Uniform Hyperpath $P^{(4)}$-Designs

Maria Di Giovanni and Mario Gionfriddo

Department of Mathematics and Comp. Sciences
University of Catania, Catania, Italy

Copyright © 2016 Maria Di Giovanni, Mario Gionfriddo. This article is distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Abstract
Given an hypergraph $H^{(4)}$, uniform of rank 4, an $H^{(4)}$-decomposition of the complete hypergraph $K^{(4)}_v$ is a collection of hypergraphs, all isomorphic to $H^{(4)}$, whose edge-sets partition the edge-set of $K^{(4)}_v$. An $H^{(4)}$-decomposition of $K^{(4)}_v$ is also called an $H^{(4)}$-design or an $H^{(4)}$-system. In this paper, we determine the spectrum in all the three cases in which $H^{(4)}$ is an hyperpath $P^{(4)}$ with two edges.

Mathematics Subject Classification: 05B05

Keywords: Path-Hypergraphs; Decompositions $K^{(4)}_v$

1 Introduction
Let $K^{(n)}_v = (X, \mathcal{E})$ be the complete hypergraph, uniform of rank $n$, defined in a vertex set $X = \{x_1, x_2, ..., x_v\}$, for $v \geq n$. This means that $\mathcal{E} = \mathcal{P}_n(X)$, the collection of all the $n$-subsets of $X$. Let $H^{(n)}$ be a subhypergraph of $K^{(4)}_v$.

An $H^{(n)}$-decomposition of $K^{(n)}_v$ is a pair $\Sigma = (X, \mathcal{B})$, where $\mathcal{B}$ is a partition of the edge set $\mathcal{P}_n(X)$ of $K^{(n)}_v$, into subsets all of which yield subhypergraphs all isomorphic to $H^{(n)}$. An $H^{(n)}$-decomposition $\Sigma = (X, \mathcal{B})$ of $K^{(n)}_v$ is also called an $H^{(n)}$-design (also an $H^{(n)}$-system) of order $v$, and the classes of the partition $\mathcal{B}$ are said to be the blocks of $\Sigma$ [1].

---

1The present research has been supported by PRIN 2012 (MIUR), FIR 2015 (Catania Univ.) and dell’INDAM-GNSAGA (Italy).
If \( n = 2 \), \( H^{(2)} \) is a graph \( G \) and \( G \)-designs have been well studied in the recent past with many results obtained by many authors.

If \( n = 3 \), \( H^{(3)} \)-designs have been studied in [2], where the spectrum has been studied in some cases of hypergraphs \( H^{(3)} \) with few edges. Recent results can be found in the references, regarding the intersection problem [5][6][8], the construction of cyclic [4] or non-cyclic [3] or balanced \( H^{(3)} \)-designs [7].

Among all the graphs, there is exactly one path with two edges and it is known as \( P_3 \). If \( x, y, z \) are the vertices of a path \( P_3 \) are the edges are \( \{x, y\}, \{y, z\} \), it will be indicate also by \( [x, (y), z] \).

Among all the hypergraphs uniform of rank 3, there are exactly two hyperpath with two edges. The number of vertices can be 4 or 5. A \( P^{(3)}(2, 4) \) will be the hyperpath having vertices \( x, y, z, t \) and edges \( \{x, y, z\}, \{y, z, t\} \), and it will be indicate by \( [x, (y, z), t] \). A \( P^{(3)}(1, 5) \) will be the hyperpath having vertices \( x, y, z, t, w \) and edges \( \{x, y, z\}, \{z, t, w\} \), and it will be indicate by \( [x, y, (z), t, w] \).

It is well-known that:

**Theorem 1.1**: There exist a \( P_3 \)-design of order \( v \) if and only if \( v \equiv 0 \) or 1 \( \mod 4 \), \( v \geq 4 \).

For \( P^{(3)}(2, 4) \) and \( P^{(3)}(1, 5) \), we have that:

**Theorem 1.2** [2]: There exist a design of type \( P^{(3)}(2, 4) \) or \( P^{(3)}(1, 5) \) of order \( v \) if and only if \( v \equiv 1 \) \( \mod 4 \) or \( v \) even, and respectively \( v \geq 4 \) or \( v \geq 5 \).

Some proof of these theorems can be found also in [1].

Let \( K^{(4)}_v = (X, \mathcal{E}) \) be the complete hypergraph, uniform of rank 4, defined in a vertex set \( X = \{x_1, x_2, ..., x_v\} \). This means that \( \mathcal{E} = \mathcal{P}_4(X) \), the collection of all the 4-subsets of \( X \).

In this paper we will consider always \( H^{(4)} \)-designs, where \( H^{(4)} \) is a subhypergraph of the complete 4-uniform hypergraph \( K^{(4)}_v \).

In what follows, we will always indicate by \( v \) the order of \( H^{(4)} \)-designs and by \( n \) the number of vertices of its blocks. Further, we consider all the possible path-hypergraphs, also hyperpaths, uniform of rank 4, with 2 edges.

In particular:

- \( P^{(4)}(1, 7) \) will indicate the hyperpath having 7 vertices and two edges with exactly one vertex in common: if \( x_1, x_2, ..., x_7 \) are the vertices and \( x_4 \) is the unique vertex in common, such an hyperpath will be indicate by \( [x_1, x_2, x_3, (x_4), \ldots] \).
Uniform hyperpath $P^{(4)}$-designs

$x_5, x_6, x_7$;

- $P^{(4)}(2, 6)$ will indicate the hyperpath having 6 vertices and two edges with exactly two vertices in common: if $x_1, x_2, ..., x_6$ are the vertices and $x_3, x_4$ are the vertices in common, such an hyperpath will be indicate by $[x_1, x_2, (x_3, x_4), x_5, x_6]$;

- $P^{(4)}(3, 5)$ will indicate the hyperpath having 5 vertices and two edges with exactly three vertices in common: if $x_1, x_2, ..., x_5$ are the vertices and $x_2, x_3, x_4$ are the vertices in common, such an hyperpath will be indicate by $[x_1, (x_2, x_3, x_4), x_5]$.

In this paper we determine the spectrum of $H^{(4)}$-designs, where $H^{(4)}$ is an hyperpath of type $P^{(4)}(1, 7)$, or $P^{(4)}(2, 6)$, or $P^{(4)}(3, 5)$.

