac{\partial f_1(\alpha, \lambda)}{\partial \lambda} = \frac{\partial f_2(\alpha, \lambda)}{\partial \alpha} = \sum_{i=1}^{n-1} x_i \frac{\exp(\lambda x_i)}{\{\exp(\lambda x_i) - \bar{x}\}^2} + \frac{2 x_n \exp(\lambda x_n)}{\{\exp(\lambda x_n) - \bar{x}\}^2},$$

and

$$\frac{\partial f_2(\alpha, \lambda)}{\partial \lambda} = -n \frac{\lambda^2}{\bar{x}^2} + \sum_{i=1}^{n-1} \frac{x_i^2 \exp(\lambda x_i)}{\{\exp(\lambda x_i) - 1\}^2} + \frac{\bar{x} x_i^2 \exp(\lambda x_i)}{\{\exp(\lambda x_i) - \bar{x}\}^2}$$

$$+ \frac{2 \bar{x} x_n^2 \exp(\lambda x_n)}{\{\exp(\lambda x_n) - \bar{x}\}^2}.$$
However, it is difficult to have the exact expression of each element of the Fisher information matrix. Hence, we want to consider the approximate asymptotic variance-covariance matrix for maximum likelihood estimators, which is

\[
Q^* = \begin{pmatrix}
Q_{11}^* & Q_{12}^* \\
Q_{21}^* & Q_{22}^*
\end{pmatrix} = \begin{pmatrix}
-\frac{\partial^2 l}{\partial \alpha^2} & -\frac{\partial^2 l}{\partial \alpha \partial \lambda} \\
-\frac{\partial^2 l}{\partial \alpha \partial \lambda} & -\frac{\partial^2 l}{\partial \lambda^2}
\end{pmatrix}^{-1}_{(\alpha, \lambda) = (\hat{\alpha}_M, \hat{\lambda}_M)},
\tag{2.7}
\]

where

\[
\frac{\partial^2 l}{\partial \alpha^2} = -\frac{n}{\alpha^2} + \sum_{i=1}^{n-1} \frac{(m + 2) \exp(-2\lambda x_i)}{(1 - \alpha \exp(-\lambda x_i))^2} + \frac{(k + 1) \exp(-2\lambda x_n)}{(1 - \alpha \exp(-\lambda x_n))^2},
\]

\[
\frac{\partial^2 l}{\partial \lambda^2} = -\frac{n}{\lambda^2} - \sum_{i=1}^{n-1} \left\{ \frac{m x_i^2 \exp(\lambda x_i)}{(\exp(\lambda x_i) - \alpha)^2} + \frac{(m + 2) \alpha x_i^2 \exp(\lambda x_i)}{(\exp(\lambda x_i) - \alpha)^2} \right\}
\]

\[
- \frac{(k - 1) x_n^2 \exp(\lambda x_n)}{(\exp(\lambda x_n) - \alpha)^2} + \frac{(k + 1) \alpha x_n^2 \exp(\lambda x_n)}{(\exp(\lambda x_n) - \alpha)^2},
\]

and

\[
\frac{\partial^2 l}{\partial \alpha \partial \lambda} = \sum_{i=1}^{n-1} \frac{(m + 2) x_i \exp(\lambda x_i)}{(\exp(\lambda x_i) - \alpha)^2} + \frac{(k + 1) x_n \exp(\lambda x_n)}{(\exp(\lambda x_n) - \alpha)^2}.
\]

To get the approximate confidence intervals for \(\alpha\) and \(\lambda\), we can use the asymptotic normality property of maximum likelihood estimators. From (2.7), they are \(\hat{\alpha}_M \pm z_\psi \sqrt{Q_{11}^*}\) and \(\hat{\lambda}_M \pm z_\psi \sqrt{Q_{22}^*}\), where \(z_\psi\) is the \(z\)-score that has an area of \(\psi\) to its right under the standard normal variate.

