Characterization of Real Hypersurface with Anti-derivatives of Structure Lie Operator in a Complex Space Form

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Abstract

In this paper, we investigate the real hypersurfaces on holomorphic distribution $D$ in a complex space form $M_n(c)$, $c \neq 0$ under the condition that $(\mathcal{L}_\xi \mathcal{L}_\xi)X + (\nabla_\xi \mathcal{L}_\xi)X = 0$, where $\mathcal{L}_\xi$ denote the structure Lie operator of $M$ in $M_n(c)$.

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1 Introduction

A complex $n$-dimensional Kaehlerian manifold of constant holomorphic sectional curvature $c$ is called a complex space form, which is denoted by $M_n(c)$. As is well-known, a complete and simply connected complex space form is complex analytically isometric to a complex projective space $P_n\mathbb{C}$, a complex Euclidean space $\mathbb{C}^n$ or a complex hyperbolic space $H_n\mathbb{C}$, according to $c > 0$, $c = 0$ or $c < 0$.

In this paper we consider a real hypersurface $M$ in a complex space form $M_n(c)$, $c \neq 0$. Then $M$ has an almost contact metric structure $(\phi, g, \xi, \eta)$ induced from the Kaehler metric and complex structure $J$ on $M_n(c)$. The
Reeb vector field $\xi$ is said to be principal if $A\xi = \alpha \xi$ is satisfied, where $A$ is the shape operator of $M$ and $\alpha = \eta(A\xi)$. In this case, it is known that $\alpha$ is locally constant ([4]) and that $M$ is called a Hopf hypersurface.

R. Takagi [15] completely classified homogeneous real hypersurfaces in such hypersurfaces as six model spaces $A_1$, $A_2$, $B$, $C$, $D$ and $E$. Berndt [1] classified all homogeneous Hopf hypersurfaces in $H_n\mathbb{C}$ as four model spaces which are said to be $A_0$, $A_1$, $A_2$ and $B$. A real hypersurface of $A_1$ or $A_2$ in $P_n\mathbb{C}$ or $A_0$, $A_1$, $A_2$ in $H_n\mathbb{C}$, then $M$ is said to be a type $A$ for simplicity.

As a typical characterization of real hypersurfaces of type $A$, the following is due to Okumura [14] for $c > 0$ and Montiel and Romero [11] for $c < 0$.

**Theorem 1** ([11, 14]) Let $M$ be a real hypersurface of $M_n(c)$, $c \neq 0$, $n \geq 2$. It satisfies $A\phi - \phi A = 0$ on $M$ if and only if $M$ is locally congruent to one of the model spaces of type $A$.

The induced operator $L_\xi$ on real hypersurface $M$ from the 2-form $L_\xi g$ is defined by $(L_\xi g)(X, Y) = g(L_\xi X, Y)$ for any vector field $X$ and $Y$ on $M$, where $L_\xi$ denotes the operator of the Lie derivative with respect to the structure vector field $\xi$. This operator $L_\xi$ is given

$$L_\xi = \phi A - A\phi$$

on $M$, and call it structure Lie operator of $M$. One of the most interesting problems in the study of real hypersurfaces $M$ in $M_n(c)$ is to investigate a geometric characterization of these model spaces. Recently, some works have studied several conditions on the structure Lie operator $L_\xi$ and given some results on the classification of real hypersurfaces of type $A$ in $M_n(c)$ ([6], [7] and [8] etc).

The Lie derivative of the shape operator, Ricci operator and Jacobi operator was investigated by Ki et al. [5], Kimura and Maeda [3], Perez et al [2]. As for the Lie derivative of structure Lie operator, Loo [10] obtained the following:

**Theorem 2** ([10]) Let $M$ be a real hypersurface of $M_n(c)$, $c \neq 0$, $n \geq 2$. It satisfies $L_\xi (\phi A - A\phi) = 0$ on $M$ if and only if $M$ is locally congruent to one of the model spaces of type $A$.

In the differential geometry, the study of real hypersurfaces whose operator is parallel is a problem of great importance. M. Ortega [13] has proved the nonexistence of real hypersurfaces in nonflat complex form with parallel structure Jacobi operator. Recently, Lim et al. ([8], [9]) has proved the following.