2 Main Definitions and Necessary Conditions

A $P^{(4)}(1, 7)$-design $\Sigma = (X, B)$ of order $v$, briefly a $P^{(4)}(1, 7)(v)$-design is a $P^{(4)}(1, 7)$-decomposition of the complete uniform hypergraph $K^{(4)}_v = (X, \mathcal{P}_4(X))$.

A $P^{(4)}(2, 6)$-design $\Sigma = (X, B)$ of order $v$, briefly a $P^{(4)}(2, 6)(v)$-design, is a $P^{(4)}(2, 6)$-decomposition of the complete uniform hypergraph $K^{(4)}_v = (X, \mathcal{P}_4(X))$.

A $P^{(4)}(3, 5)$-design $\Sigma = (X, B)$ of order $v$, briefly a $P^{(4)}(3, 5)(v)$-design, is a $P^{(4)}(3, 5)$-decomposition of the complete uniform hypergraph $K^{(4)}_v = (X, \mathcal{P}_4(X))$.

Now, we give the following existence conditions

**Theorem 2.1**: If $\Sigma = (X, B)$ is a $P^{(4)}(8 - n, n)$-design of order $v$, for $n = 5, 6, 7$, then:

1) $|B| = v(v - 1)(v - 2)(v - 3)/48$;

2) $v \equiv 0$, or $1$, or $2$, or $3$, mod $8$, $v \geq 8$.

**Proof.** It is immediate to see that the number of blocks of in any $H^{(4)}$-design $\Sigma = (X, B)$, of order $v$, where $H^{(4)}$ has two edges, is $|B| = v(v - 1)(v - 2)(v - 3)/48$.

Further, since $|B|$ is an integer, from 1) it follows 2).
The matrix $\mathcal{M}(v)$

In what follows we will use the matrix $\mathcal{M}(v)$, defined in $\mathbb{Z}_v = \{0, 1, 2, \ldots, v-1\}$. For the definition, the use and further details about this matrix see [1][4].

Respectively for $v \equiv 0 \mod 3, v \geq 6$, and for $v \equiv 1, 2 \mod 3, v \geq 4$, $\mathcal{M}(v)$ is a matrix $m \times 3$, associated with $v$, such that:

\[
\mathcal{M}(v = 3h) = \begin{bmatrix}
(1,1) & (1,v-2) & (v-2,1) \\
(1,2) & (2,v-3) & (v-3,1) \\
(...) & (...) & (...) \\
(1,v-3) & (v-3,2) & (2,1) \\
(2,2) & (2,v-4) & (v-4,2) \\
(...) & (...) & (...) \\
(2,v-5) & (v-5,3) & (3,2) \\
(...) & (...) & (...) \\
(h-1,v-1) & (h-1,v-2) & (h-2,v) \\
(h,h) & (h,h) & (h,h)
\end{bmatrix},
\]

\[
\mathcal{M}(v) = \begin{bmatrix}
(1,1) & (1,v-2) & (v-2,1) \\
(1,2) & (2,v-3) & (v-3,1) \\
(...) & (...) & (...) \\
(1,v-3) & (v-3,2) & (2,1) \\
(2,2) & (2,v-4) & (v-4,2) \\
(...) & (...) & (...) \\
(2,v-5) & (v-5,3) & (3,2) \\
(...) & (...) & (...) \\
(3,3) & (3,v-6) & (v-6,3) \\
(...) & (...) & (...) \\
(3,v-7) & (v-7,4) & (4,3) \\
(...) & (...) & (...) \\
(h,h) & (h,v-2h) & (v-2h,h) \\
(h,v-2h-1) & (v-2h-1,h+1) & (h+1,h)
\end{bmatrix}.
\]
Observe that:

1) if $v = 3h + 1$, the last row begins with the pair $(h, h)$;

2) if $v = 3h + 2$, the last row begins with the pair $(h, v - 2h - 1)$;

3) if $v = 3h$, the last row begins with the pair $(h, h)$.

Further:

1) For any triple $T = \{x, y, z\} \subseteq Z_v$ with $x < y < z$ and $y - x = a$, $z - y = b$, there exists a row of $\mathcal{M}(v)$ containing the pair $(a, b)$.

2) For any pair $(a, b)$ of $\mathcal{M}(v)$, if $T = \{x, y, z\}$ is a triple where $y - x = a$ and $z - y = b$, there is a base-triple $C = \{0, a, a + b\}$ such that $T$ can be obtained from $C$ by translation of blocks.

3) For $v = 3h$, the last row contains the three pairs $(h, h), (h, h), (h, h)$. This pairs produce the base-triple $\{0, h, 2h\}$, whose translates are exactly $h$. In this case, we say that the translates describe a short orbit.

For 2), any two pairs from the same row in the matrix $\mathcal{M}$ are said to be equivalent among them.

In what follows, fixed $v$, we will indicate by $R_i$, for every $i = 1, 2, \ldots, m_h$, the set of rows of $\mathcal{M}(v)$ having in the first column the pairs:

$$(i, i), (i, i + 1), \ldots, (v - 1 - 2i),$$

in this order.

If $|R_i| = m_i$, it is possible to calculate the number $m = m_1 + m_2 + \ldots + m_h$ of rows of $\mathcal{M}(v)$.

It is immediate to prove that:

**Theorem 3.1**: Let $v = 3h + 1$ or $v = 3h + 2$ and let $\mathcal{M}(v)$ be the matrix associated with $v$. Then:

1) $m_i = v - 3i$, for every $i = 1, 2, \ldots, h$;

2) $m = h(2v - 3h - 3)/2$;

3) if $v = 3h + 1$, then $m = h(3h - 1)/2$;

4) if $v = 3h + 2$, then $m = h(3h + 1)/2$;
4) if \( v = 3h \), then \( m = 3h(h - 1)/2 \).

4 \( P^{(4)}(3, 5)\)-designs

In this section we determine the spectrum of all \( P^{(4)}(3, 5)\)-designs. In what follows, if \( B = [d, \{a, b, c\}, e] \) is a \( P^{(4)}(3, 5) \) defined in \( Z_v \), we will say the translates of \( B \) all the hypergraphs \( B_i = [d + i, \{a + i, b + i, c + i\}, e + i] \), for every \( i \in Z_v \). We will say that the hypergraph \( B \) is a base-block having for translates the hypergraphs \( B_i \).