### 3. Bayes Estimation

We are going to estimate the parameters \(\alpha\) and \(\lambda\) under squared error loss (SEL) function, which is defined as \(L(\rho, \hat{\rho}) = (\rho - \hat{\rho})^2\) for a parameter \(\rho\). Under the assumption that the parameters \(\alpha\) and \(\lambda\) are unknown, a natural choice for the prior distributions of \(\alpha\) and \(\lambda\) would be independent gamma distributions such as

\[
\pi(\alpha, \lambda) = \pi_1(\alpha) \pi_2(\lambda),
\tag{3.1}
\]

where

\[
\pi_1(\alpha) = \frac{\beta_1^{-\theta_1}}{\Gamma(\theta_1)} \alpha^{\theta_1 - 1} \exp(-\frac{\alpha}{\beta_1}),
\]

\[
\pi_2(\lambda) = \frac{\beta_2^{-\theta_2}}{\Gamma(\theta_2)} \lambda^{\theta_2 - 1} \exp(-\frac{\lambda}{\beta_2}).
\tag{3.2}
\]
Note that $\theta_1, \theta_2, \beta_1,$ and $\beta_2$ are chosen to reflect some prior information about $\alpha$ and $\lambda$. With (2.1) and (3.2), the joint posterior density function of $\alpha$ and $\lambda$ is

$$\pi(\alpha, \lambda|x) \propto \alpha^{n+\theta_1-1} \lambda^{n+\theta_2-1} \exp\left\{-\left(\frac{\alpha}{\beta_1} + \frac{\lambda}{\beta_2}\right)\right\} \frac{\exp(-\lambda x_n)\{1 - \exp(-\lambda x_n)\}^{k-1}}{\{1 - \alpha \exp(-\lambda x_n)\}^{k+1}} \times \prod_{i=1}^{n-1} \frac{\exp(-\lambda x_i)\{1 - \exp(-\lambda x_i)\}^m}{\{1 - \alpha \exp(-\lambda x_i)\}^{m+2}}.$$  (3.3)

The logarithm of (3.3) is

$$l = \ln \pi(\alpha, \lambda|x) = \text{Constant} + (n + \theta_1 - 1) \ln \alpha + (n + \theta_2 - 1) \ln \lambda - \lambda \sum_{i=1}^{n} x_i + \sum_{i=1}^{n-1} \left[m \ln\{1 - \exp(-\lambda x_i)\} - (m + 2) \ln\{1 - \alpha \exp(-\lambda x_i)\}\right] + (k - 1) \ln\{1 - \exp(-\lambda x_n)\} - (k + 1) \ln\{1 - \alpha \exp(-\lambda x_n)\} - \frac{\alpha}{\beta_1} - \frac{\lambda}{\beta_2}. \quad (3.4)$$

It is a famous result that the Bayes estimator of a function $V = U(\alpha, \lambda)$ under the SEL function is the posterior mean of the function, which is

$$E[U(\alpha, \lambda)|x] = \hat{U}_B = \frac{\int_0^\infty \int_0^\infty U(\alpha, \lambda)L(\alpha, \lambda|x)\pi(\alpha, \lambda) d\alpha d\lambda}{\int_0^\infty \int_0^\infty L(\alpha, \lambda|x)\pi(\alpha, \lambda) d\alpha d\lambda}. \quad (3.5)$$

In general, the ratio in (3.5) cannot be simplified in a simple closed form. Hence, we need to use a numerical approximation technique such as Lindley’s approximation which Lindley (1980) developed to approximate the ratio of two integrals in (3.5). With two parameters, say $(\lambda_1, \lambda_2) = (\alpha, \lambda)$, based on Lindley’s approximation, the approximate Bayes estimator of a function $V = U(\lambda_1, \lambda_2)$, under the SEL function, leads to

$$\hat{U}_B = U(\lambda_1, \lambda_2) + \frac{1}{2} (A + \ell_{00} B_{12} + \ell_{01} B_{21} + \ell_{01} C_{12} + \ell_{02} C_{21}) + p_1 A_{12} + p_2 A_{21}, \quad (3.6)$$

where

$$A = \sum_{i=1}^{2} \sum_{j=1}^{2} V_{ij} \tau_{ij}, \quad \ell_{ij} = \frac{\partial^{i+j} l}{\partial \lambda_1^i \partial \lambda_2^j}; \quad i, j = 0, 1, 2, 3, \quad \text{with} \quad i + j = 3,$$

$$p_i = \frac{\partial p}{\partial \lambda_i}, \quad V_i = \frac{\partial V}{\partial \lambda_i}, \quad V_{ij} = \frac{\partial^2 V}{\partial \lambda_i \partial \lambda_j}, \quad p = \ln \pi(\lambda_1, \lambda_2), \quad \text{for} \quad i, j = 1, 2,$$
and for $i \neq j$,

\[
A_{ij} = V_i \tau_{ii} + V_j \tau_{ji}, \quad B_{ij} = (V_i \tau_{ii} + V_j \tau_{ij}) \tau_{ii},
\]

\[
C_{ij} = 3V_i \tau_{ii} \tau_{ij} + V_j (\tau_{ii} \tau_{jj} + 2\tau_{ij}^2).
\]