**Theorem 3** ([8]) Let $M$ be a real hypersurface of $M_n(c)$, $c \neq 0$, $n \geq 2$. Then $M$ satisfies $(\nabla_\xi L_\xi)X = (\nabla_\xi L_\xi)X$ if and only if $M$ is locally congruent to one of the model spaces of type $A$. 
Theorem 4 ([8]) Let $M$ be a real hypersurface of $M_n(c_e), \ c \neq 0, \ n \geq 2$. Then $M$ satisfies $(\mathcal{L}_\xi L_\xi)X + (\nabla_\xi L_\xi)X = 0$ if and only if $M$ is locally congruent to one of the model spaces of type A.

The holomorphic distribution $D$ of real hypersurface $M$ in $M_n(c_e)$ is defined by

$$D = \{X \in T_P(M)|g(X, \xi) = 0\}$$ (1)

In this paper, we shall study geometric characterizations of real hypersurfaces $M$ on holomorphic distribution $D$ in a non-flat complex space form $M_n(c_e)$ with Lie $\xi$-parallel and $\xi$-parallel of structure Lie operator. More specifically, we prove the following:

Main Theorem Let $M$ be a real hypersurface in a complex space form $M_n(c_e), \ c \neq 0$. Then $M$ satisfies $(\mathcal{L}_\xi L_\xi)X = (\nabla_\xi L_\xi)X$ on $D$ in $M_n(c_e)$ if and only if $M$ is locally congruent to one of the model space of type A.

All manifolds in the present paper are assumed to be connected and of class $C^\infty$ and the real hypersurfaces supposed to be oriented.

2 Preliminaries

Let $M$ be a real hypersurface immersed in a complex space form $M_n(c_e)$, and $N$ be a unit normal vector field of $M$. By $\nabla$ we denote the Levi-Civita connection with respect to the Fubini-Study metric tensor $\tilde{g}$ of $M_n(c_e)$. Then the Gauss and Weingarten formulas are given respectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \quad \tilde{\nabla}_X N = -AX$$

for any vector fields $X$ and $Y$ tangent to $M$, where $g$ denotes the Riemannian metric tensor of $M$ induced from $\tilde{g}$, and $A$ is the shape operator of $M$ in $M_n(c_e)$. For any vector field $X$ on $M$ we put

$$JX = \phi X + \eta(X)N, \quad JN = -\xi,$$

where $J$ is the almost complex structure of $M_n(c_e)$. Then we see that $M$ induces an almost contact metric structure $(\phi, g, \xi, \eta)$, that is,

$$\phi^2X = -X + \eta(X)\xi, \quad \phi \xi = 0, \quad \eta(\xi) = 1,$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi)$$ (2)

for any vector fields $X$ and $Y$ on $M$. Since the almost complex structure $J$ is parallel, we can verify from the Gauss and Weingarten formulas the followings:
\[ \nabla_X \xi = \phi AX, \]  
\[ (\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi. \]

Since the ambient manifold is of constant holomorphic sectional curvature \( c \), we have the following Gauss and Codazzi equations respectively:

\[ R(X, Y)Z = \frac{c}{4} \{ g(Y, Z)X - g(X, Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y \]
\[ -2g(\phi X, Y)\phi Z \} + g(AY, Z)AX - g(AX, Z)AY, \]  
\[ (\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4} \{ \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi \} \]

By use of (2.1), we have \((\mathcal{L}_\xi g)(X, Y) = g((\phi A - A\phi)X, Y)\) for any vector fields \( X \) and \( Y \) on \( M \), and hence the induced operator \( L_\xi \) from \( \mathcal{L}_\xi g \) is given by

\[ L_\xi X = (\phi A - A\phi)X. \]

\[ (\mathcal{L}_\xi L_\xi)X = [\xi, L_\xi X] - L_\xi [\xi, X] \]

for any vector fields \( X, Y \) and \( Z \) on \( M \), where \( R \) denotes the Riemannian curvature tensor of \( M \).

Let \( W \) be a unit vector field on \( M \) with the same direction of the vector field \( -\phi \nabla_\xi \xi \), and let \( \mu \) be the length of the vector field \( -\phi \nabla_\xi \xi \) if it does not vanish, and zero (constant function) if it vanishes. Then it is easily seen from (1) that

\[ A\xi = \alpha \xi + \mu W, \]

where \( \alpha = \eta(A\xi) \). We notice here that \( W \) is orthogonal to \( \xi \).