**Theorem 4.1**: There exist \( P^{(4)}(3, 5) \)-designs of order \( v = 8 \).

**Proof.** Let \( X = \{1, 2, ..., 8\} \). Let \( B \) be the collection of blocks:

\[
\begin{align*}
[5, (2, 3, 4), 6], [7, (2, 3, 4), 8], [6, (2, 3, 5), 7], [7, (2, 3, 6), 8], \\
[5, (2, 3, 8), 7], [6, (2, 4, 5), 7], [7, (2, 4, 6), 8], [5, (2, 4, 8), 7], \\
[7, (2, 5, 6), 8], [5, (2, 7, 8), 6], [6, (3, 4, 5), 7], [7, (3, 4, 6), 8], \\
[5, (3, 7, 8), 6], [7, (4, 5, 6), 8], [5, (4, 7, 8), 6], [1, (6, 7, 8), 5], \\
[4, (1, 2, 3), 5], [6, (1, 2, 3), 7], [6, (1, 2, 5), 7], [5, (1, 2, 4), 6], \\
[7, (1, 2, 4), 8], [3, (1, 2, 8), 5], [7, (1, 2, 6), 8], [2, (1, 7, 8), 5], \\
[7, (1, 5, 6), 8], [5, (1, 3, 4), 6], [7, (1, 3, 4), 8], [6, (1, 3, 5), 7], \\
[7, (1, 3, 8), 5], [7, (1, 3, 6), 8], [5, (1, 4, 8), 7], [6, (1, 4, 5), 7], \\
[7, (1, 4, 6), 8], [7, (3, 4, 8), 5], [7, (3, 5, 6), 8].
\end{align*}
\]

If \( B \) is the collection of all these blocks, we can constatate that for every quadruple of distinct elements \( x, y, z, t \in X \), there exists exactly one block \( B \in B \) for which \( \{x, y, z, t\} \) is an edge. Therefore, \( \Sigma = (X, B) \) is a \( P^{(4)}(3, 5) \)-design of order 8. \( \Box \)

**Theorem 4.2** - Construction \( v = 8h \rightarrow v' = 8h + 1 \): If \( \Sigma \) is a \( P^{(4)}(3, 5) \)-design of order \( v = 8h \), \( h \geq 1 \), then there exists a \( P^{(4)}(3, 5) \)-design \( \Sigma' \) of order \( v' = 8h + 1 \) embedding \( \Sigma \).
Proof. Let \( \Sigma = (X, \mathcal{B}) \) be a \( P^{(4)}(3, 5) \)-design of order \( v = 8h, h \geq 1 \). Let \( \infty \notin X, X' = X \cup \{\infty\} \).

From Theorem 1.2, for every \( v \equiv 0 \mod 8 \) there exist \( P^{(3)}(2, 4) \)-designs. Therefore, consider a \( P^{(3)}(2, 4) \)-design \( \Omega = (X, \mathcal{C}) \) of order \( 8h, h \geq 1 \), and define the following collection of hyperpaths \( P^{(4)}(3, 5) \):

\[ \Gamma = \{ [x, (\infty, y, z, t) : (x, (y, z), t) \in \mathcal{C} \} \].

If \( \mathcal{B}' = \mathcal{B} \cup \Gamma \), then \( \Sigma' = (X', \mathcal{B}') \) is a \( P^{(4)}(3, 5) \)-design of order \( v' = 8h + 1 \).

Further, \( \Sigma \) results embedded in \( \Sigma' \). Indeed, at first observe that the blocks of \( \mathcal{B} \) have edges obviously all contained in \( X \), and the blocks of \( \Gamma \) have all the edges containing \( \infty \) and three vertices of \( X \). Therefore, for any quadruple \( Q \) of \( X' \):

- if \( Q \subseteq X \), then there exists exactly one block of \( \Sigma \), and no block of \( \Gamma \), having it as edge;

- if \( Q = \{x, y, z, \infty\} \), where \( x, y, z \in X \) and \( \infty \notin X \), there exists exactly one block of \( \mathcal{C} \) having \( \{x, y, z\} \) as edge and then exactly one block of \( \Gamma \), and no block of \( \mathcal{B} \), having \( Q \) as edge. The statement is so proved. \( \square \)

Theorem 4.3 - Construction \( v=8h+1 \rightarrow v'=8h+2 \): If \( \Sigma \) is a \( P^{(4)}(3, 5) \)-design of order \( v = 8h + 1, h \geq 1 \), then there exists a \( P^{(4)}(3, 5) \)-design \( \Sigma' \) of order \( v' = 8h + 2 \) embedding \( \Sigma \).

Proof. From Theorem 1.2, also for \( v = 8h + 1, h \geq 1 \), there exist \( P^{(3)}(2, 4) \)-designs. This permits to follow the same procedure of Construction \( v = 8h \rightarrow v' = 8h + 1 \), described in Theorem 4.2, and to prove so the statement. \( \square \)

Theorem 4.4 - Construction \( v=8h+2 \rightarrow v'=8h+3 \): If \( \Sigma \) is a \( P^{(4)}(3, 5) \)-design of order \( v = 8h + 2, h \geq 1 \), then there exists a \( P^{(4)}(3, 5) \)-design \( \Sigma' \) of order \( v' = 8h + 2 \) embedding \( \Sigma \).

Proof. As in Theorem 4.3, from Theorem 1.2 \( P^{(3)}(2, 4) \)-designs of order \( v = 8h + 2 \) exist, and it is possible to follow the same way of the previous Constructions, described in Theorem 4.2, 4.3, and to prove so the statement. \( \square \)

Theorem 4.5 - Construction \( v=8h \rightarrow v'=8h+8 \): If \( \Sigma \) is a \( P^{(4)}(3, 5) \)-design of order \( v = 8h, h \geq 1 \), then there exists a \( P^{(4)}(3, 5) \)-design \( \Sigma' \) of order \( v' = 8h + 8 \) embedding both \( \Sigma \) and a \( P^{(4)}(3, 5) \)-design of order 8, without blocks in common with \( \Sigma \).
Proof. Let $\Sigma_1 = (X_1, B_1)$ be a $P^{(4)}(3,5)$-design of order $v = 8h$, $h \geq 1$. Let $\Sigma_2 = (X_2, B_2)$ be a $P^{(4)}(3,5)$-design of order 8, where $X_1 \cap X_2 = \emptyset$. Let $X_1 = \{1, 2, \ldots, 8h\}, X_2 = \{A, B, C, D, E, F, G, H\}$.