The components $\tau_{ij}$ can be obtained from the following step:

\[
\left( \begin{array}{cc}
-l_{11} & -l_{12} \\
-l_{21} & -l_{22}
\end{array} \right)^{-1} = \frac{1}{l_{11}l_{22} - (l_{12})^2} \left( \begin{array}{cc}
-l_{22} & l_{12} \\
l_{21} & -l_{11}
\end{array} \right) = \left( \begin{array}{cc}
\tau_{11} & \tau_{12} \\
\tau_{21} & \tau_{22}
\end{array} \right),
\]

(3.7)

where $l_{ij} = \frac{\partial^2 l}{\partial \lambda_i \partial \lambda_j}$. And (3.6) is evaluated at maximum likelihood estimates $\hat{\alpha}_M$ and $\hat{\lambda}_M$ of $\alpha$ and $\lambda$, respectively.

Now, we want to apply Lindley’s approximation (3.6) to our case where $(\lambda_1, \lambda_2) = (\alpha, \lambda)$. Let $H = -l_{22}$, $G = -l_{11}$, and $I = l_{12} = l_{21}$. Then $N = GH - I^2$. Then, we can rewrite this as

\[
\tau_{11} = \frac{H}{N}, \quad \tau_{12} = \tau_{21} = \frac{I}{N}, \text{ and } \tau_{22} = \frac{G}{N},
\]

(3.8)

where

\[
G = \frac{n + \theta_1 - 1}{\alpha^2} - (m + 2) \sum_{i=1}^{n-1} \frac{\exp(-2\lambda x_i)}{(1 - \alpha \exp(-\lambda x_i))^2} - \frac{(k + 1) \exp(-2\lambda x_n)}{(1 - \alpha \exp(-\lambda x_n))^2},
\]

\[
I = (m + 2) \sum_{i=1}^{n-1} \frac{x_i \exp(\lambda x_i)}{(\exp(\lambda x_i) - \alpha)^2} + \frac{(k + 1)x_n \exp(\lambda x_n)}{(\exp(\lambda x_n) - \alpha)^2},
\]

and

\[
H = \frac{n + \theta_2 - 1}{\lambda^2} - \frac{(k + 1)\alpha x_n^2 \exp(\lambda x_n)}{(\exp(\lambda x_n) - \alpha)^2} + \frac{(k - 1)x_n^2 \exp(\lambda x_n)}{(\exp(\lambda x_n) - 1)^2} - \sum_{i=1}^{n-1} \frac{(m + 2)\alpha x_i^2 \exp(\lambda x_i)}{(\exp(\lambda x_i) - \alpha)^2} - \frac{mx_i^2 \exp(\lambda x_i)}{(\exp(\lambda x_i) - 1)^2}.
\]

For $i,j=0,1,2,3$, $l^*_{ij}$ can be expressed as follows:

\[
l^*_{30} = \frac{\partial^3 l}{\partial \alpha^3} = \frac{2(n + \theta_1 - 1)}{\alpha^3} - \sum_{i=1}^{n-1} \frac{2(m + 2) \exp(-3\lambda x_i)}{(1 - \alpha \exp(-\lambda x_i))^3} - \frac{2(k + 1) \exp(-3\lambda x_n)}{(1 - \alpha \exp(-\lambda x_n))^3},
\]
\[
\begin{align*}
\ell_{21} &= \frac{\partial^3 l}{\partial \alpha^2 \partial \lambda} \\
&= -2(m + 2) \sum_{i=1}^{n-1} \frac{x_i \exp(\lambda x_i)}{\{\exp(\lambda x_i) - \bar{\alpha}\}^3} - 2(k + 1) \frac{x_n \exp(\lambda x_n)}{\{\exp(\lambda x_n) - \bar{\alpha}\}^3}, \\
\ell_{12} &= \frac{\partial^3 l}{\partial \alpha \partial \lambda^2} \\
&= -(m + 2) \sum_{i=1}^{n-1} \frac{x_i^2 \exp(\lambda x_i) \{\exp(\lambda x_i) + \bar{\alpha}\}}{\{\exp(\lambda x_i) - \bar{\alpha}\}^3} \\
&\quad - (k + 1) \frac{x_n^2 \exp(\lambda x_n) \{\exp(\lambda x_n) + \bar{\alpha}\}}{\{\exp(\lambda x_n) - \bar{\alpha}\}^3}, \\
\end{align*}
\]