We put

\[ \Omega = \{ p \in M \mid \mu(p) \neq 0 \}. \]

Then \( \Omega \) is an open subset of \( M \).
3 Some Lemmas

In this section, we prepare without proof the following Lemma 3.1, 3.2 and shall prove Lemma 3.3.

**Lemma 3.1** If $\xi$ is a principal curvature vector and the corresponding principal curvature $\alpha$ is locally constant.

**Lemma 3.2** Assume that $\xi$ is a principal curvature vector and the corresponding principal is $\alpha$. Then we have

$$A\phi A - \frac{\alpha}{2} (A\phi + \phi A) - \frac{c}{2} = 0.$$  \hfill (10)

**Lemma 3.3.** Let $M$ be a real hypersurface satisfying $(L_\xi L_\xi)X + (\nabla_\xi L_\xi)X = 0$ on holomorphic distribution $D$ in a complex space form $M_n(c)$, $c \neq 0$. Then $M$ is a Hopf hypersurface in $M_n(c)$.

**Proof.** Let $M$ be a real hypersurfaces in a complex space form $M_n(c)$, $c \neq 0$, satisfying $(L_\xi L_\xi)X + (\nabla_\xi L_\xi)X = 0$. We assume that the open set $\Omega$ given in (8) is not empty. Then the above condition together with (2) and (7) implies that

$$2(\nabla_\xi L_\xi)X = -\phi A^2\phi X - A^2X + \alpha \eta(AX)\xi + \mu \eta(AX)W$$

for any vector field $X$ on $D$. Since we have $(\nabla_\xi L_\xi)X = \nabla_\xi(L_\xi X) - L_\xi(\nabla_\xi X)$, we see from the equation above that

$$2\{\phi(\nabla_\xi A)X-(\nabla_\xi A)\phi X\} = -\phi A^2\phi X - A^2X - 3\eta(AX)A\xi + 2\{\eta(X)A^2\xi + \eta(A^2X)\xi\}. \hfill (11)$$

If we apply Codazzi equation of (6) to $\phi(\nabla_\xi A)X$, then we obtain

$$\phi(\nabla_\xi A)X = \mu \phi \nabla X W - \phi A\phi AX + \alpha\{-AX + \eta(AX)\xi\} + \frac{\phi}{2}\{-X + \eta(X)\xi\} + (X\mu)\phi W. \hfill (12)$$

As a similar argument as the above, we can also verify from $(\nabla_\xi A)\phi X$ that

$$(\nabla_\xi A)\phi X = \mu \nabla X W - A\phi A\phi X + \alpha \phi A\phi X + ((\phi X)\alpha)\xi$$

$$+ (X\mu)W + \frac{\phi}{2}\{-X + \eta(X)\xi\}. \hfill (13)$$

If we substitute (12) and (13) into (11), then we can verify that

$$2\mu\{\phi \nabla X W - \nabla \phi X W\} = \{\phi A\phi A - A\phi A\phi\}X + 2\alpha\{A + \phi A\phi\}X$$

$$-\{\phi A^2\phi + A^2\}X + 2\eta(X)A^2\xi + \{2((\phi X)\alpha) - 5\alpha \eta(AX) + 2\eta(A^2X)\}\xi$$