Define the following collections of $P^{(4)}(3,5)$s.

1) $\Omega_1$: Let $\Gamma_1 = (X_1, C_1)$ be a $P^{(3)}(2,4)$-design of order $v = 8h$, defined in $X_1$ (see Theorem 1.2). For every $[x, (y, z), t] \in C_1$ and for every $\alpha \in X_2$, consider $[x, (y, z, \alpha), t]$. Let:

$$\Omega_1 = \{[x, (y, z, \alpha), t] : [x, (y, z), t] \in C_1, \alpha \in X_2\}.$$ 

Observe that all the hyperpaths belonging to $\Omega_1$ have for edges two quadruples with three vertices in $X_1$ and one vertex in $X_2$. Further, for every quadruple $Y$, such that $|Y \cap X_1| = 3$ and $|Y \cap X_2| = 1$, there exists exactly one hyperpath of $\Omega_1$ containing it as edge.

2) $\Omega_2$: Let $\Gamma_2 = (X_2, C_2)$ be a $P^{(3)}(2,4)$-design of order 8, defined in $X_2$ (Theorem 1.2). For every $[\alpha, (\beta, \gamma), \delta] \in C_2$ and for every $x \in X_1$, consider $[\alpha, (\beta, \gamma, x), \delta]$. Let:

$$\Omega_2 = \{[\alpha, (\beta, \gamma, x), \delta] : [\alpha, (\beta, \gamma), \delta] \in C_2, x \in X_1\}.$$ 

Observe that all the hyperpaths belonging to $\Omega_2$ have for edges two quadruples with one vertex in $X_1$ and three vertices in $X_2$. Further, for every quadruple $Y$, such that $|Y \cap X_1| = 1$ and $|Y \cap X_2| = 3$, there exists exactly one hyperpath of $\Omega_2$ containing it as edge.

3) $\Pi$: Let $\Pi$ be the following collection of $P^{(4)}(3,5)$s, defined for every of distinct elements $x, y \in X_1$:

$$[B, (x, y, A), C], [D, (x, y, A), E], [F, (x, y, A), G], [A, (x, y, H), B],$$

$$[C, (x, y, B), D], [E, (x, y, B), F], [B, (x, y, G), C], [D, (x, y, C), E],$$

$$[F, (x, y, C), H], [E, (x, y, D), F], [G, (x, y, D), H], [F, (x, y, E), G],$$

$$[E, (x, y, H), F], [F, (x, y, G), H].$$

Observe that all the hyperpaths belonging to $\Pi$ have for edges two quadruples with two vertices in $X_1$ and two vertices in $X_2$. Further, for every quadruple $Y$, such that $|Y \cap X_1| = 2$ and $|Y \cap X_2| = 2$, there exists exactly one hyperpath of $\Pi$ containing it as edge.
Uniform hyperpath $P^{(4)}$-designs

If $B' = B \cup \Omega_1 \cup \Omega_2 \cup \Pi$, then $\Sigma' = (X', B')$ is a $P^{(4)}(3, 5)$-design of order $v' = 8h + 8$. Further, $\Sigma_1$ and $\Sigma_2$ result embedded in $\Sigma'$. $\square$

Collecting together Theorems 4.1, 4.2, 4.3, 4.4, 4.5 we have the spectrum of $P^{(4)}(3, 5)$-designs:

**Theorem 4.6**: There exist $P^{(4)}(3, 5)$-designs of order $v$ if and only if $v \equiv 0$, or 1, or 2, or 3, mod 8, $v \geq 8$.

5 $P^{(4)}(2, 6)$-designs

In this section we determine the spectrum of $P^{(4)}(2, 6)$-designs. In what follows, if $B=[c, d, (a, b), e, f]$ is a $P^{(4)}(2, 6)$ defined in $Z_v$, we will say translates of $B$ all hypergraphs $B_i=[c+i, d+i, (a+i, b+i), e+i, f+i]$, for every $i \in Z_v$. We will say that the hypergraph $B$ is a base-block having for translates the hypergraphs $B_i$.

**Theorem 5.1**: There exist $P^{(4)}(2, 6)$-designs of order $v = 8$.

**Proof.** Let $X = \{1, 2, ..., 8\}$. Consider the following blocks:

- $[3, 4, (1, 2), 5, 6], [3, 5, (1, 2), 4, 6], [3, 6, (1, 2), 4, 7], [3, 7, (1, 2), 5, 8], [3, 8, (1, 2), 6, 7], [4, 5, (1, 2), 6, 8], [4, 8, (1, 2), 5, 7], [4, 5, (1, 3), 6, 8], [4, 6, (1, 3), 7, 8], [4, 8, (1, 3), 5, 6], [4, 7, (1, 3), 5, 8], [5, 6, (1, 4), 7, 8], [5, 7, (1, 4), 6, 8], [5, 8, (1, 4), 6, 7], [1, 2, (7, 8), 3, 4], [1, 5, (7, 8), 2, 4], [1, 6, (7, 8), 2, 5], [2, 3, (7, 8), 4, 5], [2, 5, (4, 6), 3, 8], [2, 4, (5, 8), 6, 7], [2, 6, (7, 8), 3, 5], [1, 5, (6, 8), 2, 3], [2, 4, (6, 8), 3, 5], [2, 5, (6, 8), 4, 7], [1, 3, (6, 7), 2, 5], [1, 5, (6, 7), 2, 4], [2, 3, (6, 7), 4, 5], [2, 5, (3, 7), 6, 8], [4, 8, (5, 6), 2, 3], [2, 5, (3, 4), 6, 7], [2, 6, (3, 4), 5, 7], [2, 7, (3, 4), 5, 8], [2, 8, (3, 4), 5, 6], [1, 3, (5, 7), 2, 4], [2, 8, (3, 5), 6, 7].

If $B$ is the collection of these blocks, we can constataate that for every quadruple of distinct elements $\{x, y, z, t\} \subseteq X$, there exists exactly one block $B \in B$
for which \( \{x, y, z, t\} \) is an edge. Therefore, \( \Sigma = (X, \mathcal{B}) \) is a \( P^{(4)}(2, 6) \)-design of order 8.

**Theorem 5.2**: If \( \Sigma \) is a \( P^{(4)}(2, 6) \)-design of order \( v = 8h, h \geq 1 \), then there exists a \( P^{(4)}(2, 6) \)-design \( \Sigma' \) of order \( v' = 8h + 1 \) embedding \( \Sigma \).