and
\[
\ell_{03} = \frac{\partial^3 l}{\partial \lambda^3} = \frac{2(n + \theta_2 - 1)}{\lambda^3} - (k + 1) \frac{\bar{\alpha} x_n^3 \exp(\lambda x_n) \{\exp(\lambda x_n) + \bar{\alpha}\}}{\{\exp(\lambda x_n) - \bar{\alpha}\}^3} \\
+ (k - 1) \frac{x_n^3 \exp(\lambda x_n) \{\exp(\lambda x_n) + 1\}}{\{\exp(\lambda x_n) - 1\}^3} \\
+ \sum_{i=1}^{n-1} \frac{m x_i^3 \exp(\lambda x_i) \{\exp(\lambda x_i) + 1\}}{\{\exp(\lambda x_i) - 1\}^3} - \frac{(m + 2) \bar{\alpha} x_i^3 \exp(\lambda x_i) \{\exp(\lambda x_i) + \bar{\alpha}\}}{\{\exp(\lambda x_i) - \bar{\alpha}\}^3}.
\]

Other components we need to find are \(p_1\) and \(p_2\). To get those, we should consider
\[
p = \ln \pi(\alpha, \lambda) = \text{Constant} + (\theta_1 - 1) \ln \alpha + (\theta_2 - 1) \ln \lambda - \frac{\alpha}{\beta_1} - \frac{\lambda}{\beta_2}.
\]

Then,
\[
\begin{align*}
p_1 &= \frac{\partial p}{\partial \alpha} = \frac{\theta_1 - 1}{\alpha} - \frac{1}{\beta_1}, \\
p_2 &= \frac{\partial p}{\partial \lambda} = \frac{\theta_2 - 1}{\lambda} - \frac{1}{\beta_2}.
\end{align*}
\]

Substituting all the above components to (3.6), under the SEL function, the Bayes estimator of the function \(V = U(\alpha, \lambda)\) given in (3.5) becomes
\[
\hat{U}_B = E[U(\alpha, \lambda) | \mathcal{F}] = U(\alpha, \lambda) + \psi_0 + \psi_1 V_1 + \psi_2 V_2, \quad (3.9)
\]
where

\[ \psi_0 = \frac{1}{2N} (V_{11}H + V_{12}I + V_{21}I + V_{22}G), \]

\[ \psi_1 = \frac{r_{30}H^2}{2N^2} + \frac{r_{30}G^2}{2N^2} + \frac{r_{12}(HG + 2I^2)}{2N^2} + \frac{3r_{21}HI}{N^2} + p_1 \frac{H}{N} + p_2 \frac{I}{N}, \]

\[ \psi_2 = \frac{r_{30}HI}{2N^2} + \frac{r_{30}G^2}{2N^2} + \frac{3r_{12}GI}{2N^2} + \frac{r_{21}(HG + 2I^2)}{N^2} + p_1 \frac{I}{N} + p_2 \frac{G}{N}. \]

Using (3.9), we can deduce Bayes estimators of the parameters \( \alpha \) and \( \lambda \) of GEE distribution as follows:

If \( U(\alpha, \lambda) = \alpha \), then \( \psi_0 = 0, V_1 = 1, \) and \( V_2 = 0 \). Hence,

\[ \hat{\alpha}_B = \alpha + \psi_1. \]  

(3.10)

If \( U(\alpha, \lambda) = \lambda \), then \( \psi_0 = 0, V_1 = 0, \) and \( V_2 = 1 \). Hence,

\[ \hat{\lambda}_B = \lambda + \psi_2. \]  

(3.11)

Note that all the equations (3.9), (3.10), and (3.11) are to be evaluated at maximum likelihood estimates \( \hat{\alpha}_M \) and \( \hat{\lambda}_M \).

4. Illustrative Examples

Example 1. Simulation Data

We conduct a simulation study to assess the performance of maximum likelihood estimates (MLE) and Bayes estimates (BE) for two parameters \( \alpha \) and \( \lambda \) of GEE distribution.