$$+ \{2((\phi X)\mu) - 3\mu \eta(AX)\}W - 2(X\mu)\phi W. \hfill (14)$$
for any vector field $X$ on $D$. If we put $X = W$ into (14) and make use of (2) and (8), then we get
\[
2\mu \{ \phi \nabla W - \nabla_{\phi W} W \} = 2(\phi A\phi A - A\phi A\phi)W + 2\alpha(A + \phi A\phi)W \\
- (\phi A^2\phi + A^2)W + \{ 2((\phi W)\alpha) - 3\alpha\mu + 2\mu\gamma \} \xi \\
+ \{ 2((\phi W)\mu) - 3\mu^2 \} W - 2(W\mu)\phi W
\] (15)
where the smooth function $\gamma$ is defined by $\gamma = g(AW, W)$. Putting $X = \phi W$ into (14) and using (2) and (8), we obtain
\[
2\mu \{ \phi \nabla_{\phi W} W + \nabla_W W \} = 2\{ \phi A\phi A + A\phi A \} W \\
+ 2\alpha \{ \phi A - \phi A \} W + \{ \phi A^2 - A^2 \phi \} W \\
+ 2\{ \mu g(AW,\phi W) - (W\alpha) \} \xi \\
- 2(W\mu)W - 2((\phi W)\mu)\phi W
\] (16)
If we apply $\phi$ to (16), then we have
\[
2\mu \{ \phi \nabla W W - \nabla_{\phi W} W \} = 2\{ \phi A\phi A - A\phi A\phi \} W + 2\alpha \{ \phi A\phi + A \} W \\
- \{ \phi A^2\phi + A^2 \} W - \{ 4\mu g(A\phi W,\phi W) + \alpha\mu - \mu\gamma \} \xi \\
+ 2((\phi W)\mu)W - 2(W\mu)\phi W.
\] (17)
Comparing this equation with (15) and (17), we can verify that
\[
\{ 2((\phi W)\alpha) - 2\alpha\mu + \mu\gamma + 4\mu g(A\phi W,\phi W) \} \xi = 3\mu^2 W.
\] (18)
If we take inner product of (18) with $W$ then we obtain $\mu = 0$ and it is a contradiction. Thus the set $\Omega$ is empty, and hence $M$ is a Hopf hypersurface.

4 Result of Main Theorem

We shall prove Main Theorem given in the introduction, that is, as the characterization of Hopf hypersurface: we can state:

**Proof of Main Theorem** By Lemma 3.3, $M$ is a Hopf hypersurface in $M_n(c)$. Since $\xi$ is a Reeb vector field, the assumption $(\mathcal{L}_\xi L_\xi)X + (\nabla_\xi L_\xi)X = 0$ is equivalent to
\[
2(\nabla_\xi L_\xi)X = \phi AL_\xi X + L_\xi \phi AX
\]
for any vector field $X$ on $D$. Since we have $(\nabla_\xi L_\xi)X = \nabla_\xi (L_\xi X) - L_\xi (\nabla_\xi X)$ and make use of Codazzi equation, we see that the above equation is given by
\[
2\{ A\phi A\phi X - \phi A\phi AX \} + \phi A^2\phi X + A^2X - 2\alpha \{ AX + \phi AX \}
\]
\[- \alpha^2 \eta(X)\xi + 2((\phi X)\alpha)\xi
\] (19)
For any vector field $X$ on $D$ such that $AX = \lambda X$, it follows from (10) that

$$(\lambda - \frac{\alpha}{2})A\phi X = \frac{1}{2}(\alpha\lambda + \frac{c}{2})\phi X.$$ 

(20)

If $\lambda \neq \frac{\alpha}{2}$, then we see from (20) that $\phi X$ is also a principal direction in $D$, say $A\phi X = \mu \phi X$. From (12), we have

$$(\lambda - \mu)(\lambda + \mu - 2\alpha) = 0.$$ 

(21)

If $\lambda = \mu$, then $M$ has the two roots of the quadratic equation $\lambda^2 - 2\alpha\lambda - \frac{c}{4} = 0$ and hence $L_\xi X = (A\phi - \phi A)X = 0$ for any vector field $X$ in $D$.

In the following, if $\lambda + \mu = 2\alpha$, then we can divide into two cases (I) $c = 4$, ($M_n(c) = P_n(c)$) (II) $c = -4$, ($M_n(c) = H_n(c)$).

(I) In the case of $c = 4$, by the equation of (20), $M$ has the two roots of the quadratic equation $\lambda^2 - 2\alpha\lambda + (\alpha^2 + 1) = 0$. By the direct calculation, we know that the solution for the above equation does not exists.

(II) In the case of $c = -4$, from (20), $M$ has the two principal curvatures of the quadratic equation $\lambda^2 - 2\alpha\lambda + \alpha^2 - 1 = 0$. Therefore $M$ is locally congruent to one of model spaces of Takagi’s list and Montiel’s list. Since the model spaces $B$ do not admit when angle $= 0$ (see [7], pp 259-261).

If $\lambda = \frac{\alpha}{2}$, it is easily seen that $A\phi X = \phi AX$ for any vector field $X$ on $D$. Therefore, we have $L_\xi = \phi A - A\phi = 0$ on $D$ and $L_\xi \xi = (\phi A - A\phi)\xi = 0$. Statement of Main Theorem follows immediately from Theorem 1.

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