**Proof.** - **Construction** \( v = 8h \rightarrow v' = 8h + 1 \):

Let \( \Sigma = (X, \mathcal{B}) \) be a \( P^{(4)}(2, 6) \)-design of order \( v = 8h, h \geq 1 \), defined in \( X = \{1, 2, ..., 8h\} \). Let \( \infty \notin X \), \( X' = X \cup \{\infty\} \).

From Theorem 1.2, it is known that for every \( v \equiv 0 \mod 8 \) there exist \( P^{(3)}(1, 5) \)-designs of order \( v \). Therefore, consider a \( P^{(3)}(1, 5) \)-design \( \Omega = (X, \mathcal{C}) \) of order \( v = 8h, h \geq 1 \), and define the following collection of hyperpaths \( P^{(4)}(2, 6) \)s:

\[
\Gamma = \{[x, y, (\infty, z), t, w] : [x, y, (z), t, w] \in \mathcal{C}\}.
\]

If \( \mathcal{B}' = \mathcal{B} \cup \Gamma \), then we can prove that \( \Sigma' = (X', \mathcal{B}') \) is a \( P^{(4)}(2, 6) \)-design of order \( v' = 8h + 1 \). Further, \( \Sigma \) results embedded into \( \Sigma' \).

Indeed, at first observe that the blocks of \( \mathcal{B} \) have edges obviously all contained in \( X \), and the blocks of \( \Gamma \) have all the edges containing \( \infty \) and three vertices of \( X \). Therefore, for any quadruple \( Q \) of \( X' \):

- if \( Q \subseteq X \), then there exists exactly one block of \( \Sigma \), and no block of \( \Gamma \), having it as edge;
- if \( Q = \{x, y, z, \infty\} \), where \( x, y, z \in X \) and \( \infty \notin X \), there exists exactly one block of \( \mathcal{C} \) having \( \{x, y, z\} \) as edge and then exactly one block of \( \Gamma \), and no block of \( \mathcal{B} \), having \( Q \) as edge.

This completes the proof. \( \square \)

**Theorem 5.3** - **Construction** \( v = 8h + 1 \rightarrow v' = 8h + 2 \): If \( \Sigma \) is a \( P^{(4)}(2, 6) \)-design of order \( v = 8h + 1, h \geq 1 \), then there exists a \( P^{(4)}(2, 6) \)-design \( \Sigma' \) of order \( v' = 8h + 2 \) embedding \( \Sigma \).

**Proof.** Since \( v = 8h + 1 \) belongs to the spectrum of \( P^{(3)}(1, 5) \)-designs (Theorem 1.2), it is possible to follow the same way of Construction \( v = 8h \rightarrow v' = 8h + 1 \), described in Theorem 4.2, and to prove the statement. \( \square \)

**Theorem 5.4** - **Construction** \( v = 8h + 2 \rightarrow v' = 8h + 3 \): If \( \Sigma \) is a \( P^{(4)}(2, 6) \)-design of order \( v = 8h + 2, h \geq 1 \), then there exists a \( P^{(4)}(2, 6) \)-design \( \Sigma' \) of order \( v' = 8h + 2 \) embedding \( \Sigma \).

**Proof.** As in the previous Theorem, there exist \( P^{(3)}(1, 5) \)-designs of order \( v = 8h + 2 \) (Theorem 1.2). Therefore, it is possible to follow the same way of
the previous Constructions, described in Theorem 4.2, 4.3, and to prove the statement.

\[ \square \]

**Theorem 5.5 - Construction \( v=8h \rightarrow v'=8h+8 \):** If \( \Sigma \) is a \( P^{(4)}(2,6) \)-design of order \( v = 8h \), \( h \geq 1 \), then there exists a \( P^{(4)}(2,6) \)-design \( \Sigma' \) of order \( v' = 8h + 8 \) embedding both \( \Sigma \) and a \( P^{(4)}(2,6) \)-design of order 8 without blocks in common with \( \Sigma \).

**Proof.** Let \( \Sigma_1 = (X_1, B_1) \) be a \( P^{(4)}(2,6) \)-design of order \( v = 8h \), \( h \geq 1 \). Let \( \Sigma_2 = (X_2, B_2) \) be a \( P^{(4)}(2,6) \)-design of order 8, where \( X_1 \cap X_2 = \emptyset \). Let \( X_1 = \{1,2,\ldots,8h\} \), \( X_2 = \{A,B,C,D,E,F,G,H\} \).

Define the following collections of \( P^{(4)}(2,6) \)s.

1) \( \Omega_1 \): Let \( \Gamma_1 = (X_1, \mathcal{C}_1) \) be a \( P^{(3)}(2,4) \)-design of order \( v = 8h \), defined in \( X_1 \) (see Theorem 1.2) and \( W = \{\{A, B\}, \{C, D\}, \{E, F\}, \{G, H\}\} \) any 1-factor defined in \( X_2 \). For every \([x, (y, z), t] \in \mathcal{C}_1 \) and for every \( \{\alpha, \beta\} \in W \), consider \([x, \alpha, (y, z), \beta, t]\) and \([x, \beta, (y, z), \alpha, t]\). Let:

\[ \Omega_1 = \{[x, \alpha, (y, z), \beta, t], [x, \beta, (y, z), \alpha, t] : [x, (y, z), t] \in \mathcal{C}_1, \{\alpha, \beta\} \in W\}. \]

Observe that all the hyperpaths belonging to \( \Omega_1 \) have for edges two quadruples with three vertices in \( X_1 \) and one vertex in \( X_2 \). Further, for every quadruple \( Y \), such that \(|Y \cap X_1| = 3 \) and \(|Y \cap X_2| = 1\), there exists exactly one hyperpath of \( \Omega_1 \) containing it as edge.

2) \( \Omega_2 \): Let \( \Gamma_2 = (X_2, \mathcal{C}_2) \) be a \( P^{(3)}(2,4) \)-design of order 8, defined in \( X_2 \) (Theorem 1.2) and \( W' = \{\{1,2\}, \{3,4\}, \ldots, \{8h - 1,8h\}\} \) any 1-factor defined in \( X_1 \). For every \([R, (S, T), U] \in \mathcal{C}_2 \) and for every \( \{a, b\} \in W' \), consider \([R, a, (S, T), b, U]\) and \([R, b, (S, T), a, U]\). Let:

\[ \Omega_2 = \{[R, a, (S, T), b, U], [R, b, (S, T), a, U] : [R, (S, T), U] \in \mathcal{C}_2, \{a, b\} \in W'\}. \]

Observe that all the hyperpaths belonging to \( \Omega_2 \) have for edges two quadruples with one vertex in \( X_1 \) and three vertices in \( X_2 \). Further, for every quadruple \( Y \), such that \(|Y \cap X_1| = 1 \) and \(|Y \cap X_2| = 3\), there exists exactly one hyperpath of \( \Omega_2 \) containing it as edge.