Table 1: Averaged RMSE of MLE and BE

<table>
<thead>
<tr>
<th>( n )</th>
<th>( \hat{\alpha}_M )</th>
<th>( \hat{\lambda}_M )</th>
<th>( \hat{\alpha}_B )</th>
<th>( \hat{\lambda}_B )</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>4.783524</td>
<td>2.105552</td>
<td>3.175451</td>
<td>2.60805</td>
</tr>
<tr>
<td>8</td>
<td>4.683475</td>
<td>2.083175</td>
<td>3.065761</td>
<td>2.572613</td>
</tr>
<tr>
<td>10</td>
<td>4.664676</td>
<td>2.081126</td>
<td>3.0533</td>
<td>2.569748</td>
</tr>
</tbody>
</table>

Let \( X_L(1) = x_1, X_L(2) = x_2, \ldots, X_L(n) = x_n \) be the lower record values of size \( n \) which can be obtained from DGOS as a special case by taking \( m = -1 \) and \( k = 1 \). MLE and Bayes estimates for the parameters \( \alpha \) and \( \lambda \) of GEE distribution based on lower records are computed and compared through the Monte Carlo simulation study according to the following steps:
1. For \( \alpha = 1 \) and \( \lambda = 2 \), we take a sample of lower record values with size \( n \) \((n = 6, 8, 10)\) generated from GEE distribution. The lower record values from GEE distribution are generated using the inverse cumulative distribution function,

\[
x_i = \frac{1}{\lambda} \ln\left( \frac{u_i \alpha}{u_i - 1} \right),
\]

where \( u_i \) is the uniformly distributed random variable.

2. \( \hat{\alpha}_M \) and \( \hat{\lambda}_M \) of the parameters \( \alpha \) and \( \lambda \) are calculated using the equations (2.5) and (2.6).

3. We choose \( \theta_1 = 0.01, \beta_1 = 100, \theta_2 = 0.01, \) and \( \beta_2 = 100 \) to make both priors have big variances, since we do not know the exact prior distributions. For these given values, the Bayes estimates of \( \alpha \) and \( \lambda \) are computed from (3.10) and (3.11) with \( m = -1 \) and \( k = 1 \).

4. To evaluate the root mean squared error (RMSE) of MLE and Bayes estimates for different size \( n \), these steps are repeated 100 times. Note

\[
\text{RMSE} = \sqrt{\frac{1}{100} \sum_{i=1}^{100} (h(\hat{\delta}_0) - h(\hat{\delta}_i))^2},
\]

where \( h(\delta_0) \) is the true value and \( h(\hat{\delta}_i) \) is the \( i \)th estimate of \( h(\delta) \) evaluated at \( \hat{\delta} \).

Table 1 provides the averaged RMSE of MLE and Bayes estimates based on lower record values generated from GEE distribution. As the sample size \( n \) increases, we expect that RMSE of the estimates decreases, which is the case in our computer simulation. In the sense of comparing RMSE of the estimates, we can see that Bayes estimates of \( \alpha \) are better than MLE and MLE of \( \lambda \) are better than Bayes estimates.

**Example 2. Real Data**

Mayo Clinic performed a double-blinded randomized trial in primary biliary cirrhosis of the liver (PBC), which is a rare but fatal liver disease. The main interest of the experiment was the time to death of patients with PBC. Fleming and Harrington (1991) deals with the data from the experiment and Danish and Aslam (2016) uses survival times in days from the data with the highest category of bilirubin, which are 400, 77, 859, 71, 1037, 1427, 733, 334, 41, 51, 549, 1170, 890, 1413, 853, 216, 131, 223, 1827, 2540, 1297, 264, 797,
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930, 264, 1350, 1191, 130, 943, 974, 790. Also, Danish and Aslam (2016) shows that GEE distribution is a good fit to the survival data. Thus we want to use the data to illustrate the simulation procedure in the previous part. From the data, we have four lower record values 400, 77, 71, and 41. Under the same simulation procedure with the four lower record values, we have \( \hat{\alpha}_M = 7.619391, \hat{\lambda}_M = 7.639106, \hat{\alpha}_B = 6.008962, \) and \( \hat{\lambda}_B = 9.317161. \) If we accept that the size of the lower record values is quite small, MLE and BE are very consistent to each other. Based on the simulation result, we want to choose \( \hat{\alpha}_B = 6.008962 \) and \( \hat{\lambda}_M = 7.639106. \)

References


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