3) \( \Pi \): Let \( \Pi \) be the following collection of \( P^{(4)}(2,6) \)s, defined for every pair of distinct elements \( x, y \in X_1 \):

\[
\begin{align*}
[A, B, (x, y), C, D], & \quad [E, F, (x, y), G, H], \\
[A, C, (x, y), B, D], & \quad [E, G, (x, y), F, H],
\end{align*}
\]

Uniform hyperpath \( P^{(4)} \)-designs 3049
[A, D, (x, y), B, C], [E, H, (x, y), F, G],
[A, E, (x, y), B, F], [C, G, (x, y), D, H],

[A, H, (x, y), B, E], [C, F, (x, y), D, G],
[A, G, (x, y), B, H], [C, E, (x, y), D, F],

[A, F, (x, y), B, G], [C, H, (x, y), D, E].

Observe that all the hyperpaths belonging to Π have for edges two quadruples
with two vertices in $X_1$ and two vertices in $X_2$. Further, for every quadruple
$Y$, such that $|Y \cap X_1| = 2$ and $|Y \cap X_2| = 2$, there exists exactly one hyperpath
of Π containing it as edge.

If $B' = B \cup \Omega_1 \cup \Omega_2 \cup \Pi$, then $\Sigma' = (X', B')$ is a $P^{(4)}(2, 6)$-design of order
$v' = 8h + 8$. Further, both $\Sigma_1$ and $\Sigma_2$ are embedded in $\Sigma'$.

Collecting together Theorems 5.1, 5.2, 5.3, 5.4, 5.5 we have the spectrum of
$P^{(4)}(2, 6)$-designs:

Theorem 5.6 : There exist $P^{(4)}(2, 6)$-designs of order $v$ if and only if $v \equiv 0,$
or $1$, or $2$, or $3$, mod $8$, $v \geq 8$.

6 Partition of $P_3(X)$ into disjoint triples

In this section we give a technique to define a partition Π of $P_3(X)$, the
family of all the 3-subsets of a set $X$ of any cardinality $v$, for $v \equiv 0$ or $2,$
mod $8$, $v \geq 8$. We will use this technique in the following Section to construct
$P^{(4)}(1, 7)$-designs of order $v = 8h$, $v = 8h + 2$.

In what follows, it is always $X = Z_v = \{0, 1, 2, ..., v - 1\}$. Further,
we consider the matrix $M(v)$ and any triple $T = \{0, a, b\} \subseteq X$, where
$0 < a < b < v$. We say that $T$ is a base-triple and its translates are all
the triples $T_{i+1} = \{i, a + i, b + i\}$, for every $i = 0, 1, ..., v - 1$. We indicate by
Π $\Pi_T$ such a family of translates of $T$. Note that in the triple $T$, and in all its
translates, the differences are in the order: $a, b - a, v - b$ and the ordered pairs
$(a, b - a), (b - a, v - b), (v - b, a)$ belong to a same row $R$ of $M(v)$.

1) Case $v=8h$

1.1) If the difference $1$ does not belong to any pair of $R$, then two consecutive
translates of $T$, let $T_i, T_{i+1}$, for are always disjoint. Therefore, it is possible to
define the following partition of $\Pi_T$ into classes of two disjoint triples, as follows

$$\{T_1, T_2\}, \{T_3, T_4\}, \{T_5, T_6\}, \ldots, \{T_{8h-1}, T_{8h}\}.$$ 

**Remark:** Observe that this technique works also when $v$ is a multiple of 3. Indeed, in this case it is $v = 24k$, with $k \geq 1$, and the matrix $M(v)$ has in the last row the pair $(k, k)$ repeated three times. The previous matchings of triples must be done from $\{T_1, T_2\}$ until $\{T_{8k-1}, T_{8k}\}$.

1.2) Consider the case in which the difference 1 belongs to any pair of $R$. Without loss of generality let $a = 1$. The translates of $T$ are:

- $T_1 : (0), (1), (u), T_2 : (1), (2), (u + 1), T_3 : (2), (3), (u + 2), \ldots,$
- $T_{4h} : (4h - 1), (4h), (u + 4h - 1), T_{4h+1} : (4h), (4h + 1), (u + 4h), \ldots,$
- $T_{8h-1} : (8h - 2), (8h - 1), (u - 2), T_{8h} : (8h - 1), (0), (u - 1).$

1.2.1) If $u \neq 4h, 4h + 1$, then $u + 4h \neq 0, 1$ and the family $\Pi$ can be partitioned as follows:

$$\{T_1, T_{4h+1}\}, \{T_2, T_{4h+2}\}, \{T_3, T_{4h+3}\}, \ldots, \{T_{4h}, T_{8h}\}.$$ 

1.2.2) If $u = 4h$ or $u = 4h + 1$, the family $\Pi$ can be partitioned as follows:

$$\{T_1, T_3\}, \{T_2, T_4\}, \{T_5, T_7\}, \{T_6, T_8\}, \ldots, \{T_{8h-3}, T_{8h-1}\}, \{T_{8h-2}, T_{8h}\}.$$ 

2) Case $v = 8h + 2$

2.1) If the difference 1 does not belong to any pair of $R$, then two consecutive translates of $T$, let $T_i, T_{i+1}$, are always disjoint. Therefore, it is possible to define the following partition of $\Pi_T$ into classes of two disjoint triples:

$$\{T_1, T_2\}, \{T_3, T_4\}, \{T_5, T_6\}, \ldots, \{T_{8h}, T_{v}\}.$$ 

**Remark:** Also in this case, this technique works also when $v$ is a multiple of 3. Indeed, if $v = 24k - 6$, with $k \geq 1$, the matrix $M(v)$ has in the last row the pair $(k, k)$ repeated three times and the previous matchings of triples must be done from $\{T_1, T_2\}$ until $\{T_{8k+1}, T_{8k+2}\}$.

2.2) Consider the case in which the difference 1 belongs to any pair of $R$. Without loss of generality, let $a = 1$. The translates of $T$ are:

- $T_1 : (0), (1), (u), T_2 : (1), (2), (u + 1), T_3 : (2), (3), (u + 2), \ldots,$
2.2.1) If \( u \) partitioned as follows:

\[ T_{4h+1} : (4h), (4h+1), (u+4h), \quad T_{4h+2} : (4h+1), (4h+2), (u+4h+1), \ldots \]

\[ T_{8h+1} : (8h), (8h+1), (u-2), \quad T_{8h+2} : (8h+1), (0), (u-1). \]

2.2.1) If \( u \neq 4h + 1, 4h + 2 \), then \( u + 4h + 1 \neq 0, 1 \) and the family \( \Pi \) can be partitioned as follows:

\[ \{T_1, T_{4h+2}\}, \quad \{T_2, T_{4h+3}\}, \quad \{T_3, T_{4h+4}\}, \quad \ldots \quad \{T_{4h+1}, T_{8h+2}\}. \]

2.2.2) If \( u = 4h + 1 \) or \( u = 4h + 2 \), the family \( \Pi \) can be partitioned as follows:

\[ \{T_1, T_3\}, \quad \{T_5, T_7\}, \quad \ldots \quad \{T_{8h-3}, T_{8h-1}\}, \]

\[ \{T_4, T_6\}, \quad \{T_8, T_{10}\}, \quad \ldots \quad \{T_{8h}, T_{8h+2}\}, \]

\[ \{T_2, T_{8h+1}\}. \]

7 \( P^{(4)}(1, 7) \)-designs

In this section we determine the spectrum of \( P^{(4)}(1, 7) \)-designs.

**Theorem 7.1** : There exist \( P^{(4)}(1, 7) \)-designs of order \( v = 8 \).

**Proof.** Let \( X = \{1, 2, \ldots, 8\} \). Consider the following blocks:

\[
\begin{align*}
[2, 3, 5, (1), 4, 6, 7], & \quad [2, 3, 7, (1), 4, 5, 6], \\
[2, 4, 5, (1), 3, 6, 7], & \quad [2, 6, 7, (1), 3, 4, 5], \\
[1, 6, 8, (2), 3, 4, 5], & \quad [1, 6, 8, (3), 2, 4, 7], \\
[1, 7, 8, (3), 2, 5, 6], & \quad [1, 6, 8, (4), 2, 5, 7], \\
[1, 7, 8, (4), 2, 5, 6], & \quad [1, 6, 8, (5), 2, 3, 7], \\
[1, 6, 7, (8), 2, 3, 5], & \quad [1, 5, 7, (8), 2, 4, 6], \\
[2, 3, 4, (8), 5, 6, 7], & \quad [2, 3, 6, (7), 4, 5, 8], \\
[1, 2, 8, (4), 3, 5, 6], & \quad [1, 5, 6, (3), 4, 7, 8], \\
[1, 2, 5, (6), 3, 4, 7], & \quad [1, 4, 7, (3), 5, 6, 8], \\
[1, 4, 6, (2), 5, 7, 8], & \quad [2, 6, 8, (7), 1, 3, 5], \\
\end{align*}
\]
If \( B \) is the collection of all these blocks, we can constatate that for every quadruple of distinct elements \( x, y, z, t \in X \), there exists exactly one block \( B \in B \) for which \( x, y, z, t \) is an edge. Therefore, \( \Sigma = (X, B) \) is a \( P^{(4)}(1, 7) \)-design of order 8.

**Theorem 7.2 - Construction \( v=8h \to v'=8h+1 \):** If \( \Sigma \) is a \( P^{(4)}(1, 7) \)-design of order \( v = 8h, h \geq 1 \), then there exists a \( P^{(4)}(1, 7) \)-design \( \Sigma' \) of order \( v' = 8h + 1 \) embedding \( \Sigma \).

**Proof.** Let \( \Sigma = (X, B) \) be a \( P^{(4)}(1, 7) \)-design of order \( v = 8h, h \geq 1 \), defined in \( X = \{1, 2, ..., 8h\} \). Let \( \infty \notin X \), \( X' = X \cup \{\infty\} \).

In the previous Section, we have seen that there exists a partition \( \Pi \) of \( P_3(X) \) into classes of two disjoint triples and, since in this case \( |P_3(X)| = 4h(8h - 1)(8h - 2)/3 \), it is \( |\Pi| = 2h(8h - 1)(8h - 2)/3 \). Therefore, if 

\[
\Pi = \{C_1, C_2, ..., C_r\},
\]

where \( r = 2h(8h - 1)(8h - 2)/3 \), and

\[
C_i = \{T_{i,1}, T_{i,2}\}, \text{ for every } i = 1, 2, ..., r,
\]

it is possible to define the following family of \( P^{(4)}(1, 7) \)s:

\[
\mathcal{F} = \{[x', y', z', (\infty), x'', y'', t''] : T_{i,1} = \{x', y', z'\}, T_{i,2} = \{x'', y'', z''\}, i = 1, 2, ..., r\}.
\]

If \( B' = B \cup \mathcal{F} \), then \( \Sigma' = (X', B') \) is a \( P^{(4)}(1, 7) \)-design of order \( v' = 8h + 1 \).
Indeed, observe that the blocks of $\mathcal{B}$ have edges obviously all contained in $X$, and the blocks of $\mathcal{F}$ have all the edges containing $\infty$ and three vertices of $X$. Therefore, for any quadruple $Q$ of $X'$:
- if $Q \subseteq X$, then there exists exactly one block of $\Sigma$, and no block of $\mathcal{F}$, having it as edge;
- if $Q = \{x, y, z, \infty\}$, where $x, y, z \in X$ and $\infty \notin X$, there exists exactly one block of $\mathcal{F}$ and no block of $\mathcal{B}$, having $Q$ as edge. The statement is so proved.

\[ \square \]

**Theorem 7.3 - Construction $v=8h+2 \rightarrow v'=8h+3$:** If $\Sigma$ is a $P^{(4)}(1,7)$-design of order $v = 8h + 2, h \geq 1$, then there exists a $P^{(4)}(1,7)$-design $\Sigma'$ of order $v' = 8h + 3$ embedding $\Sigma$.

**Proof.** Let $\Sigma = (X, \mathcal{B})$ be a $P^{(4)}(1,7)$-design of order $v = 8h + 2, h \geq 1$, defined in $X = \{1, 2, \ldots, 8h + 2\}$. Let $\infty \notin X$, $X' = X \cup \{\infty\}$. Also for $v = 8h + 2$, we have seen in the previous Section that there exists a partition $\Pi$ of $P_3(X)$ into classes of two disjoint triples and, since $|P_3(X)| = (8h + 2)(8h + 1)4h/3$, it is $|\Pi| = 2h(8h + 1)(8h + 2)/3$. Therefore, if

$$\Pi = \{C_1, C_2, \ldots, C_r\},$$

where $r = 2h(8h + 1)(8h + 2)/3$, and

$$C_i = \{T_{i,1}, T_{i,2}\}, \text{ for every } i = 1, 2, \ldots, r,$$

it is possible to define the following family of $P^{(4)}(1,7)$s:

$$\mathcal{F} = \{[x', y', z', (\infty), x'', y'', t''] :$$

$$T_{i,1} = \{x', y', z'\}, T_{i,2} = \{x'', y'', z''\}, i = 1, 2, \ldots, r\}.$$

If $\mathcal{B}' = \mathcal{B} \cup \mathcal{F}$, then we can see that $\Sigma' = (X', \mathcal{B}')$ is a $P^{(4)}(1,7)$-design of order $v' = 8h + 3$.

\[ \square \]

**Theorem 7.4 - Construction $v=8h$ or $8h+2 \rightarrow v'=v+8$:** If $\Sigma$ is a $P^{(4)}(1,7)$-design of order $v = 8h$ or $v = 8h + 2$, for $h \geq 1$, then there exists a $P^{(4)}(1,7)$-design $\Sigma'$ of order $v' = v + 8$ embedding $\Sigma$ and a $P^{(4)}(1,7)$-design of order 8.

**Proof.** Let $\Sigma_1 = (X_1, \mathcal{B}_1)$ be a $P^{(4)}(1,7)$-design of order $v = 8h$ or $v = 8h + 2$, for any $h \geq 1$. Let $\Sigma_2 = (X_2, \mathcal{B}_2)$ be a $P^{(4)}(1,7)$-design of order 8. Let
Define the following collections of $P^{(4)}(1, 7)$s.

- 1) $\Gamma_1$ is a family of $P^{(4)}(1, 7)$s whose edges are quadruples with three vertices in $X_1$ and one vertex in $X_2$. To define $\Gamma_1$, we follow, for every $y \in X_2$, the same procedure seen in the previous constructions $v \to v + 1$, taking $\Sigma_1$ as starting system and $y$ as additional vertex.

- 2) $\Gamma_2$ is a family of $P^{(4)}(1, 7)$s whose edges are quadruples with one vertex in $X_1$ and three vertices in $X_2$. Also here we follow, to define $\Gamma_2$, the same procedure seen in the previous constructions $v \to v + 1$, taking $\Sigma_2$ as starting system and $x$ as additional vertex, for every $x \in X_1$.

- 3) $\Omega$ is a family of $P^{(4)}(1, 7)$s whose edges are quadruples with two vertices in $X_1$ and two vertices in $X_2$. To define $\Omega$, consider an 1-factorization $\mathcal{F} = \{F_1, F_2, \ldots, F_{h-1}\}$ of $K_8$ defined in $X_1$ and a $P_3$-design $\Pi = (X_1, \mathcal{C})$ of order $v' = 8$ defined in $X_2$ (for the existence see Theorem 1.1).

- Case $v = 8h$: Observe that, since every $F_i \in \mathcal{F}$ has cardinality $4h$, it is possible to define a partition of $F_i$ into $2h$ classes $\mathcal{P}_{i,j}$ of cardinality 2. Therefore, for every $i = 1, 2, \ldots, 8h - 1$ and $j = 1, 2, \ldots, 2h$, are defined the classes:

$$\mathcal{P}_{i,j} = \{(A, B), (C, D)\},$$

where $(A, B) \cap (C, D) = \emptyset$, because $(A, B), (C, D)$ belong to a same factor of $\mathcal{F}$. In conclusion, $\Omega$ is the family of all the $P^{(4)}(1, 7)$s so-defined:

$$[A_{i,j}, B_{i,j}, x, (y), z, C_{i,j}, D_{i,j}], [C_{i,j}, D_{i,j}, x, (y), z, A_{i,j}, B_{i,j}],$$

for every class $\mathcal{P}_{i,j} = \{\{A_{i,j}, B_{i,j}\}, \{C_{i,j}, D_{i,j}\}\}$ and for every block $[x, (y), z] \in \mathcal{C}$.

- Case $v = 8h + 2$: Consider that, in this case, the number of pairs in any factor $F_i \in \mathcal{F}$ is $4h + 1$. However, it is possible to construct the $P^{(4)}(1, 7)$s of $\Omega$ defining a partition of the pairs of $\mathcal{F}$ as follows. Observe that, for every pair of distinct factors $F', F'' \in \mathcal{F}$, there exist always two pairs $\{x', y'\} \in F'$ and $\{x'', y''\} \in F''$ such that $\{x', y'\} \cap \{x'', y''\} = \emptyset$. This permits to partition $F_i \cup F_{i+1}$, for every $i = 1, 2, \ldots, 8h - 1$, and also $F_{8h+1} = \{\{\alpha, \beta\}, \{\gamma, \delta\}, \{\mu, \nu\}\}$, into classes $\mathcal{P}_{i,j} = \{\{A, B\}, \{C, D\}\}$, of disjoint pairs $\{A, B\}, \{C, D\}$, and to define the $P^{(4)}(1, 7)$s of $\Omega$ as in the case $v = 8h$.

Finally, we add to $\Omega$ these remaining blocks:
for every $[x, (y), z] \in \mathcal{C}$.

If $X = X_1 \cup X_2$ and $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \Gamma_1 \cup \Gamma_2 \cup \Omega$, then it is possible to verify that $\Sigma = (X, \mathcal{B})$ is a $P^{(4)}(1, 7)$-design of order $v' = v + 8$. \hfill \square

Collecting together all the Theorems of this Section, we have the spectrum of $P^{(4)}(1, 7)$-designs:

**Theorem 7.5:** There exist $P^{(4)}(1, 7)$-designs of order $v$ if and only if $v \equiv 0$, or 1, or 2, or 3, mod 8, $v \geq 8$.

**References**


Received: July 27, 2016; Published: October 22, 2016